Czechoslovak Mathematical Journal

Hing Lun Chow

Maximal ideals in a semigroup of measures

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 2, 270–272

Persistent URL: http://dml.cz/dmlcz/101398

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

MAXIMAL IDEALS IN A SEMIGROUP OF MEASURES

H. L. CHOW, Hong Kong

(Received October 17, 1974)

In what follows S is a compact topological semigroup. A non-empty subset $I \subset S$ is called an ideal of S if $IS \subset I$ and $SI \subset I$. The ideal I is said to be maximal if it is proper and not properly contained in a proper ideal. Now let $P(S)$ denote the set of probability measures on S. It is well-known that $P(S)$ is a compact semigroup under convolution and the weak* topology, [2]. In this note we are concerned with maximal ideals in $P(S)$ and their intersection (which is $P(S)$ if $P(S)$ has no maximal ideal).

Let the support of a measure μ in $P(S)$ be denoted by supp μ. For $\mu_1, \mu_2 \in P(S)$, we have [2],

$$\operatorname{supp} \mu_1\mu_2 = \operatorname{supp} \mu_1 \operatorname{supp} \mu_2 .$$

Given a subset Δ of $P(S)$, let $\mathscr{S}(\Delta) = \bigcup_{\mu\in\Delta} \operatorname{supp} \mu$. It is clear that, for $\Delta_1, \Delta_2 \subset P(S)$.

$$\mathscr{S}(\Delta_1\Delta_2) = \mathscr{S}(\Delta_1)\,\mathscr{S}(\Delta_2) .$$

Therefore, if Δ is an ideal of $P(S)$, $\mathscr{S}(\Delta)$ is an ideal of S.

Proposition 1. *Every maximal ideal in $P(S)$ is dense.*

Proof. Since $P(S)$ is convex and so connected, the result follows from [5, p. 29].

Theorem 2. *Let Δ be a maximal ideal in $P(S)$. Then $\mathscr{S}(\Delta) = S$.*

Proof. Let $I = \mathscr{S}(\Delta)$ and suppose $I \neq S$. Take $a \in S \setminus I$ and let $\delta(a)$ be the unit point mass at a; then $\delta(a) \notin \tilde{I} = \{\mu \in P(S) : \operatorname{supp} \mu \cap I \neq \emptyset\}$. It is easily seen that $\tilde{I}$ is a proper ideal of $P(S)$ and $\Delta \subset \tilde{I}$. Accordingly we have $\tilde{I} = \Delta$, whence $\mathscr{S}(\tilde{I}) = I$. Pick $b \in I$ and let $\mu = \frac{1}{2}(\delta(a) + \delta(b))$. Since $\operatorname{supp} \mu = \{a, b\}$, we see that $\mu \in \tilde{I}$, giving $a \in \mathscr{S}(\tilde{I}) = I$. This contradiction proves the theorem.

Theorem 3. *Let ϕ be the intersection of all maximal ideals in $P(S)$. Then $\mathscr{S}(\phi) = S^2$.*

Proof. As shown in the first part of the proof of Corollary 3 in [4], $P(S)^2 \supset \phi$. This yields $S^2 = \mathscr{S}(P(S)^2) \supset \mathscr{S}(\phi)$. To prove the reverse inclusion, let $ab \in S^2$ where $a, b \in S$. Let $I = \mathscr{S}(\phi)$ which is evidently an ideal of S. If $a \in I$, $ab \in I$. Now suppose $a \notin I$ and we assert that $ab \in I$ also holds. Since $\delta(a) \notin \phi$, $\delta(a)$ does not belong to some maximal ideal Δ, say, of $P(S)$. Consider $\tilde{I} = \{\mu \in P(S) : \operatorname{supp} \mu \cap I \neq \emptyset\}$. Because $\delta(a) \notin \tilde{I}$, we see that $\tilde{I} \cup \Delta$ is a proper ideal of $P(S)$. It follows that $\tilde{I} \cup \Delta = \Delta$, whence $\tilde{I} \subset \Delta$. Pick $c \in I$ and let $\mu = \frac{1}{2}(\delta(b) + \delta(c))$. That $\operatorname{supp} \mu = \{b, c\}$ implies $\mu \in \tilde{I}$. By virtue of [6, Theorem 2], $\delta(a)\,\mu \in \phi$. Thus $ab \in \operatorname{supp} \delta(a)\,\mu \subset \mathscr{S}(\phi) = I$ as required.

Corollary 4. *Let F be the intersection of all maximal ideals in S. Then $\mathscr{S}(\phi) \supset \bar{F}$, where the bar denotes closure.*

Proof. Observe that $S^2 \supset F$, which implies $S^2 \supset \bar{F}$. Then apply the preceding theorem to complete the proof.

Example 5. The inclusion in the corollary above may be proper. Take the semigroup $S = \{0, 1\}$ with usual multiplication. Then $\mathscr{S}(\phi) = S^2 = S \neq \{0\} = F = \bar{F}$.

Corollary 6. *The set $\mathscr{S}(\phi)$ is an intersection of maximal ideals in S. Further, if each idempotent of S is contained in the minimal ideal of S then $\mathscr{S}(\phi)$ is the intersection of all maximal ideals of S.*

Proof. Since the intersection F of all maximal ideals of S is contained in $\mathscr{S}(\phi)$, the first part of the result is immediate from Theorem 6 of [3]. As for the second part, we note that $S^2 = F$ (see [4, Corollary 3]) and apply Theorem 3.

Proposition 7. $\mathscr{S}(\bar{\phi}) = \mathscr{S}(\phi)$.

Proof. Since $\mathscr{S}(\phi) = S^2$ by Theorem 3, we have $\mathscr{S}(\phi) = \overline{\mathscr{S}}(\phi)$. Moreover, $\overline{\mathscr{S}}(\bar{\phi}) = \overline{\mathscr{S}}(\phi)$ (cf. [2, p. 55]). It follows that $\mathscr{S}(\phi) = \overline{\mathscr{S}}(\phi) = \overline{\mathscr{S}}(\bar{\phi}) \supset \mathscr{S}(\bar{\phi}) \supset \supset \mathscr{S}(\phi)$, and the result is clear.

Following Grillet [3], we call the semigroup S intersective if the intersection F of all maximal ideals of S coincides with the minimal ideal K of S.

Proposition 8. *If $P(S)$ is intersective, then S is intersective.*

Proof. By assumption, ϕ is the minimal ideal of $P(S)$. It follows that $K \subset F \subset \subset \mathscr{S}(\phi) \subset \overline{\mathscr{S}}(\phi) = K$ (see, for example, Theorem 5 of [1]). Thus $F = K$, completing the proof.

We remark that the converse of the previous proposition is not true. For instance, consider the semigroup S given in Example 5. While S is intersective, $P(S)$ is not intersective, since $\phi = P(S) \setminus \delta(1)$ contains properly the minimal ideal $\{\delta(0)\}$ of $P(S)$.

References

[1] *H. L. Chow,* On supports of semigroups of measures, Proc. Edinburgh Math. Soc. (*2*) 19 (1974), 31—33.

[2] *I. Glicksberg,* Convolution semigroups of measures, Pacific J. Math. *9* (1959), 51—67.

[3] *P. A. Grillet,* Intersections of maximal ideals in semigroups, Amer. Math. Monthly *76* (1969), 503—509.

[4] *R. J. Koch* and *A. D. Wallace,* Maximal ideals in compact semigroups, Duke Math. J. *21* (1954), 681—685.

[5] *A. B. Paalman - de Miranda,* Topological semigroups, 2nd edition (Mathematisch Centrum, Amsterdam, 1970).

[6] *Š. Schwarz,* Prime ideals and maximal ideals in semigroups, Czech. Math. J. *19* (*94*) (1969), 72—79.

Author's address: Department of Mathematics, Chung Chi College, The Chinese University of Hong Kong.