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A NECESSARY AND SUFFICIENT CONDITION FOR CONTINUITY OF ADDITIVE FUNCTIONS

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In the sequel, a real-valued function f defined on the n-dimensional Euclidean space R^n is called to be *additive* if it satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in R^n$.

R. GER and M. KUCZMA [2] introduced the following set classes:

A set $T \subset R^n$ belongs to the class $\mathscr{B}$ if and only if each additive function upper-bounded on T is continuous.

A set $T \subset R^n$ belongs to the class $\mathscr{C}$ if and only if each additive function bounded (bilaterally) on T is continuous.

It is known that $\mathscr{B} \subset \mathscr{C}$ but $\mathscr{B} \neq \mathscr{C}$, see e.g. [2]. M. Kuczma [4] posed the problem to find some characterizations of the classes $\mathscr{B}$ and $\mathscr{C}$. The class $\mathscr{C}$ has been characterized in [5]. The main aim of the present note is to give a characterization of $\mathscr{B}$; this result is complemented by an example of a strange set belonging to $\mathscr{B}$.

Throughout the paper, the set of rational numbers will be denoted by Q. The symbols $+$, $-$ denote always the algebraic operations.

A set $A \subset R^n$ is called *Q-radial at a point* x_0 if for each $x \in R^n$ there is a real $c_x > 0$ such that $x_0 + \alpha x \in A$ whenever $\alpha \in Q$, $0 \leqq \alpha < c_x$.

A set $A \subset R^n$ is called *Q-convex* if for each $x, y \in A$, and each $\alpha \in Q$, $0 \leqq \alpha \leqq 1$, $\alpha x + (1 - \alpha) y \in A$. The Q-convex hull of a set $B \subset R^n$ (i.e. the minimal Q-convex set containing B) will be denoted by $Q(B)$.

Now we are able to prove the main result.

Theorem. *Let T be a subset of the n-dimensional Euclidean space R^n. Then each additive function $f : R^n \to R$ upper-bounded on T is continuous if and only if for each subset A of R^n, Q-radial at a certain point, the set $Q(T - A)$ contains a sphere.*

In other words, $T \in \mathscr{B}$ if and only if for each subset A of R^n, Q-radial at a point, the Q-convex hull of $T - A$ contains a ball.

P r o o f of the theorem is based on the following result of M. E. Kuczma [3]: *Let C be a Q-convex subset of R^n, Q-radial at a point; then either C contains a ball or there exists a discontinuous additive function upper-bounded on C.*

Let $T \subset R^n$ and let A be a subset of R^n, Q-radial at x_0 such that $C = Q(T - A)$ contains no ball. We may without loss of generality assume $T \neq \emptyset$. Since C is Q-convex and Q-radial (at each point of the set $T - x_0$) the above quoted result of M. E. Kuczma implies the existence of a discontinuous additive function $f : R^n \to R$ upper-bounded on $Q(T - A)$. Let a be a fixed point from A. Since $T - a \subset \subset Q(T - A)$, we conclude that f is upper-bounded on $T - a$, and consequently, by the additivity of f, f is upper-bounded on T. Thus $T \notin \mathscr{B}$.

Now assume that $Q(T - A)$ contains a ball for each subset A of R^n, Q-radial at a certain point. Let $f : R^n \to R$ be an additive function such that $f(x) < M$ for each $x \in T$. For each $y \in R^n$ let $A_y = \{\alpha y;\ \alpha \in Q, f(\alpha y) > -1\}$, and put

$$A = \bigcup_{y \in R^n} A_y .$$

Clearly, A is Q-radial at 0. For each $u \in T$, $v \in A$, $f(u - v) = f(u) - f(v) < M + 1$, thus f is upper-bounded on $T - A$ and consequently, f is bounded on $Q(T - A)$ (see e.g. [1]). Thus f is upper-bounded on a set with positive Lebesgue measure and so f is continuous (see e.g. [2]), q.e.d.

R e m a r k. It is easy to verify that in Theorem, the set $Q(T - A)$ can be replaced by $Q(T) - Q(A)$.

A set $A \subset R^n$ is called *midpoint convex* if $\frac{1}{2}(A + A) = A$. R. Ger and M. Kuczma [2] have proved the following result: *Let $T \subset R^n$. If the set $J(T) - J(T)$ has a positive inner Lebesgue measure then $T \in \mathscr{C}$ (here $J(A)$ denotes the midpoint convex hull of A).* The authors conjectured that this condition is not necessary for $T \in \mathscr{C}$. In [5] it is stated without proof that this conjecture is true. In the present note we give a somewhat stronger result, namely that this condition is not necessary for $T \in \mathscr{B}$.

E x a m p l e. Let H be a Hamel basis of the reals and let T be the set of all numbers of the form $\sum \alpha_i h_i$ (finite sum) where $h_i \in H$, and α_i are dyadic rational numbers (i.e. $\alpha_i = m_i \cdot 2^{n_i}$, where m_i, n_i are integers).

It is easy to verify that $T \in \mathscr{B}$. Clearly T is midpoint convex and so $J(T) - J(T) = T - T = T$. Now we show that the inner Lebesgue measure of T is 0.

Since H is a Hamel basis, 1 can be written uniquely (up to the order of summands) as

$$1 = \alpha_1 h_1 + \alpha_2 h_2 + \ldots + \alpha_n h_n , \tag{1}$$

where $h_i \in H$, $\alpha_i \in Q$, $i = 1, 2, \ldots, n$. Assume that $\alpha_1 = u/v$, where u, v are relatively prime integers. For each prime integer q, $q > u$, let A_q be the set $T + q^{-1}$. We show that the sets A_q are pairwise disjoint. Assume, on the contrary, that there are two prime integers $p > q$ greater than u such that $A_p \cap A_q$ is non-empty. Then $p^{-1} - q^{-1} = (p - q)/pq \in T$. On the other hand, from (1) we have

$$\frac{p-q}{pq} = \frac{p-q}{pq} \cdot \frac{u}{v} \cdot h_1 + \frac{p-q}{pq} \cdot (\alpha_2 h_2 + \ldots + \alpha_n h_n) .$$

This representation of $(p - q)/pq$ is unique so $((p - q)/pq)(u/v)$ must be a dyadic rational number. But this is impossible since $(p - q) u$ is not divisible by p. Thus the sets A_q are pairwise disjoint. Now if the inner Lebesgue measure $m_i(T)$ of T is positive then there is a finite interval $I \subset R$ and $\varepsilon > 0$ such that for each sufficiently large prime q, $m_i(I \cap A_q) > \varepsilon$. But in this case $m_i(I) = +\infty$ – a contradiction. Hence $m_i(T) = 0$, q.e.d.

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