Václav Havel

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HOMOMORPHISMS OF NETS OF FIXED DEGREE, WITH SINGULAR POINTS ON THE SAME LINE

VÁCLAV HAVEL, Brno

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Our aim is to collect our first results on projective homomorphisms of nets with fixed prescribed degree and with singular points on the same line. In § 1 we present some "synthetic" properties of images and pre-images under projective net homomorphisms whereas in § 2 we attempt to develop an algebraic method of their description.

Our results generalize a brief sketch about affine homomorphisms of finite nets of fixed degree from R. H. Bruck's paper [1], p. 102. Projective homomorphisms of arbitrary nets of degree 4 (for all three kinds of such nets: with all singular points on the same line, with just three singular points on the same line and with no three singular points on the same line) are studied by algebraic methods in our work [2].

1. GEOMETRIC PART

A *net* is defined as a triple $(\mathscr{P}, \mathscr{L}, (V_\iota)_{\iota \in \mathscr{I}})$ where $\mathscr{P}$ is a set, $\mathscr{L}$ is a set of at least two-element subsets of $\mathscr{P}$, $\mathscr{I}$ is a non-void set and $\iota \mapsto V_\iota$ is an injective mapping of $\mathscr{I}$ into $\mathscr{P}$ such that the following conditions are satisfied:

(i) $v := \{V_\iota \mid \iota \in \mathscr{I}\} \in \mathscr{L}$,

(ii) $\forall P \in \mathscr{P} \setminus v \quad \forall \iota \in \mathscr{I} \quad \exists!\, l \in \mathscr{L} \quad P, V_\iota \subset l$,

(iii) $\forall\, l \in \mathscr{L} \setminus \{v\} \quad \exists!\, \iota \in \mathscr{I} \quad V_\iota \in l$,

(iv) $\forall l_1, l_2 \in \mathscr{L} \setminus \{v\};\ l_1 \neq l_2 \quad \#(l_1 \cap l_2) = 1$.

The elements of $\mathscr{P}$ are called *points*, the elements of $\mathscr{L}$ are called *lines*[1]), the points of v are said to be *singular* or *improper*, the points of $\mathscr{P} \setminus v$ are said to be *non-singular* or *proper*, the line v is said to be *improper*, the lines of $\mathscr{L} \setminus \{v\}$ are said

[1]) If $A, B \in c$ holds for distinct $A, B \in \mathscr{P}$ and for $c \in \mathscr{L}$ so we shall write $AB := c$ and call c the *join line* of A, B. If $C \in a, b$ holds for distinct $a, b \in \mathscr{L}$ and for $C \in \mathscr{P}$ then we shall write $C =: a \sqcap b$ and call C the *intersection point* of a, b.

to be *proper*, the proper lines through the point V_ι are said to be *ι-lines* (for every $\iota \in \mathscr{I}$), the index set $\mathscr{I}$ (or its cardinality) is called *degree* of the given net. Every net is easily seen to be a regular incidence structure (for the definition of an incidence structure cf. [3], p. 6).

In the sequel we shall restrict our study to nets of fixed degree $\mathscr{I}$ with $\#\mathscr{I} \geqq 3$. If we speak of a net $\mathscr{N}$ (or $\mathscr{N}$ with a label, e.g. $\mathscr{N}'$) then we put automatically $\mathscr{N} =: (\mathscr{P}, \mathscr{L}, (V_\iota)_{\iota\in\mathscr{I}})$, $v := \{V_\iota \mid \iota \in \mathscr{I}\}$ (or with the label used, e.g. $\mathscr{N}' =: (\mathscr{P}', \mathscr{L}', (V'_\iota)_{\iota\in\mathscr{I}})$, $v' := \{V'_\iota \mid \iota \in \mathscr{I}\}$).

If $\mathscr{N}$ is a net, then $\#(l \setminus v)$ is the same for all proper lines $l \in \mathscr{L}$ and is called *order* of $\mathscr{N}$. Nets with orders 0, 1 are said to be *trivial*, other nets are *non-trivial*. It is well-known that $\#\mathscr{I} \leqq$ order of $\mathscr{N} + 1$ if $\mathscr{N}$ is non-trivial.

Let $\mathscr{N}$, $\mathscr{N}'$ be nets. By a *homomorphism* of $\mathscr{N}$ into $\mathscr{N}'$ we shall mean a mapping $\pi : \mathscr{P} \to \mathscr{P}'$ for which

(i) $V_\iota^\pi = V'_\iota \quad \forall \iota \in \mathscr{I}$, and

(ii) $\forall l \in \mathscr{L} \quad \exists \bar{l} \in \mathscr{L}' \quad l^\pi \subseteq \bar{l}$ [2]).

If in addition $V_\iota'^{\pi^{-1}} = \{V'_\iota\} \quad \forall_\iota \in \mathscr{I}$ then the homomorphism π is called *affine*. If on the other hand $\mathscr{P}^\pi = \mathscr{P}'$, we speak of *epimorphism*; in π is bijective and if also π^{-1} is epimorphism, then we speak of *isomorphism*.

If $\mathscr{N} = \mathscr{N}$ then π is said to be an *endomorphism* of $\mathscr{N}$; endomorphism which is simultaneously epimorphism is called *meromorphism*; endomorphism which is simultaneously an isomorphism is called *automorphism*.

If the given nets satisfy $\mathscr{P} \subseteq \mathscr{P}'$ then a homomorphism π of $\mathscr{N}$ into $\mathscr{N}'$ satisfying $P^\pi = P \quad \forall P \in \mathscr{P}$ (which is thus the mapping $\mathrm{id}_\mathscr{P}$) is called *embedding* and we say also that $\mathscr{N}$ is *embedded* into $\mathscr{N}'$ or that $\mathscr{N}$ is a *sub-net* of $\mathscr{N}'$.

Proposition 1. *Let π be a homomorphism of a net $\mathscr{N}$ into a net $\mathscr{N}'$. Then $(\mathscr{P}^\pi, \{l^\pi \mid l \in \mathscr{L}\}, (V'_\iota)_{\iota\in\mathscr{I}})$ is a sub-net of $\mathscr{N}'$.*

Proof. If $\#(\mathscr{P}^\pi \setminus v') = 0$ or 1 then the result is obvious. So let $\#(\mathscr{P}^\pi \setminus v') \geqq 2$. We need to verify conditions (i)–(iv):

(i) Trivial.

(ii) Let $P^\pi \notin v'$ for some $P \in \mathscr{P}$. Further let $\iota \in \mathscr{I}$. Then there exists a line $\bar{l} \in \mathscr{L}'$ such that $P^\pi, V'_\iota \in \bar{l}$ (and it is determined uniquely). This implies that there is just one $\tilde{l} \in \{l^\pi \mid l \in \mathscr{L}\}$ such that $P^\pi, V'_\iota \in \tilde{l}$.

(iii) Let $l^\pi \neq v'$ [3]) be given where $l \in \mathscr{L} \setminus \{v\}$ with $V_\alpha \in l$ for some (uniquely determined) $\alpha \in \mathscr{I}$. Thus $V'_\alpha \in l^\pi$. Suppose there exists an index $\beta \in \mathscr{I} \setminus \{\alpha\}$ so that also $V'_\beta \in l^\pi$. Then $V'_\beta = Q^\pi$ for some $Q \in l \setminus \{V_\alpha\}$ and we have $l^\pi \subseteq v'$. By hypothesis $l^\pi \neq v'$ it must be even $l^\pi \subset v'$. Thus there is an index $\gamma \in \mathscr{I} \setminus \{\alpha, \beta\}$ such that

[2]) If $\alpha : A \to A'$ is a mapping then for every $B \subseteq A$ we define $B^\alpha := \{b^\alpha \mid b \in B\}$.

[3]) It is clear that $v' = v^\pi \in \{l^\pi \mid l \in \mathscr{L}\}$.

$V_\gamma' \notin l^\pi$. Now we join every point $P \in \mathscr{P} \setminus v$ with V_γ by a line $\hat{l}$ which intersects l at the point $\hat{P}$. As $\hat{P}^\pi \in v' \setminus \{V_\gamma'\}$ it follows $\hat{l}^\pi \subseteq v'$ and consequently $\mathscr{P}^\pi = v'$, a contradiction. We have proved that V_α' is the only singular point on l^π. (Fig. 1.)

(iv) Let l_1^π, l_2^π be distinct sets not equal to v' for some $l_1, l_2 \in \mathscr{L} \setminus \{v\}$. Then $l_1 \neq l_2$. For $P := l_1 \sqcap l_2$ we have $P^\pi \in l_1^\pi \cap l_2^\pi$. Suppose there is another point $Q' \in l_1^\pi \cap l_2^\pi$. Then l_1^π, l_2^π are contained in the same line $l' \in \mathscr{L}'$.

If $l' \neq v'$ then l_1^π, l_2^π contain two different singular points P^π Q' which contradicts (iii).

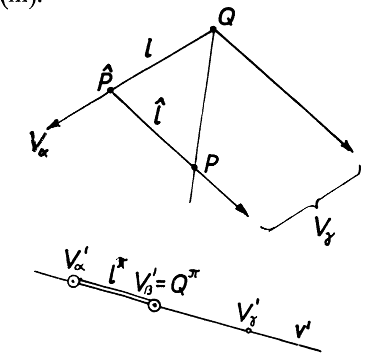

Fig. 1.

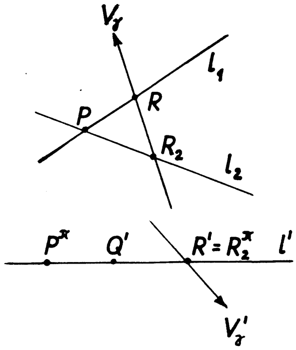

Fig. 2.

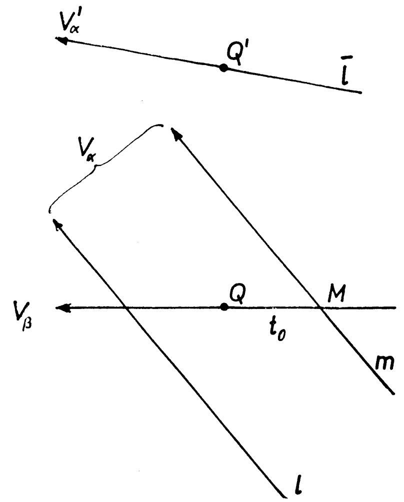

Fig. 3.

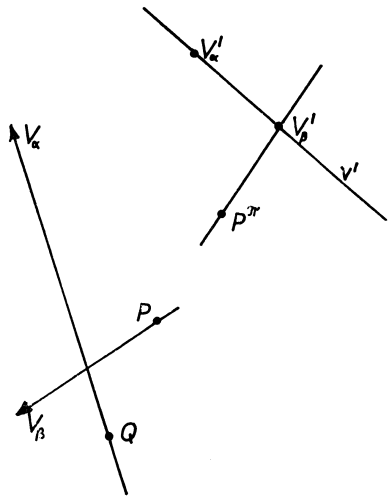

Fig. 4.

If $l' \neq v'$ then there is a point R' on one of l_1^π, l_2^π which does not belong to the other; say $R' \in l_1^\pi \setminus l_2^\pi$. Now take points $R \in R'^{\pi^{-1}} \cap l_1$, $V_\gamma \in v \setminus (l_1 \cup l_2)$ and observe the point $R_2 := RV_\gamma \sqcap l_2$. Then $R_2^\pi = R'$ contrary to the assumption about R'. (Fig. 2.) ■

Proposition 2. *Let $\mathcal{N}$, $\mathcal{N}'$ be non-trivial nets and π an epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$. Then $l \in \mathcal{L} \Rightarrow l^\pi \in \mathcal{L}'$.*

Proof. Let $l \in \mathcal{L}$. If $l = v$ so it is at once $l^\pi = v'$. Therefore we may assume $l \neq v$. Then there is an $\alpha \in \mathcal{I}$ with $V_\alpha \in l$ and a line $\bar{l} \in \mathcal{L}'$ with $l^\pi \subseteq \bar{l}$. Let $\bar{l} = v'$. Then assume there exists a $\beta \in \mathcal{I} \setminus \{\alpha\}$ such that $V'_\beta \notin l^\pi$ and take a point $P \in \mathcal{P}$ with $P^\pi \notin v'$. Then $(PV_\beta \sqcap l)^\pi \in v' \setminus \{V'_\beta\}$ and consequently $(PV_\beta)^\pi \subseteq v'$, a contradiction to $P^\pi \notin v'$. (Fig. 3.)

Let $\bar{l} \neq v'$. Then $V_\alpha \in \bar{l}$. Suppose there is a point $Q' \in \bar{l} \setminus l^\pi$. Thus $Q'^{\pi^{-1}} \cap l = \emptyset$. Choose an arbitrary point $Q \in Q'^{\pi^{-1}}$. If $t \in \mathcal{L}$, $Q \in t$ then $t^\pi \subseteq \bar{l}$ because of $Q^\pi = Q' \in \bar{l}$, $(t \sqcap l)^\pi \in \bar{l} \setminus \{Q'\}$. This is valid especially for the line $t_0 = QV_\beta$ for some $\beta \in \mathcal{I} \setminus \{\alpha\}$. Now investigate all $m \in \mathcal{L} \setminus \{v\}$ with $V_\alpha \in m$. Then for $M := t \sqcap m$ it follows $M^\pi \neq V'_\alpha$: Indeed, if the contrary case $M^\pi = V'_\alpha$ occurs then $t_0^\pi \subseteq v'$ because of $M^\pi = V'_\alpha \neq V_\beta^\pi = V'_\beta$, in contradiction to $Q^\pi \in v'$. Thus $m^\pi \subseteq \bar{l}$ and consequently $\mathcal{P}^\pi \subseteq \bar{l}$ which contradicts the definition of a net. So $\bar{l} = l^\pi$. (Fig. 4.) ∎

Proposition 3. *Let π be a bijective epimorphism of a net $\mathcal{N}$ onto a net $\mathcal{N}'$. Then π is an isomorphism.*

Proof. If the nets $\mathcal{N}$, $\mathcal{N}'$ are both trivial then the conclusion is clear. So let $\mathcal{N}$, $\mathcal{N}'$ be non-trivial. It is easily seen that $v'^{\pi^{-1}} = v$. Investigate any line $l' \in \mathcal{L} \setminus \{v'\}$. Does it exist a line $\bar{l} \in \mathcal{L}$ such that $l'^{\pi^{-1}} \subseteq \bar{l}$?

Let $V'_\alpha \in l'$ for some $\alpha \in \mathcal{I}$. Then $V'^{\pi^{-1}}_\alpha = V_\alpha$. Further choose a point $Q' \in l' \setminus \{V'_\alpha\}$ and denote $Q := Q'^{\pi^{-1}}$, $l := QV_\alpha$. We know that $l^\pi = l'$ and since π is bijective it must be also $l = l'^{\pi^{-1}}$. The proof is complete. (Fig. 5.) ∎

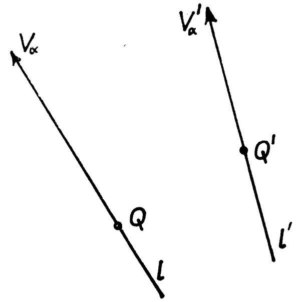

Fig. 5.

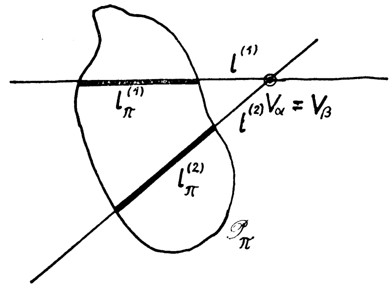

Fig. 6.

Proposition 4. *Let $\mathcal{N}$, $\mathcal{N}'$ be non-trivial nets and π an epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$. Define $\mathcal{P}_\pi := \{X \in \mathcal{P} \mid X^\pi \in v'\} \cup \{V_\iota \mid \iota \in \mathcal{I}\}$, $\mathcal{L}_\pi := \{l \cap \mathcal{P}_\pi \mid l \in \mathcal{L}, \#(l \cap \mathcal{P}_\pi) \geqq 2\}$, $\mathcal{N}_\pi := (\mathcal{P}_\pi, \mathcal{L}_\pi, (V_\iota)_{\iota \in \mathcal{I}})$. Then $\mathcal{N}_\pi$ is a sub-net of $\mathcal{N}$ and $\pi|_{\mathcal{P}_\pi}$ is an affine epimorphism of $\mathcal{N}_\pi$ onto $\mathcal{N}'$.*

Proof. We shall verify the fulfillment of conditions (i)–(iv) from the definition of a net for $\mathcal{N}_\pi$:

(i) is trivial;

(ii) follows immediately from the definition of $\mathcal{P}_\pi$ and $\mathcal{N}_\pi$;

(iii) is also an immediate corollary of definition of $\mathcal{P}_\pi$ and $\mathcal{N}_\pi$.

The only non-obvious condition is (iv): Let us have distinct $l_\pi^{(1)}, l_\pi^{(2)} \in \mathcal{L}_\pi \setminus \{v\}$ where $l_\pi^{(1)} = l^{(1)} \cap \mathcal{P}_\pi$, $l_\pi^{(2)} = l^{(2)} \cap \mathcal{P}_\pi$ are at least two-element sets for distinct $l^{(1)}, l^{(2)} \in \mathcal{L} \setminus \{v\}$ and $V_\alpha \in l_\pi^{(1)}$, $V_\beta \in l_\pi^{(2)}$ for $\alpha, \beta \in \mathcal{I}$. If $\alpha = \beta$ then $l_\pi^{(1)} \cap l_\pi^{(2)} =$

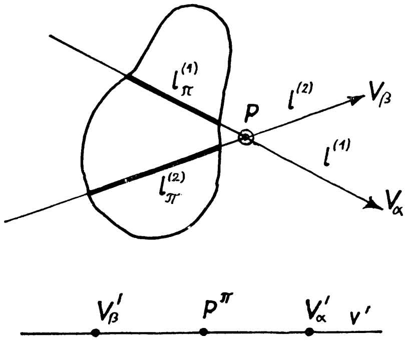

Fig. 7.

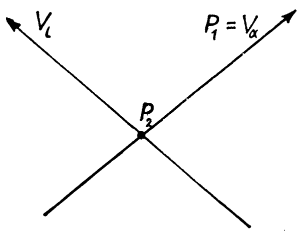

Fig. 8.

$= \{V_\alpha\}$. (Fig. 6.) If $\alpha \neq \beta$ then assume $l_\pi^{(1)} \cap l_\pi^{(2)} = \emptyset$. We know that then $l^{(1)} \cap l^{(2)} = \{P\}$ for a point $P \in v'^{\pi^{-1}}$. But either $P^\pi \neq V'_\alpha$ or $P^\pi \neq V'_\beta$; let e.g. $P^\pi \neq V'_\beta$. Then consequently $l^{(2)\pi} = v'$ and $\#(l^{(2)} \cap \mathcal{P}_\pi) = 1$, a contradiction. Thus $\#(l_\pi^{(1)} \cap l_\pi^{(2)}) \geqq 1$. Since $\#(l_\pi^{(1)} \cap l_\pi^{(2)}) \leqq \#(l^{(1)} \cap l^{(2)}) \leqq 1$, we have $\#(l_\pi^{(1)} \cap l_\pi^{(2)}) = 1$ as expected. (Fig. 7.)

The conclusion about the sub-net $\mathcal{N}_\pi$ and about the affine epimorphism $\pi|_{\mathcal{P}_\pi}$ of $\mathcal{N}_\pi$ onto $\mathcal{N}'$ is obvious. ■

Corollary. *If $\mathcal{N}$, $\mathcal{N}'$ from Proposition* 4 *are simultaneously projective planes then π is necessarily an isomorphism.*

Proof. Assume that $P_1^\pi = P_2^\pi$ for distinct $P_1, P_2 \in \mathcal{P}$. If $P_1^\pi = P_2^\pi$ is an improper point V'_α, $\alpha \in \mathcal{I}$, so it can be assumed without loss of generality that $P_1 = V_\alpha$ and P_2 is a proper point. Then $(P_2 V_\iota)^\pi = v'$ for all $\iota \in \mathcal{I} \setminus \{\alpha\}$ and consequently $\mathcal{P}^\pi = v'$, which is a contradiction. (Fig. 8.) If $P_1^\pi = P_2^\pi \notin v'$ then take any distinct $\alpha, \beta \in \mathcal{I}$. Then $(P_1 V_\alpha)^\pi = P_1^\pi V'_\alpha$, $(P_2 V_\beta)^\pi = P_2^\pi V'_\beta$ implies $(P_1 V_\alpha \sqcap P_2 V_\beta)^\pi = P_1^\pi$. Thus $Q^\pi = P_1^\pi$ for all $Q \in \mathcal{P} \setminus (v \cup P_1 P_2)$. Replacing P_2 by a proper point from $P_1 P_2$ we get $Q_1^\pi = P_1^\pi$ for all $Q \in \mathcal{P} \setminus v$. (Fig. 9.) But this contradicts the hypothesis that $\mathcal{N}'$ is non-trivial. ■

Proposition 5. *Let $\mathcal{N}$, $\mathcal{N}'$ be non-trivial nets and π an affine epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$. For every point $X' \in \mathcal{P}' \setminus v'$ define $\mathcal{P}_{X'} := X'^{\pi^{-1}} \cup v$, $\mathcal{L}_{X'} := \{l \cap \mathcal{P}_{X'} \mid l \in \in \mathcal{L},\ \#(l \cap \mathcal{P}_{X'}) \geqq 2\}$, $\mathcal{N}_{X'} := (\mathcal{P}_{X'}, \mathcal{L}_{X'}, (V_\iota)_{\iota \in \mathcal{I}})$. Then $\{\mathcal{N}_{X'} \mid X' \in \mathcal{P}' \setminus v'\}$ is a set of sub-nets of $\mathcal{N}$ having the same order.*

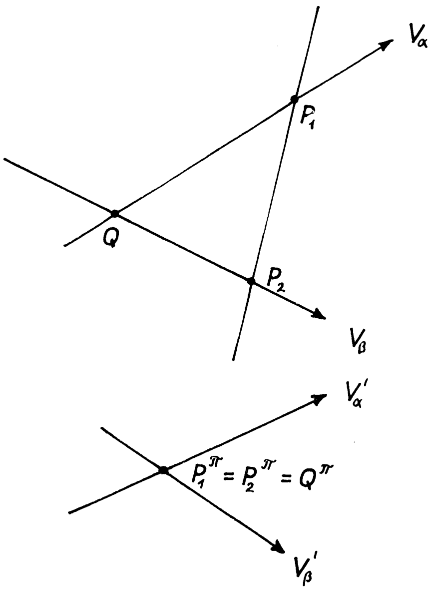

Fig. 9.

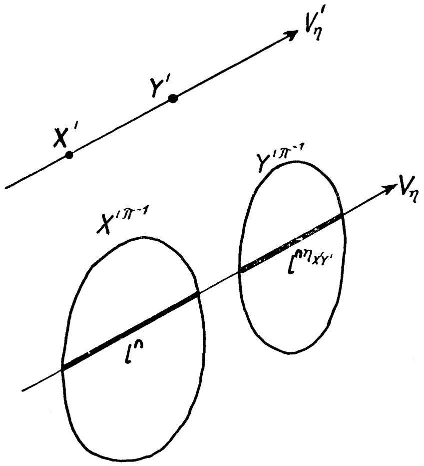

Fig. 10.

Proof. Let $X' \in \mathcal{P}' \setminus v'$. Then $\#(\mathcal{P}_{X'} \setminus v) \geqq 1$ and we verify that the conditions (i)–(iv) are satisfied from the definition of a net for $\mathcal{N}_{X'}$:

(i) is obvious.

(ii) If $X \in \mathcal{P}_{X'} \setminus v$ then for every $\iota \in \mathcal{I}$ we have $(XV_\iota)^\pi = X'V'_\iota$ and thus $XV_\iota \cap \mathcal{P}_{X'}$ is the desired (and unique, as it can be easily shown) element of $\mathcal{L}_{X'}$ through V_ι which contains X.

(iii) Let $l^\cap \in \mathcal{L}_{X'} \setminus \{v\}$. Then $l^\cap = l \cap \mathcal{P}_{X'}$ for a unique $l \in \mathcal{L}$ and consequently the improper point of l is the unique improper point on $l^\cap$.

(iv) Let $l_1^\cap$, $l_2^\cap$ be distinct elements of $\mathcal{L}_{X'} \setminus \{v\}$ with their improper points V_α, V_β for suitable $\alpha, \beta \in \mathcal{I}$. Denote by l_1, l_2 the lines from $\mathcal{L}$ such that $l_1^\cap \subseteq l_1$, $l_2^\cap \subseteq l_2$. Then l_1^π, l_2^π go through X'. If $\alpha = \beta$ then obviously V_α is the unique common point of $l_1^\cap, l_2^\cap$. If $\alpha \neq \beta$ then $V'_\alpha \in l_1^\pi$, $V'_\beta \in l_2^\pi$. Since $(l_1 \sqcap l_2)^\pi = l_1^\pi \sqcap l_2^\pi = X'$ it follows also $l_1^\cap \cap l_2^\cap = \{l_1 \sqcap l_2\}$. The first part of the assertion is proved.

Now about the order of various $\mathcal{N}_{X'}$: Choose two (auxiliary) indices $\xi, \eta \in \mathcal{I}$; $\xi \neq \eta$. If X', Y' are proper points of $\mathcal{N}'$ lying on the same η-line of $\mathcal{N}'$ we see that any η-line of $\mathcal{N}$ intersects either both sets $X'^{\pi^{-1}}$, $Y'^{\pi^{-1}}$ or no one of them. Thus we establish a bijection $\eta_{X',Y'}$ of the set $\{l^\cap \in \mathcal{L}_{X'} \setminus \{v\} \mid V_\eta \in l^\cap\}$ onto $\{l^\wedge \in \mathcal{L}_{Y'} \setminus \{v\} \mid V_\eta \in l^\wedge\}$ by the requirement that $l^\cap$, $l^{\cap \eta_{X',Y'}}$ are contained in the same η-line of $\mathcal{N}$ for every $l^\cap \in \mathcal{L}_{X'} \setminus \{v\}$ passing through V_η. (Fig. 10.)

Similarly we establish a bijection $\xi_{X',Y'}$ of $\{l^\cup \in \mathcal{L}_{X'} \setminus \{v\} \mid V_\xi \in l^\cup\}$ onto $\{l^\vee \in \mathcal{L}_{Y'} \setminus \{v\} \mid V_\xi \in l^\vee\}$ for any two proper points X', Y' of $\mathcal{N}'$ lying on the same ξ-line of $\mathcal{N}'$. Using these bijections we verify easily that all $\mathcal{N}_{X'}$ have the same order. ∎

Proposition 6. *Let $\mathcal{N}'$ be a non-trivial net and for all $X' \in \mathcal{P}' \setminus v'$ let $\mathcal{N}_{X'} = (\mathcal{P}_{X'}, \mathcal{L}_{X'}, (V_\iota)_{\iota \in \mathcal{I}})$ be a net of the same order $\nu \geqq 1$ such that $(\mathcal{P}_{X'} \setminus \{V_\iota \mid \iota \in \mathcal{I}\}) \cap (\mathcal{P}_{Y'} \setminus \{V_\iota \mid \iota \in \mathcal{I}\}) = \emptyset$ for all distinct X', $Y' \in \mathcal{P}' \setminus v'$. Then there is a net $\mathcal{N}$ and an affine epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$ with $X'^{\pi^{-1}} = \mathcal{P}_{X'}$ for all $X' \in \mathcal{P}' \setminus v'$.*

Proof. For every $X' \in \mathcal{P}' \setminus v'$, $\iota \in \mathcal{I}$, choose a bijection $\lambda_{X',\iota}$ of a set S with cardinality ν onto the set of just all ι-lines of $\mathcal{N}_{X'}$. Now define

$$\mathcal{P} := \bigcup_{X' \in \mathcal{P}' \setminus v'} \mathcal{P}_{X'}, \quad \mathcal{L} := \Big(\bigcup_{X' \in l' \setminus v'} s^{\lambda_{X',\iota}} \;\Big|\; V'_\iota \in l' \in \mathcal{L}' \setminus \{v'\},\ \iota \in \mathcal{I},$$

$$s \in S\Big\} \cup \{V_\iota \mid \iota \in \mathcal{I}\}, \quad \mathcal{N} := (\mathcal{P}, \mathcal{L}, (V_\iota)_{\iota \in \mathcal{I}}).$$

We shall verify the condition (i)–(iv) from the definition of a net: (i) is obvious and (ii)–(iii) follow from the definition of $\mathcal{P}$ and $\mathcal{L}$. To verify (iv) assume that l_1, l_2 are distinct elements of $\mathcal{L} \setminus \{\{V_\iota \mid \iota \in \mathcal{I}\}\}$ with $V_\alpha \in l_1$, $V_\beta \in l_2$ for uniquely determined $\alpha, \beta \in \mathcal{I}$. If $\alpha = \beta$ then $l_1 \cap l_2 = \{V_\alpha\}$ immediately by the definition of $\mathcal{P}$ and $\mathcal{L}$. If $\alpha \neq \beta$ then write l_1, l_2 in the explicit form $l_1 = \bigcup_{X' \in l_1' \setminus v'} s_1^{\lambda_{X',\alpha}}$, $l_2 = \bigcup_{X' \in l_2' \setminus v'} s_2^{\lambda_{X',\beta}}$ for some $s_1, s_2 \in S$; $l'_1, l'_2 \in \mathcal{L}' \setminus \{v'\}$. Here l'_1, l'_2 are distinct and have a one-point intersection in $\mathcal{N}'$ so that also $\#(l_1 \cap l_2) = 1$. Thus $\mathcal{N}$ is a net. From the definition of nets $\mathcal{N}_{X'}$ it follows at once that $\mathcal{N}_{X'}$ is a sub-net of $\mathcal{N}$ for all $X' \in \mathcal{P}' \setminus v'$.

Now define the mapping $\pi : \mathcal{P} \to \mathcal{P}'$ by $X^\pi = X'$ for all $X \in \mathcal{P}_{X'} \setminus \{V_\iota \mid \iota \in \mathcal{I}\}$ and by $V_\iota^\pi = V'_\iota$ for all $\iota \in \mathcal{I}$. Then π is an affine epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$ which follows also immediately from the definition of $\mathcal{P}$ and $\mathcal{L}$. ∎

Remark 1. A special case of Proposition 6 occurs if $\mathcal{N}_{X'}$ are mutually isomorphic.[4]) Then we can introduce bijections $\lambda_{X',\iota}$ as follows: Take a point $O' \in \mathcal{P}' \setminus v'$ and an isomorphism $\pi_{X'}$ of $\mathcal{N}_{O'}$ onto $\mathcal{N}_{X'}$ for all $X' \in \mathcal{P}' \setminus v'$. Let S be a set with cardinality ν and let bijections $\lambda_{O',\iota}$ be chosen arbitrarily for all $\iota \in \mathcal{I}$. Then define $\lambda_{X',\iota}$ for all $X' \in \mathcal{P}' \setminus v'$, $\iota \in \mathcal{I}$ in such a way that to every $s \in S$ the corresponding line is $(s^{\lambda_{O',\iota}})^{\pi_{X'}}$.

Remark 2. We can ask whether an affine epimorphism of a non trivial net onto another is uniquely determined by the full pre-image of one proper point. It would

[4]) This special case was studied independently by J. Bôrik (Brno).

be also interesting to know what are the full pre-images of improper points under an epimorphism of a non trivial net onto an other. We postpone these problems to the algebraic part of our investigations.

2. ALGEBRAIC PART

Let $\mathfrak{I}$ be a fixed non void set with one prominent index Θ (shortly: an index set.). Then an $\mathfrak{I}$-loop is a quadruple $(S, 0, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ where S is a set, 0 a distinguished element of S and for all $\iota \in \mathfrak{I}$, σ_ι is a permutation of S with $0^{\sigma_\iota} = 0$ and $+_\iota$ is a loop operation over S with the neutral element 0 such that

(i) $\sigma_\Theta = \mathrm{id}_S$,

(ii) $\forall \alpha, \beta \in \mathfrak{I};\ \alpha \neq \beta \quad \forall b, c \in S \quad \exists!\, a \in S \quad a^{\sigma_\alpha} +_\alpha b = a^{\sigma_\beta} +_\beta c.$

If $\mathfrak{L}$ is an $\mathfrak{I}$-loop then we shall write $\mathfrak{L} =: (S, 0, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ and if $\mathfrak{L}$ has a label then the same label will be used for the symbols in the brackets on the right hand side.

It can be readily shown that any $\mathfrak{I}$-loop $\mathfrak{L}$ satisfies $\#\mathfrak{I} + 1 \leqq \#S$ whenever $\#S > 1$.

We shall not introduce here the concept of sub-$\mathfrak{I}$-loop of a given $\mathfrak{I}$-loop. To give at least partial answer to the questions posed in Remark 2 we shall manage with the usual concept of sub-loop and normal subloop (cf. [1], pp. 60–61).

Now let $\mathfrak{L}$, $\mathfrak{L}'$ be $\mathfrak{I}$-loops. Under a *place* from $\mathfrak{L}$ onto $\mathfrak{L}'$ we shall mean a mapping Θ of a set $\mathrm{Dom}\,\Theta \subseteq S$ onto S with the following properties (where $\uparrow a :\Leftrightarrow a \in \mathrm{Dom}\,\Theta$ and $\downarrow a :\Leftrightarrow a \in S \setminus \mathrm{Dom}\,\Theta$):

(i) $\uparrow a, \uparrow b \Rightarrow a^{\sigma_\iota} +_\iota b \in ((a^\Theta)^{\sigma_\iota'} +_\iota' b^\Theta)^{\Theta^{-1}}$,

(ii) $\uparrow a, \downarrow b \vee \downarrow a, \uparrow b \Rightarrow \downarrow(a^{\sigma_\iota} +_\iota b) \quad \forall \iota \in \mathfrak{I}$,

(iii) $\downarrow a, \downarrow a^{\sigma_\alpha} +_\alpha b = a^{\sigma_\beta} +_\beta c$ for some $\alpha, \beta \in \mathfrak{I};\ \alpha \neq \beta \ \Rightarrow\ \downarrow b \vee \downarrow c$.

If, in addition, $\mathrm{Dom}\,\Theta = S$ then Θ is said to be an *epimorphism* of $\mathfrak{L}$ onto $\mathfrak{L}'$. If Θ is a bijective epimorphism then it is called *isomorphism*.

Proposition 7. *Let Θ be a place from an $\mathfrak{I}$-loop $\mathfrak{L}$ onto an $\mathfrak{I}$-loop $\mathfrak{L}'$. Then*

(iv) $0 \in 0'^{\Theta^{-1}}$,

(v) $\uparrow x \Leftrightarrow \uparrow x^{\sigma_\iota} \ \forall \iota \in \mathfrak{I}$,

(vi) $\uparrow(a^{\sigma_\iota} +_\iota b) \Rightarrow \uparrow a, \uparrow b \ \vee \ \downarrow a, \downarrow b.$

Proof. If $S' = \{0\}$ then (iv) is clear. Thus suppose there exists an element $e' \in S' \setminus \{0'\}$. Then there is also an element $e \in e'^{\Theta^{-1}}$. If $\downarrow 0$, then, by (ii), $\uparrow e$, $\downarrow 0$ implies $\downarrow 0 +_\Theta e = e$, a contradiction. Thus $\uparrow 0$ and, by (i), $0 +_\Theta 0 \in (0^\Theta +_\Theta' 0^\Theta)^{\Theta^{-1}} = 0'^{\Theta^{-1}}$. Suppose $\uparrow x$. Then, by (i) we have for every $\iota \in \mathfrak{I}$ $x^{\sigma_\iota} = x^{\sigma_\iota} +_\iota 0 \in ((x^\Theta)^{\sigma_\iota} +_\iota' 0^\Theta)^{\Theta^{-1}} = ((x^\Theta)^{\sigma_\iota})^{\Theta^{-1}}$ so that $\uparrow x^{\sigma_\iota}$. Suppose $\downarrow x$. Then, by (ii), $\downarrow x$, $\uparrow 0$ implies $\downarrow x^{\sigma_\iota} +_\iota 0 = x^{\sigma_\iota}$ for all $\iota \in \mathfrak{I}$. Condition (vi) is only a reformulation of (ii). ∎

Proposition 8. *Let Θ be a place from an $\mathfrak{J}$-loop $\mathfrak{L}$ onto an the loop $\mathfrak{L}'$. Then* $\mathbf{L}_1 := (\mathrm{Dom}\ \Theta,\ +_\Theta|_{(\mathrm{Dom}\,\theta)^2})$ *is a sub-loop of the loop* $\mathbf{L} := (S,\ +_\Theta)$ *and* $\mathbf{L}_2 := := (0'^{\Theta^{-1}},\ +_\Theta|_{(0'^{\Theta^{-1}})^2})$ *is a normal sub-loop of* $\mathbf{L}_1$.

Proof. If $a, b \in \mathrm{Dom}\ \Theta$ then $a +_\Theta b \in \mathrm{Dom}\ \Theta$ and if $a, b \in 0'^{\Theta^{-1}}$ then $a +_\Theta b \in \in 0'^{\Theta^{-1}}$ (by condition (i) from the definition of a place). If $a, b \in S$; $a, a +_\Theta b \in \in \mathrm{Dom}\ \Theta$ then also $b \in \mathrm{Dom}\ \Theta$ by condition (ii) from the definition of a place. Similarly $a, b \in S$; $b, a +_\Theta b \in \mathrm{Dom}\ \Theta \Rightarrow a \in \mathrm{Dom}\ \Theta$. Thus $\mathbf{L}_1$ is a loop. Analogously for $\mathbf{L}_2$. Now $\mathbf{L}_2$ is normal in $\mathbf{L}_1$ because the set of all elements of $\mathbf{L}_2$ is the full pre-image of the neutral element $0'$ of the loop $\mathbf{L}' = (S', +'_\Theta)$ under the epimorphism Θ of $\mathbf{L}_1$ onto $\mathbf{L}$. By [1b], p. 61, the decomposition of $\mathrm{Dom}\ \Theta$ onto full pre-images of elements of S' under the epimorphism Θ of $\mathbf{L}_1$ onto $\mathbf{L}'$ is either $\{0'^{\Theta^{-1}} +_\Theta x \mid x \in \mathrm{Dom}\ \Theta\}$ or $\{x +_\Theta 0'^{\Theta^{-1}} \mid x \in \mathrm{Dom}\ \Theta\}$.

In what follows either $\mathfrak{J}$ or $\mathscr{I}$ is fixed depending on the fact whether we start with an $\mathfrak{J}$-loop or a net. Let $\mathfrak{L}$ be an $\mathfrak{J}$-loop. Set $\mathscr{P} := S^2 \cup \mathfrak{J} \cup \{\xi, \eta\}$ (with disjoint summands), $\mathscr{L} := \{\{(x, y) \mid x = a\} \cup \{\xi\} \mid a \in S\} \cup \{\{(x, y) \mid y = b\} \cup \{\eta\} \mid b \in S\} \cup \cup \{\{(x, y) \mid y = x^{\sigma_\iota} +_\iota c\} \cup \{\iota\} \mid c \in S,\ \iota \in \mathfrak{J}\} \cup \{\mathfrak{J} \cup \{\xi, \eta\}\}$, $\mathscr{I} := \mathfrak{J} \cup \{\xi, \eta\}$, $\mathscr{N}_{\mathfrak{L},\xi,\eta} := (\mathscr{P}, \mathscr{L}, \mathscr{I})$. Then $\mathscr{N}_{\mathfrak{L},\xi,\eta}$ is a net (called the net *over* $\mathfrak{L}$). The proof is only a routine verification of axioms of a net for $\mathscr{N}_{\mathfrak{L},\xi,\eta}$.

Let $\mathscr{N}$ be a net of order $\geqq 1$, O its distinguished proper point; (ξ, η, ζ) a triple of pairwise distinct indices from $\mathscr{I}$. Let $S := OV_\xi \setminus \{V_\xi\}$. Define a bijection Π of $\mathscr{P} \setminus v$ onto S^2 which carries every proper point P onto a couple $((PV_\xi \sqcap OV_\zeta)\ V_\eta \sqcap OV_\xi,\ PV_\eta \sqcap OV_\xi)$. (Fig. 11.)

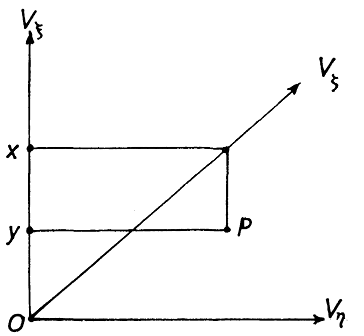

Fig. 11.

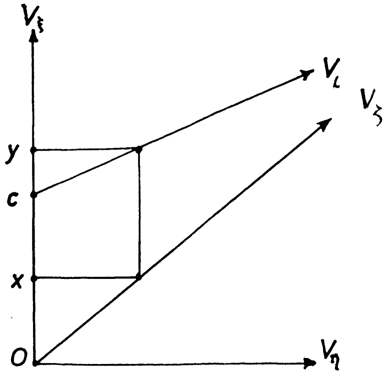

Fig. 12.

We shall introduce now an index set $\mathfrak{J} := \mathscr{I} \setminus \{\xi, \eta\}$ with a prominent index $\Theta := \zeta$. For all $\iota \in \mathfrak{J}$ define a permutation σ_ι of the set S in such a way that $\{(x, y)^{\Pi^{-1}} \mid y = x^{\sigma_\iota}\} \cup \{V_\iota\} = OV_\iota$ and a binary operation $+_\iota$ over S so that $\{(x, y)^{\Pi^{-1}} \mid y = x^{\sigma_\iota} +_\iota c\} \cup \{V_\iota\} = cV_\iota$. (Fig. 12.) Then it can be readily verified that $\mathfrak{L}_{\mathscr{N},O,\xi,\eta,\zeta} := (S, O, (\sigma_\iota)_{\iota \in \mathfrak{J}}, (+_\iota)_{\iota \in \mathfrak{J}})$ is an $\mathfrak{J}$-loop. Its name will be the *coordinatizing* $\mathfrak{J}$*-loop* of $\mathscr{N}$.

Proposition 9. a) *Let* $\mathfrak{L}$ *be an* $\mathfrak{J}$*-loop,* $\mathcal{N} := \mathcal{N}_{\mathfrak{L},\xi,\eta}$ *a net over* $\mathfrak{L}$ *and* $\mathfrak{L}' := := \mathfrak{L}_{\mathcal{N},(0,0),\xi,\eta,\Theta}$ *a coordinatizing loop of* $\mathcal{N}$. *Then* $\mathfrak{L}$ *and* $\mathfrak{L}'$ *are isomorphic.* b) *Let* $\mathcal{N}$ *be a net of order at least* 1, $\mathfrak{L} := \mathfrak{L}_{\mathcal{N},O,\xi,\eta,\zeta}$ *one of its coordinatizing* $\mathfrak{J}$*-loops and* $\mathcal{N}' := \mathcal{N}_{\mathfrak{L},\xi,\eta}$ *a net over* $\mathfrak{L}$. *Then* $\mathcal{N}$ *and* $\mathcal{N}'$ *are isomorphic.*

The proof is omitted.

Proposition 10. *Let* $\mathcal{N}, \mathcal{N}'$ *be non-trivial nets and* π *an epimorphism of* $\mathcal{N}$ *onto* $\mathcal{N}'$. *Choose a coordinatizing* $\mathfrak{J}$*-loop* $\mathfrak{L} := \mathfrak{L}_{\mathcal{N},O,\xi,\eta,\zeta}$ *and, respectively,* $\mathfrak{L}' := := \mathfrak{L}_{\mathcal{N}',O^\pi,\xi,\eta,\zeta}$. *Then* π *induces a place* $\hat{\pi}$ *from* $\mathfrak{L}$ *onto* $\mathfrak{L}'$.

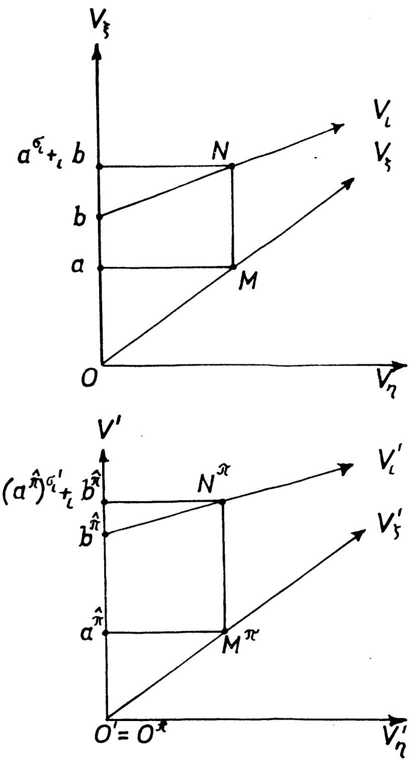

Fig. 13.

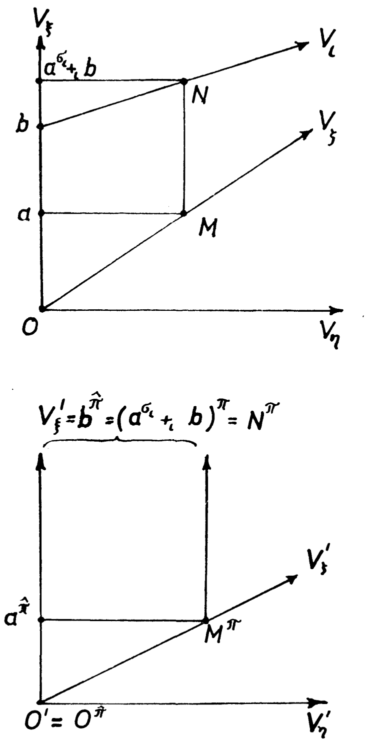

Fig. 14.

Proof. Define a mapping $\hat{\pi}$ of the set Dom $\hat{\pi} := S \setminus V_\xi^{\pi^{-1}}$ onto S' [5]) by $x \mapsto x^\pi$ for every $x \in$ Dom $\hat{\pi}$. We assert that $\hat{\pi}$ is a place from $\mathfrak{L}$ onto $\mathfrak{L}'$ so that we have to verify axioms (i)—(ii) from the definition of the place for $\hat{\pi}$ instead of Θ.

First we shall consider condition (i): Let $a, b \in$ Dom $\hat{\pi}$ and $\iota \in \mathfrak{J}$. Then $(aV_\eta)^\pi = a^{\hat{\pi}}V_\eta'$, $(OV_\zeta)^\pi = O'V_\zeta'$, $(aV_\eta \sqcap OV_\zeta)^\pi = a^{\hat{\pi}}V_\eta' \sqcap O'V_\zeta'$, $(bV_\iota)^\pi = b^{\hat{\pi}}V_\iota'$,

[5]) Recall that $S := OV_\xi \setminus \{V_\xi\}$, $S' := O^\pi V_\xi' \setminus \{V_\xi'\}$.

$((aV_\eta \sqcap OV_\zeta)\, V_\xi)^\pi = (a^{\hat\pi} V'_\eta \sqcap O'V'_\zeta)\, V'_\xi$, $(bV_\iota \sqcap (aV_\eta \sqcap OV_\zeta)\, V_\xi)^\pi = b^{\hat\pi} V'_\iota \sqcap (a^{\hat\pi} V'_\eta \sqcap O'V'_\zeta)\, V'_\xi$, $((bV_\iota \sqcap (aV_\eta \sqcap OV_\zeta)\, V_\xi)\, V_\eta)^\pi = (b^{\hat\pi} V'_\iota \sqcap (a^{\hat\pi} V'_\eta \sqcap O'V'_\zeta)\, V'_\xi)\, V'_\eta$, $(OV_\xi)^\pi = O'V'_\xi$ and finally $((bV_\iota \sqcap (aV_\eta \sqcap OV_\zeta)\, V_\xi)\, V_\eta \sqcap OV_\xi)^\pi = (b^{\hat\pi} V'_\iota \sqcap (a^{\hat\pi} V'_\eta \sqcap O'V'_\zeta)\, V'_\xi)\, V'_\eta \sqcap O'V'_\xi$. (Fig. 13.) It follows that $a^{\sigma_\iota} +_\iota b \in \mathrm{Dom}\,\hat\pi$ and $(a^{\sigma_\iota} +_\iota b)^{\hat\pi} = (a^{\hat\pi})^{\sigma_\iota} +'_\iota b^{\hat\pi}$. Thus (i) is verified.

Now investigate condition (ii). Suppose that $a \in \mathrm{Dom}\,\hat\pi$, $b^\pi = V'_\xi$, $\iota \in \mathfrak{J}$. (See Fig. 14.)

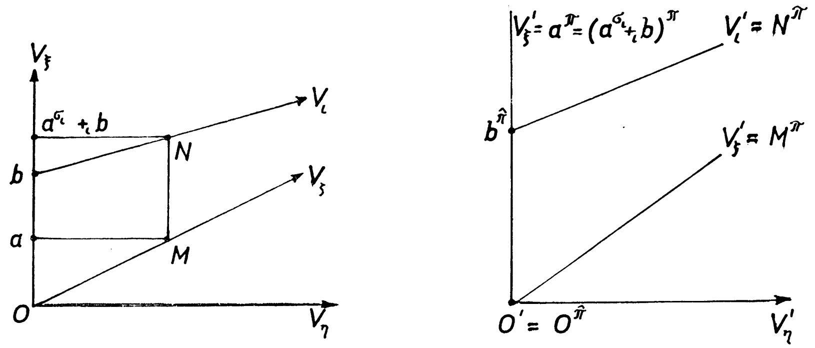

Fig. 15.

Here $(aV_\eta \sqcap OV_\zeta)^\pi = a^{\hat\pi} V'_\eta \sqcap O'V'_\zeta$ while $(bV_\iota)^\pi = V'_\xi V'_\iota = v'$ so that a similar argument as above implies $(a^{\sigma_\iota} +_\iota b)^\pi = V'_\xi$. Analogously for $a \in S \setminus \mathrm{Dom}\,\hat\pi$, $b \in \mathrm{Dom}\,\hat\pi$. (Fig. 15). Thus (ii) is verified, too.

Finally, let $a \in \mathrm{Dom}\,\hat\pi$, $a^{\sigma_\alpha} +_\alpha b = a^{\sigma_\beta} +_\beta c \in V_\xi'^{\pi^{-1}}$ for $\alpha, \beta \in \mathfrak{J}$, $\alpha \neq \beta$. We need to prove that then $\uparrow b$, $\uparrow c$ is not possible. Suppose on the contrary that $\uparrow b$, $\uparrow c$. Then the point $(bV_\alpha \sqcap cV_\beta)^\pi$ must be proper, but this contradicts the fact that $((a^{\sigma_\alpha} +_\alpha b)\, V_\eta)^\pi = v'$. (Fig. 16.)

Proposition 11. *Let $\mathfrak{L}$, $\mathfrak{L}'$ be $\mathfrak{J}$-loops and Θ a place from $\mathfrak{L}$ onto $\mathfrak{L}'$. Then there is an epimorphism $\bar\Theta$ of the net $\mathcal{N} := \mathcal{N}_{\mathfrak{L},\xi,\eta}$ [6]) onto the net $\mathcal{N}' := \mathcal{N}_{\mathfrak{L}',\xi,\eta}$. Its "proper" part $\bar\Theta|_{(\mathrm{Dom}\Theta)^2}$ is determined uniquely by Θ.*

Proof. Define a mapping $\bar\Theta : \mathcal{P} \to \mathcal{P}'$ as follows[7]):

(1) For $\uparrow a$, $\uparrow b$ let $(a, b)^{\bar\Theta} = (a^\Theta, b^\Theta)$.

(2) For $\uparrow a$, $\downarrow b$ let $(a, b)^{\bar\Theta} = \xi$.

(3) For $\downarrow a$, $\uparrow b$ let $(a, b)^{\bar\Theta} = \eta$.

[6]) Hence the improper points of this net may be denoted in two ways: either directly by indices ι or symbols V_ι. Similarly for the image net.

[7]) We shall use the abbreviation $\uparrow x :\Leftrightarrow x \in \mathrm{Dom}\,\Theta$, $\downarrow x :\Leftrightarrow x \in S \setminus \mathrm{Dom}\,\Theta$ as in the definition of the place.

(4) For $\downarrow a, \uparrow b, b = a^{\sigma_\iota} +_\iota c, \uparrow c$ (where $\iota \in \mathfrak{I}$) let $(a, b)^{\bar{\Theta}} = \iota$.

(5) For $\downarrow a, \downarrow b, b = a^{\sigma_{\iota_1}} +_{\iota_1} c_1 = a^{\sigma_{\iota_2}} +_{\iota_2} c_2, \downarrow c_1, \downarrow c_2$ with distinct ι_1, ι_2 from $\mathfrak{I}$ let $(a, b)^{\Theta}$ be an arbitrary improper point.[8])

(6) $\iota^{\bar{\Theta}} = \iota$ for all $\iota \in \mathfrak{I} \cup \{\xi, \eta\}$.

These requirements are consistent (cf. condition (iv) from the definition of the place) and $\bar{\Theta}$ is easily seen to be a mapping of $\mathscr{P}$ onto $\mathscr{P}'$. Now verify that $\forall l \in \mathscr{L} \;\; \exists \mathfrak{l} \in \mathscr{L}'$ $l^{\bar{\Theta}} \subseteq \mathfrak{l}$.

Let $l \in \{(x, b) \mid x \in S\} \cup \{\eta\}$ with $\uparrow b$. Then for $\uparrow x$ we have[9]) $(x, b)^{\bar{\Theta}} = (x^{\Theta}, b^{\Theta})$, for $\downarrow x$ we have $(x, b)^{\bar{\Theta}} = \eta$ and finally $\eta^{\bar{\Theta}} = \eta$. Thus $l^{\bar{\Theta}} \subseteq \{(x', b^{\Theta}) \mid x' \in S'\} \cup \{\eta\}$. Let $l = \{(x, b) \mid x \in S\} \cup \{\eta\}$ with $\downarrow b$. Then for $\uparrow x$ it follows $(x, b)^{\bar{\Theta}} = \xi$, for $\downarrow x$ it follows $(x, b)^{\bar{\Theta}} \in v'$ and finally $\eta^{\bar{\Theta}} = \eta$. Thus $l^{\bar{\Theta}} \subseteq v'$.

Let $l = \{(a, y) \mid y \in S\} \cup \{\xi\}$ with $\uparrow a$. Then for $\uparrow y$ we have $(a, y)^{\bar{\Theta}} = (a^{\Theta}, y^{\Theta})$, for $\downarrow y$ it is $(a, y)^{\bar{\Theta}} = \xi$ and finally $\xi^{\bar{\Theta}} = \xi$. Thus $l^{\bar{\Theta}} \subseteq \{(a^{\Theta}, y') \mid y' \in S'\} \cup \{\xi\}$. Let $l = \{(a, y) \mid a \in S\} \cup \{\xi\}$ with $\downarrow a$. Then for $\uparrow y$ it follows $(a, y)^{\bar{\Theta}} = \eta$, for $\downarrow y$ it is $(a, y)^{\bar{\Theta}} \in v'$ and finally $\xi^{\bar{\Theta}} = \xi$. Thus $l^{\bar{\Theta}} \subseteq v'$.

Let $l = \{x, x^{\sigma_\iota} +_\iota c) \mid x \in S\} \cup \{\iota\}$ for $\iota \in \mathfrak{I}$ and for $\uparrow c$. Then if $\uparrow x$ we have $(x, x^{\sigma_\iota} +_\iota c)^{\bar{\Theta}} = (x^{\Theta}, (x^{\Theta})^{\sigma'_\iota} +'_\iota c^{\Theta})$, for $\downarrow x$ we have $(x, x^{\sigma_\iota} +_\iota c)^{\bar{\Theta}} = \iota$ by (4) and finally $\iota^{\bar{\Theta}} = \iota$, so that $l^{\bar{\Theta}} \subseteq \{(x', x'^{\sigma'_\iota} +'_\iota c^{\Theta}) \mid x' \in S'\} \cup \{\iota\}$. Let $l = \{x, x^{\sigma_\iota} +_\iota c) \mid x \in S\} \cup \{\iota\}$ for $\iota \in \mathfrak{I}$ and for $\downarrow c$. If $\uparrow x$ then $(x, x^{\sigma_\iota} +_\iota c)^{\bar{\Theta}} = \xi$ by (2). If $\downarrow x$ then $(x, x^{\sigma_\iota} +_\iota c)^{\bar{\Theta}} \in v'$ and, finally, $\iota^{\bar{\Theta}} = \iota$. Thus $l^{\bar{\Theta}} \subseteq v'$. The last case $v^{\bar{\Theta}} \subseteq v'$ is quite trivial. The proof is complete. ■

Remark 3. As a consequence of Propositions 8–11 we have (at least partly) an answer to the questions from Remark 2: An affine epimorphism of a non-trivial net $\mathscr{N}$ onto a non-trivial net $\mathscr{N}'$ is uniquely determined by the full pre-image of one proper point. The full preimages of improper points under an epimorphism of a non-trivial not $\mathscr{N}$ onto a non-trivial net $\mathscr{N}'$ are situated as described in the proof of Proposition 11. We shall not go into details here.

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Author's address: 602 00 Brno, Hilleho 6, ČSSR (Vysoké učení technické).

[8]) Here the construction is not uniquely determined.

[9]) We shall not remark, in general, which of the rules (1) to (6) will be used in our reasoning because this can be seen easily.