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THE LATTICE OF TOPOLOGIES OF TOPOLOGICAL L-GROUPS

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On a lattice ordered group G (l-group G) we can consider lattices $\mathfrak{F}(G)$ of all topologies ($\mathfrak{Y}(G)$ of all topologies where the group operation in G is continuous, $\mathfrak{L}(G)$ of all topologies where the group and lattice operations in G are continuous) with the same underlying set $|G|$. If all topologies in $\mathfrak{F}(G)$ ($\mathfrak{Y}(G)$, $\mathfrak{L}(G)$) are T_0-topologies, then we use the notation $\mathfrak{F}_0(G)$ ($\mathfrak{Y}_0(G)$, $\mathfrak{L}_0(G)$, respectively). In this paper relations and properties of those lattices, namely complementarity, modularity and distributivity are investigated. The main results are restricted to abelian groups.

Topological lattice ordered group (notation: *tl-group*) is an l-group G with a topology in which both group and lattice operations are continuous. In this paper every topology is considered in the sense of Bourbaki and is usually given by a basis Σ^* of open sets (neighbourhood basis). This topology is denoted by $\tau(\Sigma^*)$, the topological space on a set N with the topology $\tau(\Sigma^*)$ is denoted by (N, Σ^*) and for every $M \subseteq N$ the closure of M in $\tau(\Sigma^*)$ is denoted by $\overline{M}_{\Sigma^*}$. In case that the group operation in a group G is continuous in a certain topology we can give this topology by a basis Σ of open sets containing zero in G (neighbourhood basis of zero). This topology is denoted by $\tau(\Sigma)$, the topological group G with the topology $\tau(\Sigma)$ is denoted by (G, Σ) and for every $M \subseteq G$ the closure of M in $\tau(\Sigma)$ is denoted by $\overline{M}_\Sigma$. The next two theorems are fundamental for our work (see [3]):

Theorem A. *Let (G, Σ) be a tl-group. Then Σ fulfils the following conditions:*

1. *The intersection of two arbitrary sets of Σ contains a set of Σ.*
2. *For any set $U \in \Sigma$ there exists a set $V \in \Sigma$ such that $V - V \subseteq U$.*
3. *For any set $U \in \Sigma$ and any element $u \in U$ there exists a set $V \in \Sigma$ such that $V + u \subseteq U$.*
4. *For any set $U \in \Sigma$ and any element $g \in G$ there exists a set $V \in \Sigma$ such that $-g + V + g \subseteq U$.*
5. *For any set $U \in \Sigma$ and any element $g \in G$ there exists a set $V \in \Sigma$ such that $(V - g^+) \vee (V + g^-) \subseteq U$.*

Theorem B. *Let G be an l-group. Let Σ be a system of subsets of G fulfilling the conditions* $1-5$ *of Theorem A. Then (G, Σ) is a tl-group.*

Remark. $\bigcap\Sigma = \bigcap\{U : U \in \Sigma\}$.

1.

1.1. Definition. Let $\tau_1, \tau_2 \in \mathfrak{F}(G)$. Then we shall say that τ_1 *is stronger than* τ_2 (τ_2 *is weaker than* τ_1) if there exist neighbourhood bases Σ_1^* in τ_1 and Σ_2^* in τ_2 such that $\Sigma_1^* \supseteq \Sigma_2^*$. We shall write $\tau_1 \geqq \tau_2$.

Remark. The relation $\geqq$ introduced in Definition 1.1 is a partial order on the set $\mathfrak{F}(G)$.

1.2. *Let $\tau(\Sigma_1), \tau(\Sigma_2) \in \mathfrak{Y}(G)$. Then the following assertions are equivalent:*

1. $\tau(\Sigma_1) \geqq \tau(\Sigma_2)$.
2. *For any set $M \subseteq G$ it holds $\overline{M}_{\Sigma_1} \subseteq \overline{M}_{\Sigma_2}$.*
3. *For any neighbourhood $U \in \Sigma_2$ there exists a neighbourhood $V \in \Sigma_1$ such that $U \supseteq V$.*
4. *For systems Σ^1 and Σ^2 of all open sets containing zero in G in $\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ it holds $\Sigma^1 \supseteq \Sigma^2$.*

1.3. Theorem. *The set $\mathfrak{L}(G)$ of all topologies of tl-groups on $|G|$ is a complete lattice with the greatest element $\tau(\Sigma^0)$, where $\Sigma^0 = \{X \subseteq G : 0 \in X\}$ and the smallest element $\tau(\Sigma_0)$, where $\Sigma_0 = \{G\}$.*

Proof. If $\tau(\Sigma_i) \in \mathfrak{L}(G)$, $i \in I$, then $\tau(\Sigma^0) \geqq \tau(\Sigma_i) \geqq \tau(\Sigma_0)$. Let $Q = \{\bigcap U_i : U_i \in \Sigma_i, i \in I, \text{card}\{i \in I : U_i \neq G\} < \aleph_0\}$ and let us prove by virtue of Theorem B that $\bigvee_{\mathfrak{L}(G)}\{\tau(\Sigma_i) : i \in I\} = \tau(Q)$:

Let $W_1 = \bigcap_{i \in I} U_i^1$, $W_2 = \bigcap_{i \in I} U_i^2$, where only for finite number of indices $i \in I$, U_i^1 and U_i^2 are different from G. Hence $W_1 \cap W_2 = \bigcap_{i \in I} (U_i^1 \cap U_i^2) \supset \bigcap_{i \in I} U_i$, where $U_i = G$ for such $i \in I$ that $U_i^1 \cap U_i^2 = G$ and $U_i \subseteq U_i^1 \cap U_i^2$, $U_i \in \Sigma_i$ for such $i \in I$ that $U_i^1 \cap U_i^2 \neq G$. Then $\bigcap_{i \in I} U_i \in Q$.

Now, let $W = \bigcap_{i \in I} U_i \in Q$. Then there exists a set $I_0 \subseteq I$, card $I_0 < \aleph_0$ such that $U_i \neq G$ for $i \in I_0$ and $U_i = G$ for $i \in I \setminus I_0$. For arbitrary elements $w \in W$, $g \in G$, $i \in I_0$ there exists a neighbourhood $V_i \in \Sigma_i$ with the property $V_i - V_i \subseteq U_i$ (or $V_i + w \subseteq U_i$, $-g + V_i + g \subseteq U_i$, $(V_i + g^-) \vee (V_i - g^+) \subseteq U_i$) for $i \in I_0$ – see Theorem A. For $i \in I \setminus I_0$ these relations hold for $V_i = G$. It means $\bigcap_{i \in I} V_i - \bigcap_{i \in I} V_i \subseteq$

$\subseteq \bigcap_{i\in I} (V_i - V_i) \subseteq \bigcap_{i\in I} U_i$ $(\bigcap_{i\in I} V_i + w = \bigcap_{i\in I} (V_i + w) \subseteq \bigcap_{i\in I} U_i$, $-g + \bigcap_{i\in I} V_i + g = \bigcap_{i\in I} (-g + V_i + g) \subseteq \bigcap_{i\in I} U_i$, $(\bigcap_{i\in I} V_i - g^+) \vee (\bigcap_{i\in I} V_i + g^-) \subseteq \bigcap_{i\in I} [(V_i - g^+) \vee (V_i + g^-)] \subseteq \bigcap_{i\in I} U_i$, respectively), where only for $i \in I_0$, $I_0 \subseteq I$, $\operatorname{card} I_0 < \aleph_0$ it is $V_i \neq G$ and thus $\bigcap_{i\in I} V_i \in Q$. With regard to Theorem B, $\tau(Q) \in \mathfrak{L}(G)$.

Finally, $\bigcup_{i\in I} \Sigma_i \subseteq Q$ and $\tau(Q) \geqq \tau(\Sigma_i)$ for $i \in I$. If there exists $\tau(\Sigma) \in \mathfrak{L}(G)$ such that $\tau(\Sigma) \geqq \tau(\Sigma_i)$, $i \in I$, then $\Sigma \supseteq \bigcup_{i\in I} \Sigma_i$ and also $\Sigma \supseteq Q$, i.e., $\tau(\Sigma) \geqq \tau(Q)$.

1.4. Corollary. *If $\tau(\Sigma_i) \in \mathfrak{L}(G)$, $i \in I$, then it holds $\bigvee_{\mathfrak{F}(G)} \tau(\Sigma_i)$ $(i \in I) = \bigvee_{\mathfrak{Y}(G)} \tau(\Sigma_i)$ $(i \in I) = \bigvee_{\mathfrak{L}(G)} \tau(\Sigma_i)$ $(i \in I) = \tau(Q)$, where $Q = \{\bigcap U_i \ (i \in I) : U_i \in \Sigma_i, \ \operatorname{card} \{i \in I : U_i \neq G\} < \aleph_0\}$.*

2.

2.1. Definition. Let $\tau(\Sigma_1), \tau(\Sigma_2) \in \mathfrak{Y}(G)$. We recall that $\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are *permutable* if for any $U \in \Sigma_1$, $V \in \Sigma_2$ there exist $U_1, U_2 \in \Sigma_1$, $V_1, V_2 \in \Sigma_2$ such that $U + V \supseteq V_1 + U_1$, $V + U \supseteq U_2 + V_2$.

2.2. Theorem. *If $\tau(\Sigma_1), \tau(\Sigma_2) \in \mathfrak{Y}(G)$, $\Sigma = \{U + V : U \in \Sigma_1, \ V \in \Sigma_2\}$, $\Sigma' = \{V + U : U \in U \in \Sigma_1, \ V \in \Sigma_2\}$ then the following assertions are equivalent:*

1. *$\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are permutable topologies.*
2. *$\tau(\Sigma) = \tau(\Sigma')$.*
3. *$\tau(\Sigma_1) \wedge_{\mathfrak{Y}(G)} \tau(\Sigma_2) = \tau(\Sigma)$.*

Proof. $1 \Rightarrow 3$: First, we prove that the system Σ fulfils all conditions of the neighbourhood basis of zero of a topology from $\mathfrak{Y}(G)$:

1. For any $U + V$, $U_1 + V_1 \in \Sigma$ it is $(U + V) \cap (U_1 + V_1) \supseteq (U \cap U_1) + (V \cap V_1) \supseteq U_2 + V_2$, where $U_2 \in \Sigma_1$, $U_2 \subseteq U \cap U_1$, $V_2 \in \Sigma_2$, $V_2 \subseteq V \cap V_1$.

2. For any $U + V \in \Sigma$ there exist $U' \in \Sigma_1$, $V' \in \Sigma_2$ such that $U \supseteq U' + U'$, $V \supseteq V' + V'$ and because $U' + V' \in \Sigma$ there exist $U'' \in \Sigma_1$, $V'' \in \Sigma_2$ such that $U' + V' \supseteq V'' + U''$, $U'' + V'' \in \Sigma$, $U'' \subseteq U'$, $V'' \subseteq V'$ and $(U'' + V'') + (U'' + V'') = U'' + (V'' + U'') + V'' \subseteq U'' + (U' + V') + V'' \subseteq (U' + U') + (V' + V') \subseteq U + V$. Further, $V''' \in \Sigma_2$, $U''' \in \Sigma_1$ exist such that $-V''' \subseteq V''$, $-U''' \subseteq U''$, $U''' + V''' \in \Sigma$ and $-(U''' + V''') = -V''' - U''' \subseteq V'' + U'' \subseteq U' + V' \subseteq U + V$.

3. For any $U + V \in \Sigma$ and any $u + v \in U + V$ there exist $U' \in \Sigma_1$, $V' \in \Sigma_2$, $V'' \in \Sigma_2$ such that $U' + u \subseteq U$, $V' + v \subseteq V$, $-u + V'' + v \subseteq V'$. Hence $U' + V'' \in \Sigma$, $(U' + V'') + (u + v) = U' + (V'' + u) + v \subseteq U' + (u + V') + v = (U' + u) + (V' + v) \subseteq U + V$.

4. For any $U + V \in \Sigma$, $g \in G$ there exist $U_1 \in \Sigma_1$, $V_1 \in \Sigma_2$ such that $-g + U_1 + g \subseteq U$, $-g + V_1 + g \subseteq V$ and therefore $-g + (U_1 + V_1) + g = (-g + U_1 + g) + (-g + V_1 + g) \subseteq U + V$, $U_1 + V_1 \in \Sigma$.

Together, $\tau(\Sigma) \in \mathfrak{Y}(G)$.

Now, we prove that $\tau(\Sigma_1) \wedge_{\mathfrak{Y}(G)} \tau(\Sigma_2) = \tau(\Sigma)$. Clearly $\tau(\Sigma) \leqq \tau(\Sigma_i)$, $i = 1, 2$ and if there exists $\tau(\Sigma_0) \in \mathfrak{Y}(G)$, $\tau(\Sigma_0) \leqq \tau(\Sigma_i)$, $i = 1, 2$, then for any neighbourhood $U_0 \in \Sigma_0$ there exists $W_0 \in \Sigma_0$ such that $U_0 \supseteq W_0 + W_0$. Further, there exist $U_1 \in \Sigma_1$, $V_1 \in \Sigma_2$, $W_0 \supseteq U_1 \cup V_1$ and therefore $U_0 \supseteq U_1 + V_1$, $U_1 + V_1 \in \Sigma$, i.e., $\tau(\Sigma_0) \leqq \leqq \tau(\Sigma)$ – see 1.2.

$3 \Rightarrow 1$: For any $U \in \Sigma_1$, $V \in \Sigma_2$ there exist $U_1 \in \Sigma_1$, $V_1 \in \Sigma_2$, $\pm U_1 \subseteq U$, $\pm V_1 \subseteq V$, $U + V \supseteq (U_1 + V_1) + (U_1 + V_1) \supseteq V_1 + U_1$. Neighbourhoods $U_2 \in \Sigma_1$, $V_2 \in \Sigma_2$ exist such that $U_1 + V_1 \supseteq -(U_2 + V_2)$ and $V + U \supseteq -V_1 - U_1 = -(U_1 + V_1) \supseteq U_2 + V_2$. It means that $\tau(\Sigma)$ and $\tau(\Sigma')$ are permutable.

$1 \Leftrightarrow 2$ follows from Definition 2.1 and from 1.2.

2.3. Corollary. *If G is an abelian group, $\tau(\Sigma_1), \tau(\Sigma_2) \in \mathfrak{Y}(G)$, then $\tau(\Sigma_1) \wedge_{\mathfrak{Y}(G)} \wedge_{\mathfrak{Y}(G)} \tau(\Sigma_2) = \tau(\Sigma)$, where $\Sigma = \{U + V : U \in \Sigma_1, V \in \Sigma_2\}$.*

2.4. Definition (see [6]). Let $(M, \geqq, \tau)$ be a partially ordered set with a topology $\tau = \tau(\Sigma^*)$. We shall call the partial order $\geqq$ *continuous with respect to the topology* τ if it holds: If $a, b \in M$, a non $\leqq b$ then there exist $U, V \in \Sigma^*$, $a \in U$, $b \in V$ such that for any $u \in U$, $v \in V$ it is u non $\leqq v$.

2.5. *Let $(G, \leqq, \Sigma)$ be a partially ordered topological group. Then the partial order $\leqq$ is continuous with respect to $\tau(\Sigma)$ if and only if for any $g \in G$, g non $\leqq 0$ there exists a neighbourhood $U \in \Sigma$ with the property g non $\leqq u$ for any $u \in U$.*

Proof. If $\leqq$ is continuous with respect to $\tau(\Sigma)$, then $U \in \Sigma$ exists such that for any $u, u_1 \in U$ it is $g + u_1$ non $\leqq u$ and also g non $\leqq u$.

On the contrary, if $a, b \in G$, a non $\leqq b$ exist and for any $U \in \Sigma$ there exist elements $u_1, u_2 \in U$ such that $a + u_1 > b + u_2$, then $g = -b + a > u_2 - u_1$. But according to the condition from the proposition $U_0 \in \Sigma$ exists such that g non $\leqq n$ for any $u \in U_0$. If we choose $U \in \Sigma$ such that $U_0 \supseteq U - U$, we get a contradiction.

2.6. *If (G, Σ) is a tl-group, then its lattice order is continuous with respect to $\tau(\Sigma)$ if and only if $\tau(\Sigma)$ is a T_0-topology.*

Proof. $\Rightarrow$: It follows from [6], L.2.

$\Leftarrow$: Let g non $\leqq 0$ and $W_{g \vee 0} = \{x \in G : g \vee 0 \text{ non } \leqq x \text{ non } \leqq -(g \vee 0)\}$. Then with regard to [6], § 2 the set $W_{g \vee 0}$ is open in $\tau(\Sigma)$ and hence $W \in \Sigma$ exists such that $W \vee W \subseteq W_{g \vee 0}$. Now, the existence of an element $w \in W$, $g \leqq w$ leads to a contradiction, because $g \vee 0 \leqq w \vee 0 \in W \vee W \subseteq W_{g \vee 0}$.

2.7. Definition. Let $(G, \geqq, \Sigma)$ be a partially ordered topological group. The topology $\tau(\Sigma)$ is called *locally convex* if for any $U \in \Sigma$ there exists $V \in \Sigma$, $V \subseteq U$, V being a convex set in order $\geqq$.

The topology $\tau(\Sigma)$ is called *weakly locally convex* if for any $U \in \Sigma$ there exists $V \in \Sigma$ with the property: $v_1, v_2 \in V, g \in G, v_1 \geqq g \geqq v_2 \Rightarrow g \in U$.

2.8. *An abelian tl-group (G, Σ) with a T_0-topology $\tau(\Sigma)$ is a uniform ordered space with a locally convex topology $\tau(\Sigma)$.*

Proof. In order to establish the fact that (G, Σ) is a uniform ordered space it is sufficient to prove the next two assertions (see [2], Prop. 12): 1° $G^+ = \{g \in G : g \geqq 0\}$ is a closed set in $\tau(\Sigma)$; this is evident;

2° For any $U \in \Sigma$ there exists $V \in \Sigma$ with the property $0 \leqq x \leqq y, y \in V \Rightarrow x \in U$.

This assertion is also valid, because the existence of $U \in \Sigma$ such that for any $V \in \Sigma$ there exist $x \in G \setminus U, y \in V, 0 \leqq x \leqq y$ implies the existence of $U_1, V \in \Sigma, V \subseteq U_1 \subseteq \subseteq U, \pm U_1 \pm U_1 \subseteq U, x^- \vee (V - x^+) \subseteq U$, the validity of $y - x = 0 \vee (y - x) \in \in x^- \vee (V - x^+) \subseteq U_1, x \in -U_1 + y \subseteq -U_1 + U_1 \subseteq U$ and together a contradiction. The local convexity of $\tau(\Sigma)$ follows from [2], Prop. 9.

2.9. *Let (G, Σ) be an abelian topological group with a T_0-topology and let G be an l-group. Then (G, Σ) is a tl-group if and only if it holds:*

(i) *$\tau(\Sigma)$ is locally convex,*

(ii) *for any $U \in \Sigma$ there exists $V \in \Sigma$ such that $V \vee 0 \subseteq U$.*

Proof. $\Rightarrow$: see 2.8.

$\Leftarrow$: If $g \in G$, $U \in \Sigma$, then $V_i \in \Sigma$, $i = 1, 2, 3, 4$ exist such that $U \supseteq V_1$, V_1 is a convex set, $\pm V_4 \subseteq V_3$, $V_3 \subseteq V_2$, $V_3 \vee 0 \subseteq V_2$, $\pm V_2 \subseteq V_1$. Hence for any $v \in V_4$ it is $v^+ \in V_4 \vee 0 \subseteq V_1$, $v^- = -(-v \vee 0) \in -(V_3 \vee 0) \subseteq V_1$, $v^+ = = v^+ + (-g^+ \vee g^-) = (v^+ - g^+) \vee (v^+ + g^-) \geqq (-g^+) \vee (v + g^-) \geqq (v^- - - g^+) \vee (v^- + g^-) = v^-$. Hence $-g^+ \vee (V_4 + g^-) \subseteq V_1 \subseteq U$ and the rest follows from [4], 1.1.

Remark. If (G, Σ) is a topological group, then for any $u \in U$ there exists $V_u \in \Sigma$ such that $V_u + u \subseteq U$ and therefore $\bigcap \Sigma + U \subseteq \bigcup\{V_u + u : u \in U\} \subseteq U$.

2.10. *If (G, Σ) is an abelian tl-group, then $\tau(\Sigma)$ is locally convex.*

Proof. For $\tau(\Sigma) \in \mathfrak{L}_0(G)$ the proposition follows from 2.8. If $\tau(\Sigma) \in \mathfrak{L}(G) \setminus \mathfrak{L}_0(G)$, then $\bigcap\Sigma \neq \{0\}$ is a closed l-ideal in G (see [4], 1.4) and $G/\bigcap\Sigma$ is an abelian tl-group with a T_0-topology $\tau(\Sigma/\bigcap\Sigma)$, where $\Sigma/\bigcap\Sigma = \{(U + \bigcap\Sigma)/\bigcap\Sigma : U \in \Sigma\}$ and $\tau(\Sigma/\bigcap\Sigma)$ is locally convex (see 2.8). It means that for any $U \in \Sigma$ there exists $V \in \Sigma$ such that $(V + \bigcap\Sigma)/\bigcap\Sigma$ is a convex set in an l-factorgroup $G/\bigcap\Sigma$. If $v_1, v_2 \in V, x \in G, v_1 \geqq x \geqq \geqq v_2$, then $v_1 + \bigcap\Sigma \geqq x + \bigcap\Sigma \geqq v_2 + \bigcap\Sigma$ in $G/\bigcap\Sigma$ and $x + \bigcap\Sigma \subseteq V + \bigcap\Sigma$. Consequently $x \in V + \bigcap\Sigma = V$ (see Remark) and V is a convex set.

2.11. *If (G, Σ_i) are tl-groups with locally convex topologies $\tau(\Sigma_i)$, $i = 1, 2$, then $\tau(\Sigma)$ is weakly locally convex, where $\Sigma = \{U + V : U \in \Sigma_1, V \in \Sigma_2\}$.*

Proof. If $U \in \Sigma_1$, $V \in \Sigma_2$ are arbitrary neighbourhoods, then there exist convex neighbourhoods $U_1 \in \Sigma_1$, $V_1 \in \Sigma_2$, $U_1 + U_1 \subseteq U$, $V_1 + V_1 \subseteq V$ and neighbourhoods $U_2, U' \in \Sigma_1$, $V_2, V' \in \Sigma_2$ such that $\pm U' \pm U' \subseteq U_2$, $U_2 \vee U_2 \subseteq U_1$, $\pm V' \pm \pm V' \subseteq V_2$, $V_2 \wedge V_2 \subseteq V_1$. For any $u_1, u_2 \in U'$, $v_1, v_2 \in V'$, $g \in G$, $u_1 + v_1 \geqq g \geqq u_2 + v_2$ it is $-u_2 + u_1 \geqq -u_2 + g - v_1 \geqq v_2 - v_1$, $-u_2 + u_1 \in U_2$, $v_2 - v_1 \in V_2$ and if we denote $m = -u_2 + g - v_1$, it is $(-u_2 + u_1)^+ \geqq m^+ \geqq (v_2 - v_1)^+ \geqq 0$, $0 \geqq (-u_2 + u_1)^- \geqq m^- \geqq (v_2 - v_1)^-$, $(-u_2 + u_1)^+ \in U_1$, $(v_2 - v_1)^- \in V_1$, too. Together $m^+ \in U_1$, $m^- \in V_1$, $m = m^+ + m^- \in U_1 + V_1$ and $g \in u_2 + U_1 + V_1 + v_1 \subseteq (U_1 + U_1) + (V_1 + V_1) \subseteq U + V$.

2.12. Theorem. *If G is an abelian l-group, then the lattice $\mathfrak{L}(G)$ is a sublattice in the lattice $\mathfrak{Y}(G)$.*

Proof. With regard to 1.4, 2.3 and [4], 1.1 it is sufficient to prove the fact that for any $\tau(\Sigma_i) \in \mathfrak{L}(G)$, $i = 1, 2$, the system $\Sigma = \{U + V : U \in \Sigma_1, V \in \Sigma_2\}$ fulfils the property: For any $g \in G$, $U + V \in \Sigma$ there exists $U_0 + V_0 \in \Sigma$ such that $-g^+ \vee \vee (U_0 + V_0 + g^-) \subseteq U + V$.

To this aim, let $g \in G$, $U + V \in \Sigma$ be arbitrarily chosen. Then there exists $U' + + V' \in \Sigma$ such that $U + V \supseteq U' + V'$ and for any $u_1, u_2 \in U'$, $v_1, v_2 \in V'$, $g \in G$ from $u_1 + v_1 \geqq g \geqq u_2 + v_2$ it follows $g \in U + V$ (see 2.11). Further, there exist $U_0 \in \Sigma_1$, $V_0 \in \Sigma_2$, $U_0 \subseteq U'$, $V_0 \subseteq V'$, $V_0 \vee 0 \subseteq V'$, $V_0 \wedge 0 \subseteq V'$, $-g^+ \vee (U_0 + + g^-) \subseteq U'$ and therefore for any $u_0 \in U_0$, $v_0 \in V_0$ there exist $u \in U'$, $v, v' \in V'$ such that $u + v = [-g^+ \vee (u_0 + g^-)] + (v_0 \vee 0) = (-g^+ + v_0) \vee (-g^+) \vee (u_0 \vee (u_0 + g^- + v_0) \vee (u_0 + g^-) \geqq (-g^+) \vee (u_0 + v_0 + g^-) \geqq [(-g^+ + v_0) \vee \vee (u_0 + v_0 + g^-)] \wedge [(-g^+) \vee (u_0 + v_0 + g^-)] \wedge [(u_0 + g^-) \vee (-g^+ + v_0)] \wedge \wedge [(u_0 + g^-) \vee (-g^+)] = [(-g^+ + v_0) \wedge (-g^+)] \vee [(u_0 + v_0 + g^-) \wedge (u_0 + + g^-)] = [(-g^+) + (v_0 \wedge 0)] \vee [u_0 + g^+ + (v_0 \wedge 0)] = [-g^+ \vee (u_0 + g^-)] + + (v_0 \wedge 0) = u + v'$. It means that $-g^+ \vee (u_0 + v_0 + g^-) \in U + V$, for any $u_0 \in U_0$, $v_0 \in V_0$ and $\tau(\Sigma) \in \mathfrak{L}(G)$, $\tau(\Sigma) = \tau(\Sigma_1) \wedge_{\mathfrak{L}(G)} \tau(\Sigma_2)$.

2.13. *If G is an abelian fully ordered group, then it holds*: 1. *$\mathfrak{L}(G)$ is a chain.*

2. *If $\tau \in \mathfrak{L}_0(G)$, τ is no discrete topology, then τ is the interval topology.*

3. *The interval topology in G is a dual atom in $\mathfrak{L}(G)$.*

Proof. If $\tau(\Sigma) \in \mathfrak{L}_0(G)$, $\tau(\Sigma') \in \mathfrak{L}(G) \setminus \mathfrak{L}_0(G)$, then $\bigcap \Sigma = \{0\}$, there exists an element s, $0 \leqq s \in \bigcap \Sigma'$ and $\bigcap \Sigma' \neq \{0\}$ is an l-ideal in G (see [4], 1.2). Clearly, $U \in \Sigma$, $s \notin U$ exists and according to [4], 2.2 $V \in \Sigma$ exists such that for any $v \in V$ it is $s > |v|$. It means that $V \subseteq \bigcap \Sigma'$ and $\tau(\Sigma) > \tau(\Sigma')$ – see 1.2.

Let now $\tau(\Sigma_1), \tau(\Sigma_2) \in \mathfrak{L}(G) \setminus \mathfrak{L}_0(G)$, $\tau(\Sigma_1) \parallel \tau(\Sigma_2)$. If $\bigcap \Sigma_1 \nsubseteq \bigcap \Sigma_2$, then there exists $s_2 \in \bigcap \Sigma_2 \setminus \bigcap \Sigma_1$, $0 < s_2$ and according to [4], 2.2 there exists $V \in \Sigma_1$ such that for any $v \in V$ it is $s_2 > |v|$, i.e., $V \subseteq \bigcap \Sigma_2$. Hence and from 1.2 it is $\tau(\Sigma_1) \leqq \tau(\Sigma_2)$, a contradiction. Similarly we prove that the case $\bigcap \Sigma_2 \nsubseteq \bigcap \Sigma_1$ is impossible. Therefore

$s_1 \in \bigcap\Sigma_1 \setminus \bigcap\Sigma_2$, $0 < s_1$, $s_2 \in \bigcap\Sigma_2 \setminus \bigcap\Sigma_1$, $0 < s_2$ exist and again [4], 2.2 implies the existence of neighbourhoods $V_1 \in \Sigma_1$, $V_2 \in \Sigma_2$ such that for any $v_1 \in V_1$, $v_2 \in V_2$ it holds $s_1 > |v_2|$, $s_2 > |v_1|$. It means that $V_1 \subseteq \bigcap\Sigma_2$, $V_2 \subseteq \bigcap\Sigma_1$ and $\tau(\Sigma_1) = \tau(\Sigma_2)$.

Finally, if $\tau(\Sigma) \in \mathfrak{L}_0(G)$, $\tau(\Sigma)$ is no discrete topology, then the sets $W_g = \{x \in G : : |g| > x > -|g|\}$ are open in $\tau(\Sigma)$ for any $g \in G$. Hence $\tau(\Sigma) \geqq \iota$, where ι is the interval topology in G. On the other hand, for any $U \in \Sigma$ there exists $V \in \Sigma$, $V \subseteq U$, V being a convex set in G (see 2.8) and there exists an element $v \in V$, $0 < v$, $-v \in V$ (see [4], 2.1). Then the set $W_v = \{x \in G : v > x > -v\} \subseteq V \subseteq U$ and $\iota \geqq \tau(\Sigma)$, i.e., $\tau(\Sigma) = \iota$.

Remark. In [1] an example is given with the property that a set $\mathfrak{Y}_0(G)$ is no lattice but only a $\vee$-semilattice in $\mathfrak{Y}(G)$. In that case G is a fully ordered abelian group.

3.

In the end of this paper let us deal with the complementarity of topologies on groups in lattices $\mathfrak{F}$, $\mathfrak{Y}$ and $\mathfrak{L}$ and modularity and distributivity of lattices $\mathfrak{Y}$ and $\mathfrak{L}$.

3.1. Theorem. *If G is an abelian group, then $\mathfrak{Y}(G)$ is a modular lattice.*

Proof. Let $\tau(\Sigma_i) \in \mathfrak{Y}(G)$, $i = 1, 2, 3$, $\tau(\Sigma_1) \leqq \tau(\Sigma_2)$. We can suppose that Σ_i are formed by all open sets in $\tau(\Sigma_i)$ containing zero in G $(i = 1, 2, 3)$. Let us denote $\tau' = \tau(\Sigma') = \tau(\Sigma_1) \vee_{\mathfrak{Y}(G)} [\tau(\Sigma_2) \wedge_{\mathfrak{Y}(G)} \tau(\Sigma_3)]$, $\tau'' = \tau(\Sigma'') = [\tau(\Sigma_1) \vee_{\mathfrak{Y}(G)} \tau(\Sigma_3)] \wedge_{\mathfrak{Y}(G)} \tau(\Sigma_2)$. According to Theorems 1.3 and 2.2 $\Sigma' = \{U_1 \cap (U_2 + U_3) : U_i \in \Sigma_i,\ i = 1, 2, 3\}$ and $\Sigma'' = \{(U_1 \cap U_3) + U_2 : U_i \in \Sigma_i,\ i = 1, 2, 3\}$. If $U'' \in \Sigma''$ is an arbitrary neighbourhood, then $U'' = (U_1 \cap U_3) + U_2$, $U_i \in \Sigma_i$, $i = 1, 2, 3$ and there exist $U_1^0 \in \Sigma_1$, $U_2^0 \in \Sigma_2$ such that $-U_1^0 + U_1^0 \subseteq U_1$, $U_2^0 \subseteq U_1^0 \cap U_2$ because $U_1^0 \in \Sigma_1 \subseteq \Sigma_2$. Hence for any $u' \in U' = U_1^0 \cap (U_2^0 + U_3)$, $U' \in \Sigma'$, it holds $u' = u_2 + u_3 \in U_1^0$, where $u_2 \in U_2^0$, $u_3 \in U_3$. It means that $u_3 = -u_2 + u' \in -U_2^0 + U_1^0 \subseteq -U_1^0 + U_1^0 \subseteq U_1$, i.e., $u' \in U_2 + (U_1 \cap U_3)$, $U' \subseteq U''$, $\tau' \geqq \tau''$ (see 1.2). It is clear that $\tau' \leqq \tau''$ and together $\mathfrak{Y}(G)$ is a modular lattice.

Example. If G is an abelian group, $G = A_1 \times A_2 = A_1 \times A_3$ are direct products, $A_2 \neq A_3$, $A_2 \neq \{0\} \neq A_3$, then for the topologies $\tau(\Sigma_i)$, where $\Sigma_i = \{X \subseteq G : A_i \subseteq X\}$ it holds $\tau(\Sigma_i) \in \mathfrak{Y}(G)$, $i = 1, 2, 3$ and $\tau(\Sigma_1)$ is a complement to $\tau(\Sigma_2)$ and $\tau(\Sigma_3)$ – see 3.4, $\tau(\Sigma_2) \neq \tau(\Sigma_3)$, i.e., $\mathfrak{Y}(G)$ is no distributive lattice.

3.2. Lemma. *If G is an l-group, $a, b, c \in G$, $a, b, c \geqq 0$, then*

$$a \wedge (b + c) \leqq (a \wedge b) + (a \wedge c).$$

Proof. $(a \wedge b) + (a \wedge c) = [(a \wedge b) + a] \wedge [(a \wedge b) + c] = 2a \wedge (b + a) \wedge (a + c) \wedge (b + c) \geqq a \wedge (b + c)$.

3.3. Theorem. *If G is an abelian l-group, then $\mathfrak{L}(G)$ is a distributive lattice.*

Proof. Let $\tau(\Sigma_i) \in \mathfrak{L}(G)$, $i = 1, 2, 3$ and let us denote $\tau' = \tau(\Sigma') = \tau(\Sigma_1) \vee_{\mathfrak{L}(G)} \vee_{\mathfrak{L}(G)} [\tau(\Sigma_2) \wedge_{\mathfrak{L}(G)} \tau(\Sigma_3)]$, $\tau'' = \tau(\Sigma'') = [\tau(\Sigma_1) \vee_{\mathfrak{L}(G)} \tau(\Sigma_2)] \wedge_{\mathfrak{L}(G)} [\tau(\Sigma_1) \vee_{\mathfrak{L}(G)} \vee_{\mathfrak{L}(G)} \tau(\Sigma_3)]$. It is clear that $\tau' \leqq \tau''$ and we proceed to prove $\tau' \geqq \tau''$:

Theorem 1.3, 2.2 and 2.12 imply $\Sigma' = \{U_1 \cap (U_2 + U_3) : U_i \in \Sigma_i,\ i = 1, 2, 3\}$, $\Sigma'' = \{(U_1 \cap U_2) + (U_1' \cap U_3) : U_i \in \Sigma_i,\ U_1' \in \Sigma_1,\ i = 1, 2, 3\}$. If $U'' \in \Sigma''$ is an arbitrary neighbourhood, then there exist $V, W \in \Sigma''$ such that $\pm V \subseteq W$, $\pm W \subseteq U''$, W, V are convex sets (see 2.10). Hence $V = (U_1 \cap U_2) + (U_1' \cap U_3)$, $U_i \in \Sigma_i$, $i = 1, 2, 3$, $U_1' \in \Sigma_1$ and with regard to Theorem A $U_i^0 \in \Sigma_i$ exist, $U_i^0 \subseteq U_i$, $U_1^0 \subseteq U_1'$, U_i^0 are convex sets, $i = 1, 2, 3$ (see 2.10) and $U_i^1 \in \Sigma_i$ exist, $U_i^1 \subseteq U_i^0$, $U_1^1 \subseteq U_1'$ such that $|U_i^1| = \{|u| : u \in U_i^1\} \subseteq U_i^1 \vee -U_i^1 \subseteq U_i^0$. It means that $|U_1^1| \wedge |U_2^1| \subseteq \subseteq U_1^0 \cap U_2^0$, $|U_1^1| \wedge |U_3^1| \subseteq U_1^0 \cap U_3^0$.

Now, if $U' = U_1^1 \cap (U_2^1 + U_3^1)$, then $U' \in \Sigma'$ and for any element $u \in U'$ it holds $u = u_2 + u_3 \in U_1^1$, where $u_2 \in U_2^1$, $u_3 \in U_3^1$ are suitable elements. Hence $0 \leqq |u| = |u| \wedge |u_2 + u_3| \leqq |u| \wedge (|u_2| + |u_3|) \leqq (|u| \wedge |u_2|) + (|u| + |u_3|) \in (|U_1^1| \wedge |U_2^1|) + (|U_1^1| \wedge |U_3^1|) \subseteq (U_1^0 \cap U_2^0) + (U_1^0 \cap U_3^0) \subseteq V$ (see L. 3.2), i.e., $|u| \in \in V \subseteq W$, $-|u| \in W$, $u \in W \subseteq U''$. Together $U' \subseteq U''$, $\tau' \geqq \tau''$ (see 1.2). Hence $\tau' = \tau''$ and $\mathfrak{L}(G)$ is a distributive lattice.

3.4. Theorem. *Let G be a group, $\tau(\Sigma_1), \tau(\Sigma_2) \in \mathfrak{Y}(G)$, ($\tau(\Sigma_1), \tau(\Sigma_2)$ are permutable topologies in $\mathfrak{Y}(G)$). Then $\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are complementary topologies in the lattice $\mathfrak{F}(G)$ ($\mathfrak{Y}(G)$) if and only if $\bigcap\Sigma_1 \in \Sigma_1$, $\bigcap\Sigma_2 \in \Sigma_2$ and $\bigcap\Sigma_1, \bigcap\Sigma_2$ are complementary direct factors in G.*

Proof. $\Leftarrow$: If $\bigcap\Sigma_1 \in \Sigma_1$, $\bigcap\Sigma_2 \in \Sigma_2$ then $\bigcap\Sigma_1 \cap \bigcap\Sigma_2 = \{0\}$, $\bigcap\Sigma_1 + \bigcap\Sigma_2 = G$ and thus $\tau(\Sigma_1) \vee_{\mathfrak{F}(G)} \tau(\Sigma_2)$ is a discrete topology and $\tau(\Sigma_1) \wedge_{\mathfrak{F}(G)} \tau(\Sigma_2) = \tau(\{G\})$.

$\Rightarrow$: With regard to [1], Theorem 3.5, the fact that $\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are complementary in $\mathfrak{F}(G)$ implies the existence of a neighbourhood basis of zero $\Sigma_1^0 \subseteq \Sigma_1$, $\Sigma_2^0 \subseteq \Sigma_2$ such that any $U \in \Sigma_1^0$ and any $V \in \Sigma_2^0$ are complementary direct factors in G. This implies $\Sigma_1^0 = \{\bigcap\Sigma_1^0\} = \{\bigcap\Sigma_1\}$, $\Sigma_2^0 = \{\bigcap\Sigma_2^0\} = \{\bigcap\Sigma_2\}$ and $\bigcap\Sigma_1 \in \Sigma_1$, $\bigcap\Sigma_2 \in \Sigma_2$, $\bigcap\Sigma_1$, $\bigcap\Sigma_2$ are complementary direct factors in G.

The rest for permutable topologies follows from the fact $\tau(\Sigma_1) \wedge_{\mathfrak{Y}(G)} \tau(\Sigma_2) = \tau(\Sigma)$, where $\Sigma = \{U + V : U \in \Sigma_1,\ V \in \Sigma_2\}$ (see 2.2).

3.5. Corollary. *Let G be an abelian l-group. Then the following assertions are equivalent:*

1. *$\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are complementary in the lattice $\mathfrak{F}(G)$.*
2. *$\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are complementary in the lattice $\mathfrak{Y}(G)$.*
3. *$\tau(\Sigma_1)$ and $\tau(\Sigma_2)$ are complementary in the lattice $\mathfrak{L}(G)$.*
4. *$\bigcap\Sigma_1 \in \Sigma_1$, $\bigcap\Sigma_2 \in \Sigma_2$ and $\bigcap\Sigma_1, \bigcap\Sigma_2$ are complementary direct factors in G.*

Proof. $1 \Leftrightarrow 2$ (see 3.2), $2 \Leftrightarrow 3$ (see 2.12), $4 \Rightarrow 1$ follows from Theorem 3.4.

$1 \Rightarrow 4$: According to [4], 1.2 $\bigcap \Sigma_1$, $\bigcap \Sigma_2$ are l-ideals in G and the rest follows from Theorem 3.4.

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