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EXTENSIONS OF ORDERED SEMIGROUPS*)

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By an *ordered semigroup* we mean a system $(S, \circ, <)$ where $S(\circ)$ is a semigroup, $S(<)$ is a totally ordered set, and the *monotone condition* – $a \leqq b$ and x in S imply $ax \leqq bx$ and $xa \leqq xb$ – is satisfied.

An (*ideal*) *extension* of a semigroup S by a semigroup T with zero is a semigroup Σ containing S as an ideal such that Σ/S, the Rees factor semigroup, is isomorphic with T.

Let S and T be disjoint semigroups, T having a zero element 0, and let $T^* = T\backslash 0$. Let φ be a single-valued mapping of T^* into S satisfying $tt' \neq 0$ implies $(tt')\,\varphi = t\varphi t'\varphi$, and let $\Sigma = S \cup T^*$. Let $\circ$ be defined on Σ as follows $(t, t' \in T^*;\ s, s' \in S)$:

(M1) $$t \circ t' = \begin{cases} tt' & \text{if} \quad tt' \neq 0\,, \\ t\varphi t'\varphi & \text{if} \quad tt' = 0\,; \end{cases}$$

(M2) $$t \circ s = (t\varphi)\, s\,;$$

(M3) $$s \circ t = s(t\varphi)\,;$$

(M4) $$s \circ s' = ss'\,.$$

Then Σ is an extension of S by T. [1, Theorem 4.1] (A necessary and sufficient condition for every ideal extension of S to be determined in this manner is that S have an identity element.)

The object of the present paper is to initiate an investigation of the following problem: If Σ is an extension of one ordered semigroup S by another ordered semigroup T with zero, ascertain if it is possible to define a monotone order on Σ which extends the given orders on S and T^*; and, if so, to describe all possible ways of doing so. The results established are limited to the case when Σ is defined as above in terms of a partial homomorphism φ of T^* into S.

Let M be the set of all *annihilators* of T; i.e., $t \in M$ if and only if $t \neq 0$ and $tt' = 0 = t't$ for all t' in T. Throughout we shall assume that $M \subseteq T^2$, in which case we shall call T an *essential semigroup*.

*) This work was begun while the author held a National Science Foundation Science Faculty Fellowship.

We call $(<)$ an *extending order* on Σ if $(<)$ is a total, monotone order on Σ which extends, or preserves, the existing orders on S and T^*. In Section 1 we show that every extending order on Σ carries with it a certain "null decomposition" of T^*, and we determine the structure of all such decompositions. In Example 1 we show that for certain semigroups S, Σ will admit no extending order, regardless of the conditions imposed on either T^* or φ.

The second section is devoted to establishing necessary and sufficient conditions for the existence of an extending order on Σ. In this section, also, we introduce the first of three special types of extending orders to be considered herein: the "close" extending orders. We show that if Σ admits an extending order, then it admits one which is close: and we tell how to describe such an order in terms of its null decomposition.

Section 2 ends with an example which, among other things, illustrates the other two special types of orders mentioned above: "φ-admissible" and "ϱ-separating" extending orders. These orders are the subjects of sections 3 and 4, respectively. We determine necessary and sufficient conditions for the existence of such orders, and describe each in terms of its null decomposition. We also show that if S is weakly reductive, then every extending order on Σ is φ-admissible; and if T^* has no annihilators, then every extending order on Σ is ϱ-separating.

The reader is referred to [1] and [2] for all concepts not defined in this paper.

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1. Null decompositions of T^*.

By a *null decomposition* of T^* we mean a pair $(X, Y) = (Y, X)$ of complementary subsets of T^* which satisfy the following conditions:

(N1) If $X \neq \emptyset \neq Y$, then $XY = YX = \{0\}$.

(N2) $X^2 \subseteq X \cup \{0\}$ and $Y^2 \subseteq Y \cup \{0\}$.

In the sequel we shall often make use of the following evident lemma.

1.1 Lemma. *Let (X, Y) be a null decomposition of T^*. If $tt' \neq 0$, then t, t' and tt' either all belong to X or all to Y.*

If we assume, as we do throughout this paper, that T is an essential ordered semigroup, the null decompositions of T^* are easy to describe.

1.2 Theorem. *If T is an essential ordered semigroup, then (T^-, T^+) and $(T^*, \emptyset)$, where $T^- = \{t \in T^* : t < 0\}$ and $T^+ = \{t \in T^* : t > 0\}$, are the only null decompositions of T^*.*

Proof. It is clear that if a and b both belong to $T^-\backslash M$, then there exists an element t of T^- such that either $at \neq 0 \neq bt$ or $ta \neq 0 \neq tb$. In either case, if one of the elements, a or b, also belongs to X, then Lemma 1.1 implies that the other one belongs to X, too. From this we conclude that if $X \cap T^- \backslash M \neq \emptyset$, then $T^- \backslash M \subseteq X$. Since $M \subseteq T^2$, if $x \in X \cap T^- \cap M$, then there exist elements t_1 and t_2 in T^- such that $x = t_1 t_2$. By Lemma 1.1, $t_1 \in X$. Thus $X \cap T^- \cap M \neq \emptyset$ implies $X \cap T^- \backslash M \neq \emptyset$. It follows that if $X \cap T^- \neq \emptyset$, then $T^- \backslash M \subseteq X$. In this case $T^- \cap M \subseteq (T^-)^2 \backslash 0 = (T^- \backslash M)^2 \backslash 0 \subseteq X^2 \backslash 0 \subseteq X$. We have thus shown that if $X \cap T^- \neq \emptyset$, then $T^- \subseteq X$. A similar argument leads to the conclusion that if $X \cap T^+ \neq \emptyset$, then $T^+ \subseteq X$. Therefore, $(X, Y) = (T^-, T^+)$ or $(X, Y) = (T^*, \emptyset)$.

A monotone order on Σ will be called an *extending order* if it coincides with the given orders on S and T^*. The relationship between extending orders on Σ and null decompositions of T^* is the subject of Lemma 1.3.

1.3 Lemma. *If* $(<)$ *is an extending order on* Σ *then* (X, Y), *where* $X = \{t \in T^* : t (<) t\varphi\}$ *and* $Y = \{t \in T^* : t (>) t\varphi\}$, *is a null decomposition of* T^*.

Proof. Since $(<)$ is a total order it is clear that $T^* = X \cup Y$ and $X \cap Y = \emptyset$. Suppose that neither X nor Y is empty and let $t_1 \in X$, $t_2 \in Y$. Then $t_1 (<) t_1\varphi$ and $t_2\varphi (<) t_2$. Since $(<)$ is monotone, $t_1\varphi t_2\varphi = t_1 \circ (t_2\varphi) (\leqq) t_1 \circ t_2 (\leqq) (t_1\varphi) \circ t_2 = t_1\varphi t_2\varphi$. Thus, $t_1 \circ t_2 = t_1\varphi t_2\varphi$. From the definition of $\circ$ in Σ it follows that $t_1 t_2 = 0$ (in T^*). Similarly $t_2 t_1 = 0$. Thus (N1) is satisfied. To prove that (N2) also holds, let $t_1, t_2 \in X$. Then $t_1 (<) t_1\varphi$, $t_2 (<) t_2\varphi$, and $t_1 \circ t_2 (\leqq) (t_1\varphi) \circ t_2 = t_1\varphi t_2\varphi$. Hence, if $t_1 t_2 \neq 0$, then $t_1\varphi t_2\varphi = (t_1 t_2)\varphi$ and $t_1 t_2 (<) (t_1 t_2)\varphi$. We conclude that $t_1 t_2 = 0$ or $t_1 t_2 \in X$. Likewise, $Y^2 \subseteq Y \cup \{0\}$.

Although there always is a null decomposition of T^*, the following example shows that it is not true that every extension admits an extending order.

Example 1. Let S be an unbounded right zero semigroup ($ab = b$ for all $a, b \in S$) and let x and y be elements of T^* such that $xy \neq 0$. We claim that no extension of S by T determined by a partial homomorphism admits an extending order. Suppose, by way of contradiction, that there exists such an extension Σ which does admit an extending order, and let $(<)$ be one such order. Assume that $x (<) x\varphi$. By hypothesis S contains an element a such that $a < x\varphi$. If $x (<) a < x\varphi$, then $x\varphi = a(x\varphi) = a \circ x \leqq a^2 = a$, a contradiction. Suppose $a (<) x (<) x\varphi$. Then $y\varphi = a(y\varphi) = a \circ y (<) xy (<) (x\varphi) \circ y = x\varphi y\varphi = y\varphi$, another contradiction. Since S also contains no greatest element, we again arrive at a contradiction if we assume that $x\varphi (<) x$. We thus conclude that our claim is valid.

The reader should observe that in Example 1, T may be any essential semigroup and φ, any partial homomorphism.

2. Extending orders on Σ and $L'G'$-decompositions of T^*. Define the relation ϱ on S as follows: $s_1 \varrho s_2$ *if and only if* $s_1 x = s_2 x$ *and* $x s_1 = x s_2$ *for all* x *in* S. Then ϱ is a convex congruence on S, and we can order the semigroup $\bar{S} = S/\varrho$ in the usual manner. i.e., $\bar{s}_1 < \bar{s}_2$ *if and only if* $\bar{s}_1 \neq \bar{s}_2$ *and* $s_1 < s_2$. (We wirte $\bar{s}$ for $s\varrho$.) In the following evident lemma we give an explicit description of this order on $\bar{S}$.

2.1 Lemma. *$\bar{s}_1 < \bar{s}_2$ if and only if there exists an element x of S such that either $s_1 x < s_2 x$ or $x s_1 < x s_2$. Also $\bar{s}_1 \leqq \bar{s}_2$ if and only if $s_1 x \leqq s_2 x$ and $x s_1 \leqq x s_2$ for all x in S.*

If, for each t in T^*, we let $t\bar{\varphi} = \overline{t\varphi}$, then $\bar{\varphi}$ is a partial homomorphism of T^* into $\bar{S}$.

2.2 Lemma. *If Σ admits an extending order, then $\bar{\varphi}$ is order preserving.*

Proof. Let $t_1 < t_2$. Then, for each x in S,

$$(t_1\varphi)\, x = t_1 \circ x \leqq t_2 \circ x = (t_2\varphi)\, x \quad \text{and} \quad x(t_1\varphi) = x \circ t_1 \leqq x \circ t_2 = x(t_2\varphi)\,.$$

It follows from Lemma 2.1 that $t_1\bar{\varphi} \leqq t_2\bar{\varphi}$.

Example 1 shows that the converse of Lemma 2.2 does not hold.

Let $[X, Y]$ be a null decomposition of T^*. For each t in T^* define A_t and B_t as follows: $A_t = \{s \in S : x\varphi \leqq s$ *for some* $x \in X$ *with* $t \leqq x\}$, $B_t = \{s \in S : y\varphi \geqq s$ *for some* $y \in Y$ *with* $t \geqq y\}$. Then A_t is an upper class and B_t, a lower class in S. We note for future reference that if φ is order preserving, then $t\varphi$ is the least [greatest] element in A_t [B_t] if $t \in X$ [$t \in Y$].

Example 2 below shows that this need not be true if φ is not order preserving.

Example 2. Let S and T be given by $a < b < a^2 < ab < ba < b^2 < 0$ and $m < n < m^2 < nm < mn < n^2 < 0$, respectively, with any product involving three or more factors equal to 0. Let φ be the partial homomorphism from T^* into S such that $m\varphi = a$ and $n\varphi = b$. Clearly, φ is not order preserving. Taking $[X, Y] = [T^*, \emptyset]$ and $t = nm$, we see that $A_t = \{ab, ba, b^2, 0\}$ and that $t\varphi = (nm)\,\varphi = ba$ is not the least element in A_t.

2.3 Lemma. *Let $(<)$ be an extending order on Σ, and let $[X, Y]$ be its null decomposition. Then*

(a) *$t\,(<)\,s$ implies $t\bar{\varphi} \leqq \bar{s}$;*

(b) *$t\,(>)\,s$ implies $t\bar{\varphi} \geqq \bar{s}$;*

(c) *if $s_1\,(<)\,t\,(<)\,s_2$ and $t \notin M$, then $\bar{s}_1 < \bar{s}_2$;*

(d) *for each t in T^*, $B_t\,(<)\,t\,(<)\,A_t$.*

Proof. (a) Let $t\,(<)\,s$ and let $x \in S$. Then $(t\varphi)\,x = t \circ x \leqq sx$ and $x(t\varphi) = x \circ t \leqq xs$. Thus $t\bar{\varphi} \leqq \bar{s}$. (b) is the dual of (a). (c) We may assume that $tt' \neq 0$ for some t' in T^*. Then $s_1\,(<)\,t\,(<)\,s_2$ implies $s_1(t'\varphi)\,(<)\,tt'\,(<)\,s_2(t'\varphi)$. Hence, by Lemma 2.1, $\bar{s}_1 < \bar{s}_2$.

(d) Let $s \in A_t$. Let x in X be such that $t \leqq x$ and $x\varphi \leqq s$. By definition of X, $x\,(<)\,x\varphi$. Therefore $t\,(<)\,s$. Dually, if $s \in B_t$, then $s\,(<)\,t$.

The next two corollaries are immediate consequences of Lemma 2.3.

2.4 Corollary. *Let $(<)$ be an extending order on Σ and let $[X, Y]$ be its null decomposition. Then*

(a)

$$\left.\begin{array}{l} t\bar{\varphi} < s;\ or \\ t\bar{\varphi} = \bar{s}\ and\ t \in X\backslash M;\ or \\ t\bar{\varphi} = \bar{s},\ t \in X \cap M\ and\ s \in A_t \end{array}\right\} implies\ t\,(<)\,s;\ and$$

(b)

$$\left.\begin{array}{l} t\bar{\varphi} > \bar{s};\ or \\ t\bar{\varphi} = \bar{s}\ and\ t \in Y\backslash M;\ or \\ t\bar{\varphi} = \bar{s},\ t \in Y \cap M\ and\ s \in B_t \end{array}\right\} implies\ t\,(>)\,s\,.$$

Remark. Throughout this paper statements such as (a) and (b) will be considered to be order duals.

2.5 Corollary. *If $(<_1)$ and $(<_2)$ are two extending orders on Σ sharing the same null decomposition, then $t\ (<_1)\,s$ and $s\,(<_2)\ t$ only if $t \in M$, $t\bar{\varphi} = \bar{s}$ and $B_t < s < A_t$.*

By a *close extending order* on Σ we mean an extending order which places each element t of T^* as close to its image $t\varphi$ as Corollary 2.4 permits. It is clear that if such an order exists, it is uniquely determined (among close extending orders) by its null decomposition $[X, Y]$ as follows:

$$\textbf{2.6.} \qquad t\,(<)\,s\ if\ and\ only\ if \begin{cases} t\bar{\varphi} < \bar{s};\ or \\ t\bar{\varphi} = \bar{s}\ and\ t \in X\backslash M;\ or \\ t\bar{\varphi} = \bar{s},\ t \in X \cap M\ and\ s \in A_t;\ or \\ t\bar{\varphi} = \bar{s},\ t \in Y \cap M\ and\ s \notin B_t. \end{cases}$$

Remark. To define an extending order on Σ we tell when an element t of T^* is less than an element s of S; then $t\,(>)\,s$ means $t\,(<)\,s$ is not true. In the case of a close extending order, for example, we obtain $t\,(>)\,s$ from (2.6) by replacing $<$, X, A_t, Y and B_t by $>$, Y, B_t, X and A_t, respectively.

Our immediate goal (Theorem 2.13 below) is to show that if Σ admits an extending order, it admits a close extending order, and to give necessary and sufficient conditions for this to occur. Let (X, Y) be a null decomposition of T^*. The ordered pair $[X, Y]$ will be called an *$L'G'$-decomposition* of T^* if X satisfies Condition L′ given below, and Y satisfies its order dual, Condition G′.

Condition L′

(L′1) If $tt' \in X$ $[t't \in X]$ and $\bar{s} < t\bar{\varphi}$, then $s(t'\varphi) \notin A_{tt'}$, $[(t'\varphi)\,s \notin A_{t't}]$.

(L′2) If $tt' \in X$ $[t't \in X]$, $t_1 < t$ and $t_1\bar{\varphi} = t\varphi$, then $t_1t' \in X$ $[t't_1 \in X]$.

Together Lemma 2.2 and Lemma 2.7 below prove half of the first statement in Theorem 2.13.

2.7 Lemma. *Let* $(<)$ *be an extending order on* Σ *and let* $[X, Y]$ *be its null decomposition. Then* $[X, Y]$ *is an* $L'G'$*-decomposition of* T^*.

Proof. Since Conditions L′ and G′ are order duals, we need only show that X satisfies Condition L′. Furthermore we need only consider the "right half" of that condition.

(L′1) Let $tt' \in X$ and $\bar{s} < t\bar{\varphi}$. By Corollary 2.4, $s\,(<)\,t$. Hence if $x \in X$ and $tt' \leqq x$, then $s(t'\varphi) = s \circ t'\,(<)\,tt' \leqq x\,(<)\,x\varphi$. It follows from the definition of $A_{tt'}$ that $s(t'\varphi) \notin A_{tt'}$.

(L′2) Let $tt' \in X$, $t_1 < t$ and $t_1\bar{\varphi} = t\bar{\varphi}$. Then $t_1 \circ t' \leqq tt'$ and $tt'\,(<)\,(tt')\,\varphi = t\varphi t'\varphi$. Since $t_1\bar{\varphi} = t\bar{\varphi}$, $t\varphi t'\varphi = t_1\varphi t'\varphi$. Thus $t_1 \circ t'\,(<)\,t_1\varphi t'\varphi$; i.e., $t_1 t' \in X$.

2.8 Lemma. *Let* $\bar{\varphi}$ *be order preserving. Let* $(<)$ *be a total order on the set* Σ *which preserves the existing orders on* S *and* T^* *and such that* $t\,(<)\,s$ *implies* $t\bar{\varphi} \leqq \bar{s}$ *and* $t\,(>)\,s$ *implies* $t\bar{\varphi} \geqq \bar{s}$. *Let* $X = \{t \in T^* : t\,(<)\,t\varphi\}$ *and* $Y = \{t \in T^* : t\,(>)\,t\varphi\}$. *If* X *satisfies* (L′2) *and* Y *satisfies* (G′2), *then* $(<)$ *satisfies the right monotone condition if and only if* (i) $s\,(<)\,t$ *and* $tt' \in X$ *implies* $s(t'\varphi)\,(<)\,tt'$, *and* (ii) $s\,(>)\,t$ *and* $tt' \in Y$ *implies* $s(t'\varphi)\,(>)\,tt'$.

Proof. In the course of this proof Lemmas 1.1 and 2.1, as well as the definition of $(\circ)$ in Σ, are often used. Specific reference to these properties will not be made. The necessity of (i) and (ii) is evident, so let us assume (i) and (ii). We consider four major cases.

(a) $s_1 < s_2$. Using the monotonicity of $<$ in S, we have $s_1 \circ a \leqq s_2 \circ a$ for all a in Σ.

(b) $s\,(<)\,t$. If $a \in S$ or if $a \in T^*$ and $ta = 0$, then $s \circ a \leqq t \circ a$ since $s\,(<)\,t$ implies $\bar{s} \leqq t\bar{\varphi}$. If $tt' \in Y$, then $(tt')\,\varphi\,(<)\,tt'$ and $s(t'\varphi) \leqq t\varphi t'\varphi\,(<)\,tt'$. The case when $s\,(<)\,t$ and $tt' \in X$ is condition (i) of this lemma.

(c) $s\,(>)\,t$. This is the order dual of (b).

(d) $t_1 < t_2$. Since $\bar{\varphi}$ is order preserving, $t_1\bar{\varphi} \leqq t_2\bar{\varphi}$. Thus, if $a \in S$ or $a \in T^*$ and $t_1 a = 0 = t_2 a$, then $t_1 \circ a \leqq t_2 \circ a$. Suppose $t_1 t' = 0$ and $t_2 t' \neq 0$. If $t_2 t' \in Y$, then $t_1 \circ t' = t_1\varphi t'\varphi \leqq t_2\varphi t'\varphi\,(<)\,t_2 t'$. If $t_2 t' \in X$, then $t_1\bar{\varphi} < t_2\bar{\varphi}$ since X satisfies (L′2). Thus $t_1\varphi\,(<)\,t_2$ and, applying (i), we conclude that $t_1 \circ t' = t_1\varphi t'\varphi\,(<)\,t_2 t'$. The subcase $t_1 t' \neq 0$ and $t_2 t' = 0$ is dual to the one just considered, and makes use of (ii) and the fact that Y satisfies (G′2). Finally, if $t_1 t' \neq 0 \neq t_2 t'$, then $t_1 t' \leqq t_2 t'$ in T^*.

As noted earlier, to define an extending order on Σ we tell when $t\,(<)\,s$, and let $t\,(>)\,s$ mean that $t\,(<)\,s$ is not true. To prove that a relation defined in this manner is an order on the *set* Σ extending those on S and T^*, we need only show that it is transitive; i.e., $a\,(<)\,b$ and $b\,(<)\,c$ imply $a\,(<)\,c$. Moreover, since $<$ is transitive in S and in T^*, it suffices to consider the six cases when exactly two of the elements a, b, c belong to S or to T^*. This list of cases can be further reduced by observing that the three cases that occur when two of the elements belong to S are equivalent. Similarly, the three cases that occur when two of the elements belong to T^* are equivalent. We summarize the foregoing remarks in Lemma 2.9.

2.9 Lemma. *A relation* $(<)$ *defined on* Σ *as in the above remark is transitive if and only if these two conditions are satisfied*: (i) $s_1 < s_2$ *and* $s_2\,(<)\,t$ *imply* $s_1\,(<)\,t$; (ii) $t_1 < t_2$ *and* $t_2\,(<)\,s$ *imply* $t_1\,(<)\,s$.

2.10 Lemma. *Let* $\bar{\varphi}$ *be order preserving. If* $t_1 \in T^- \cap M$ *and* $t_2 \in T^+ \cap M$, *then* $t_1\varphi \leqq t_2\varphi$.

Proof. Since $M \subseteq T^2$, $t_1 = a_1 b_1$ and $t_2 = a_2 b_2$ with $a_1 < a_2$ and $b_1 < b_2$. Since $\bar{\varphi}$ is order preserving, $a_1\bar{\varphi} \leqq a_2\bar{\varphi}$ and $b_1\bar{\varphi} \leqq b_2\bar{\varphi}$. Applying Lemma 2.1, we get $a_1\varphi b_1\varphi \leqq a_2\varphi b_2\varphi$ and $a_2\varphi b_1\varphi \leqq a_2\varphi b_1\varphi$; hence $t_1\varphi \leqq t_2\varphi$.

2.11 Lemma. *Let* $\bar{\varphi}$ *be order preserving and let* $[X, Y]$ *be an* $L'G'$*-decomposition of* T^*. *If* $t_1 \in Y$, $t_2 \in X$ *and* $t_1 < t_2$, *then* $t_1\varphi < t_2\varphi$. *If, in addition, either* $t_1 \notin M$ *or* $t_2 \notin M$, *then* $t_1\bar{\varphi} < t_2\bar{\varphi}$.

Proof. Suppose $t_2 \in X \backslash M$ and let $t_2 t \in X$. Since X satisfies (L′2), the conditions $\bar{\varphi}$ is order preserving and $t_1 < t_2$ imply that either $t_1\bar{\varphi} < t_2\bar{\varphi}$ or $t_1 t \in X$. Since the latter implies $t_1 \in X$ (Lemma 1.1), we conclude that $t_1\bar{\varphi} < t_2\bar{\varphi}$. Similarly, since Y satisfies (G′2), if $t_1 \notin M$, then $t_1\bar{\varphi} < t_2\bar{\varphi}$. To complete the proof of this lemma we need only show that if both t_1 and t_2 are annihilators, then $t_1\varphi < t_2\varphi$.

By Theorem 1.2, $Y = T^-$ and $X = T^+$. Since $M \subseteq T^2$, if $t_1 \in M$ and $t_2 \in M$, then $t_1 = a_1 b_1$ and $t_2 = a_2 b_2$ with $a_1, b_1 \in T^- \backslash M$ and $a_2, b_2 \in T^+ \backslash M$. Applying the results of the preceding paragraph, we see that $a_1\bar{\varphi} < a_2\bar{\varphi}$ and $b_1\bar{\varphi} < b_2\bar{\varphi}$. Since X satisfies (L′1), the conjunction of $a_1\bar{\varphi} < a_2\bar{\varphi}$ and $a_2 b_2 \in X$ implies $a_1\varphi b_2\varphi <$ $< a_2\varphi b_2\varphi$. Y satisfies (G′1); hence $b_1\bar{\varphi} < b_2\bar{\varphi}$ and $a_1 b_1 \in Y$ imply $a_1\varphi b_1\varphi <$ $< a_1\varphi b_2\varphi$. Therefore $t_1\varphi = (a_1 b_1)\,\varphi < (a_2 b_2)\,\varphi = t_2\varphi$.

2.12 Lemma. *If* $\bar{\varphi}$ *is order preserving and* $[X, Y]$ *is an* $L'G'$*-decomposition of* T^*, *then the following hold*:

(1) *If* $tt' \in X \backslash M$ $[t't \in X \backslash M]$ *and* $\bar{s} < t\bar{\varphi}$, *then* $\bar{s}(t'\bar{\varphi}) < (tt')\,\bar{\varphi}$ $[(t'\bar{\varphi})\,\bar{s} < (t't)\,\bar{\varphi}]$.

(2) *If* $tt' \in Y \backslash M$ $[t't \in Y \backslash M]$ *and* $\bar{s} > t\bar{\varphi}$, *then* $\bar{s}(t'\bar{\varphi}) > (tt')\,\bar{\varphi}$ $[(t'\bar{\varphi})\,\bar{s} > (t't)\,\bar{\varphi}]$.

(3) *If* $t_1 < t_2$, *then*

(*a*) $A_{t_1} \supseteq A_{t_2}$;

(b) *if* $t_1 \in X \cap M$ *and* $t_2 \in Y \cap M$, *then* $S \backslash B_{t_2} \subseteq A_{t_1}$;

(c) *if* $t_1 \in Y \cap M$ *and* $t_2 \in X \cap M$, *then* $A_{t_2} \subseteq S \backslash B_{t_1}$;

(d) $B_{t_1} \subseteq b_{t_2}$.

Proof. As before we despose of both (1) and (2) by proving the "right half" of (1) and appealing to duality for the remaining parts of the proof. We shall make frequent use of Lemma 2.1 and the fact that X satisfies (L′1).

If $tt' \in X \backslash M$, then for some a in T^* either $tt'a \in X$ or $att' \in X$. If $tt'a \in X$, then $\bar{s} < t\bar{\varphi}$ implies that $s(t'\varphi)\,(a\varphi) = s(t'a)\,\varphi < (tt'a)\,\varphi = (tt')\,\varphi(a\varphi)$, which in turn implies that $\bar{s}(t'\bar{\varphi}) < (tt')\,\bar{\varphi}$. If, one the other hand, $att' \in X$, then either $a^2t \in X$ or $tt't' \in X$. The latter case is handled as above. If we assume that $a^2t \in X$, we may conclude that $\overline{(a\varphi)\,s} < (at)\,\bar{\varphi}$ by using the left dual of the previous arument. Since $(at)\,t' \in X$, this implies that $(a\varphi)\,s(t'\varphi) = [(a\varphi)\,s]\,(t'\varphi) < [(at)\,t']\,\varphi = (a\varphi)\,(tt')\,\varphi$. Thus, $\bar{s}(t'\bar{\varphi}) < (tt')\,\bar{\varphi}$. This concludes the proof of parts (1) and (2).

Now for part (3). (a) and (d) are true by definition of A_t and B_t, respectively. By Lemma 2.10 if the conditions in (b) hold, then $t_1\varphi \leqq t_2\varphi$. If $s \notin B_{t_2}$, then $t_2\varphi < s$. Since $t_1 \in X$ and $t_1\varphi < s$, $s \in A_{t_1}$, which proves (b). The conditions in (c) and Lemma 2.11 imply that $y\varphi < x\varphi$ for all $y \in Y$ and $x \in X$. (Note that from the conditions given we have $X = T^+$ and $Y = T^-$.) Thus, if $x \in X$, $t_2 \leqq x$ and $x\varphi \leqq s$, then $y\varphi < s$; i.e., $s \notin B_{t_1}$.

We are now prepared to state and prove our first main theorem.

2.13 First Main Theorem. *Σ admits an extending order if and only if $\bar{\varphi}$ is order preserving and T^* admits an $L'G'$-decomposition.*

If $\bar{\varphi}$ is order preserving, then there is a one-to-one correspondence between the set of $L'G'$-decomposition of T^* and the set of close extending orders on Σ such that if $[X, Y]$ and $(<)$ correspond, then $[X, Y]$ is that decomposition of T^* given in Lemma 1.3 *and $(<)$ is given by* (2.6).

Remark. By Theorem 1.2, Σ may admit at most four close extending orders.

Proof. First, assume that Σ admits an extending order $(<)$, and let $[X, Y]$ be its null decomposition (Lemma 1.3). Then $\bar{\varphi}$ is order preserving (Lemma 2.2) and $[X, Y]$ is an L′G′-decomposition of T^* (Lemma 2.7).

Conversely, let $\bar{\varphi}$ be order preserving, let $[X, Y]$ be an L′G′-decomposition and let $(<)$ be defined by (2.6). To prove that $(<)$ is transitive we follow Lemma 2.9. Let $s_1 < s_2$ and $s_2\,(<)\,t$. Then $\bar{s}_1 \leqq \bar{s}_2$ and $\bar{s}_2 \leqq t\bar{\varphi}$. If strict inequality holds in either place, then $\bar{s}_1 < t\bar{\varphi}$. By definition of $(<)$, $s_1\,(<)\,t$. Suppose $\bar{s}_1 = \bar{s}_2$ and $\bar{s}_2 = t\bar{\varphi}$. By (2.6), $t \in Y \cup M$. If $t \in Y\backslash M$, then $\bar{s}_1 = t\bar{\varphi}$ implies $s_1\,(<)\,t$. If $t \in X \cap M$, then $s_2 \notin A_t$. Since A_t is an upper class in S, $s_1 \notin A_t$; hence $s_1\,(<)\,t$. Suppose $t \in Y \cap M$ so that $s_2 \in B_t$. Then, because B_t is a lower class in S, $s_1 \in B_t$. Again $s_1\,(<)\,t$ by (2.6).

To complete the proof that $(<)$ is transitive, we must show that if $t_1 < t_2$ and $t_2\,(<)\,s$, then $t_1\,(<)\,s$. Since $\bar{\varphi}$ is order preserving, $t_1\bar{\varphi} \leqq t_2\bar{\varphi}$. Also, by definition of $(<)$, $t_2\bar{\varphi} \leqq \bar{s}$. If $t_1\bar{\varphi} < t_2\bar{\varphi}$ or if $t_2\bar{\varphi} < \bar{s}$, then $t_1\,(<)\,s$. Hence we may assume that $t_1\bar{\varphi} = t_2\bar{\varphi}$ and $t_2\bar{\varphi} = \bar{s}$.

The conditions $t_2\,(<)\,s$ and $t_2\bar{\varphi} = \bar{s}$ imply $t_2 \in X \cup M$, which in turn implies that $t_1 \in X \cup M$ (Lemma 2.11). If $t_1 \in X\backslash M$, then $t_1\,(<)\,s$, by definition. Suppose $t_1 \in M$. We consider several cases. Let $t_1 \in X \cap M$. If $t_2 \in X \cap M$, then $s \in A_{t_2} \subseteq A_{t_1}$ (Lemma 2.12 (a)). If $t_2 \in Y \cap M$, then $s \in S\backslash B_{t_2} \subseteq A_{t_1}$ (Lemma 2.12 (b)). Let $t_1 \in Y \cap M$. If $t_2 \in X \cap M$, then $s \in A_{t_2} \subseteq S\backslash B_{t_1}$ (Lemma 2.12 (c)). If $t_2 \in Y \cap M$, then $s \in S\backslash B_{t_2} \subseteq S\backslash B_{t_1}$ (Lemma 2.12 (d)). In either of the above we conclude that $t_1\,(<)\,s$.

It remains to prove that $(<)$ is monotone. We need show only that $(<)$ is right monotone. Let $s\,(<)\,t$ and $tt' \in X$. Since $tt' \in X$ implies $t \in X \backslash M$ $s\,(<)\,t$ implies $\bar{s} < t'\bar{\varphi}$ by (2.6). Since X satisfies (L'1), $\bar{s}(t'\bar{\varphi}) < t\bar{\varphi}t'\bar{\varphi}$ if $tt' \notin M$, and $s(t'\varphi) \notin A_{tt'}$ otherwise. In either case $s(t'\varphi)\,(<)\,tt'$ by definition of $(<)$. Dually, if $s\,(>)\,t$ and $tt' \in Y$, then $s(t'\varphi)\,(>)\,tt'$. Applying Lemma 2.8, we conclude that $(<)$ satisfies the right monotone condition.

The remainder of the theorem follows immediately.

Although a given close extending order is uniquely determined by its null decomposition, it may be that a given L'G'-decomposition of T^* is associated with more than one extending order.

Example 3. Let T be the cyclic semigroup $t < t^2 < 0$, let S be the commutative semigroup $s < a < s^2 < 0$ with $ax = s^2x = 0x = 0$ for all x in S; and let $t\varphi = s$, $t^2\varphi = s^2$. Then $[T^*, \emptyset]$ is the null decomposition for both of the following extending orders, the first of which is the close order determined by $[T^*, \emptyset]$:

(3a) $t\,(<)\,s < a\,(<)\,t^2\,(<)\,s^2 < 0$;

(3b) $t\,(<)\,s\,(<)\,t^2\,(<)\,a < s^2 < 0$.

Note that the ϱ-classes of S are $\{s\}$ and $\{a, s^2, 0\}$.

As we saw earlier (Lemma 2.3), if $(<)$ is an extending order on Σ and if $[X, Y]$ is its null decomposition of T^*, then, thinking in terms of placing the elements of T^* among those of S, each t in X must be placed immediately below the ϱ-class $t\bar{\varphi}$ or else in $t\bar{\varphi}$ (below A_t). Dual remarks hold for elements of Y. We thus see that among all possible extending orders there are two extreme cases:

(1) each element t of T^* is placed next to $t\varphi$ (relative to S, allowing for other elements of T^* to be placed between t and $t\varphi$);

(2) each element t of T^* is placed as far away from $t\varphi$ as possible; namely, adjacent to $t\bar{\varphi}$. Examples (3a) and (3b) illustrate these two extremes; (3a) corresponds to (1); while (3b) corresponds to (2). In the next two sections we consider these special orders.

3. φ-Admissible extending orders on Σ and LG-decompositions of T^*. An extending order $(<)$ on Σ will be called *φ-admissible* if $t\,(<)\,s$ implies $t\varphi \leqq s$ and $t\,(>)\,s$ implies $t\varphi \geqq s$. If $(<)$ is such an order and if $[X, Y]$ is its null decomposition, it is clear that $(<)$ is given by

3.1. $$t\,(<)\,s \textit{ if and only if } \begin{cases} t\varphi < s\,, & \textit{or} \\ t\varphi = s & \textit{and} \quad t \in X\,. \end{cases}$$

Evidenlty, a φ-admissible extending order is close.

3.2 Lemma. *If Σ admits a φ-admissible extending order, then φ is order preserving.*

Proof. Let $(<)$ be φ-admissible with null decomposition $[X, Y]$. Let $t_1 < t_2$. In view of Theorem 2.13 and Lemmas 2.10 and 2.11 it remains to show that $t_1\varphi \leqq$

$\leqq t_2\varphi$ when all of the following conditions hold: $t_1 \in X$, $t_2 \in Y$, $t_1 \notin M$ or $t_2 \notin M$, and $t_1\bar{\varphi} = t_2\bar{\varphi}$. By Corollary 2.4; the conjunction of $t_1\bar{\varphi} = t_2\bar{\varphi}$ and $t_2 \in Y\backslash M$ implies $t_1\varphi\,(<)\,t_2$; similarly, if $t_1 \in X\backslash M$, then $t_1\,(<)\,t_2\varphi$. In either case, since $(<)$ is φ-admissible, $t_1\varphi \leqq t_2\varphi$.

Let $[X, Y]$ be a null decomposition of T^*. The ordered pair $[X, Y]$ will be called an *LG-decomposition* of T^* if X satisfies Condition L given below; and Y satisfies its order dual, Condition G.

Condition L.

(L1) If $tt' \in X$ $[t't \in X]$ and $s < t\varphi$, then $s(t'\varphi) < t\varphi t'\varphi$ $[(t'\varphi)\,s < t'\varphi t\varphi]$.

(L2) If $tt' \in X$ $[t't \in X]$, $t_1 < t$ and $t_1\varphi = t\varphi$, then $t_1t' \in X$ $[t't_1 \in X]$.

3.3 Lemma. *If φ is order preserving, then Condition $L[G]$ implies Condition $L'[G']$.*

Proof. Assume that X satisfies Condition L. Let $tt' \in X$. If $\bar{s} < t\bar{\varphi}$, then $s < t\varphi$. By (L1), $s(t'\varphi) < t\varphi t'\varphi$. Since φ is order preserving $(tt')\,\varphi$ is the least element in $A_{tt'}$. Thus X satisfies (the right half of) (L'1). Next, if $t_1 < t$ and $t_1\bar{\varphi} = t\bar{\varphi}$, then $t_1\varphi = t\varphi$; otherwise, $t_1\varphi < t\varphi$ and $t_1\varphi t'\varphi < t\varphi t'\varphi$ by (L1), contradicting $t_1\bar{\varphi} = t\bar{\varphi}$. From (L2) we get $t_1t' \in X$.

3.4 Lemma. *If φ is order preserving and $[X, Y]$ is an LG-decomposition, then the relations defined by (3.1) and (2.6) are the same.*

Proof. Denote the relation given by (3.1) by $<_\varphi$ and that given by (2.6) by $<_c$. Since $<_\varphi$ and $<_c$ agree with $<$ on S and T^* and since $t\,(>)\,s$ is dual to $t\,(<)\,s$ in each case, it suffices to show that $t <_\varphi s$ implies $t <_c s$.

Let $t <_\varphi s$. Since Y satisfies (G1), if $t\varphi < s$, then either $t\bar{\varphi} < \bar{s}$ or $t \in X \cup M$. If $t\bar{\varphi} < \bar{s}$, then $t <_c s$ by definition. Hence we may assume that $t\bar{\varphi} \geqq \bar{s}$ and $t \in X \cup M$. If $t \in X\backslash M$, then, because φ is order preserving, $t\varphi < s$ implies $s \in A_t$ if $t \in X \cap M$ and $s \notin B_t$ if $t \in Y \cap M$. In either case, $t <_c s$. Next, suppose that $t\varphi = s$ and $t \in X$. If $t \in X\backslash M$, then $t <_c s$. If $t \in X \cap M$, then $s \in A_t$. Thus $t <_c s$.

3.5 Second Main Theorem. *Σ admits a φ-admissible extending order if and only if φ is order preserving and T^* admits an LG-decomposition.*

If φ is order preserving, then there is a one-to-one correspondence between the set of LG-decompositions of T^* and the set of φ-admissible extending orders on Σ such that if $[X, Y]$ and $(<)$ correspond, then $[X, Y]$ is that decomposition of T^* given in Lemma 1.3 and $(<)$ is given by (3.1).

Proof. Let φ be order preserving and let $[X, Y]$ be an LG-decomposition of T^*. Then $\bar{\varphi}$ is order preserving and $[X, Y]$ is an L'G'-decomposition (Lemma 3.3). It follows from Theorem 2.13 that (2.6) defines an extending order on Σ. However, by Lemma 3.4, in this case (2.6) and (3.1) are the same. We conclude that Σ admits a φ-admissible extending order.

Conversely, suppose that $(<)$ is φ-admissible. Then φ is order preserving (Lemma 3.2), and $[X, Y]$, its null decomposition, is an L′G′-decomposition (Theorem 2.13). Clearly, X satisfies (L2), since it satisfies (L′2). To show that X satisfies (the right half of) (L1), let $tt' \in X$ and $s < t\varphi$. Then, by (3.1), $s\,(<)\,t\,(<)\,t\varphi$. Therefore $s(t'\varphi) = s \circ t'\,(<)\,tt'\,(<)\,t\varphi t'\varphi$. Dually, Y satisfies Condition G.

The remainder of the theorem follows immediately.

A semigroup S is *weakly reductive* if ϱ is the identity relation on S; i.e., if $ax = bx$ and $xa = xb$ for all x in S imply $a = b$. In this situation all extending orders on Σ are determined by Theorem 3.5. To prove the first statement in Corollary 3.6, we need only replace $\bar{\varphi}$ by φ and $\bar{s}$ by s in parts (a) and (b) of Lemma 2.3. The remainder of the corollary follows from Theorems 3.5 and 1.2.

3.6 Corollary. *If S is weakly reductive, then every extending order on Σ is φ-admissible. In this case Σ admits an extending order if and only if φ is order preserving and T^* admits an LG-decomposition; hence Σ admits at most four extending orders.*

If the semigroup S is cancellative, it is weakly reductive. In this case, also, $[T^-, T^+]$ is an LG-decomposition. In view of these facts, the next corollary is obvious.

3.7 Corollary. *If S is cancellative, then Σ admits an extending order (necessarily φ-admissible) if and only if φ is order preserving.*

Remark. Note that in Example 1, S is left cancellative, yet Σ admits no extending order regardless of the conditions imposed on T or φ.

4. ϱ-Separating extending orders on Σ and $\overline{\text{LG}}$-decompositions of T^*. In this section we discuss the second of the two extreme cases among all possible extending orders mentioned at the end of Section 2. If the extending order $(<)$ on Σ permits no t in T^* to "split" a ϱ-class; i.e., if $s_1\,(<)\,t\,(<)\,s_2$ implies $\bar{s}_1 < \bar{s}_2$; $(<)$ will be called *ϱ-separating.* If $(<)$ is such an order and if $[X, Y]$ is its null decomposition, it is clear that $(<)$ is given by

4.1. $$t\,(<)\,s \text{ if and only if } \begin{cases} t\bar{\varphi} < \bar{s}, & \text{or} \\ t\bar{\varphi} = \bar{s} & \text{and} \quad t \in X\,. \end{cases}$$

It follows from this and from Theorem 1.2 that Σ admits at most four ϱ-separating extending orders.

Since $\bar{\varphi}$ is a partial homomorphism from T^* to $\bar{S}$, we can consider the extension $\bar{\Sigma} = \bar{S} \cup T^*$ of $\bar{S}$ by T determined by $\bar{\varphi}$. An LG-decomposition of T^* relative to $\bar{\varphi}$ and $\bar{S}$ will be called an $\overline{\text{LG}}$-decomposition of T^*. Thus, Condition $\bar{\text{L}}$ may be obtained from Condition L by replacing φ and s in the latter by $\bar{\varphi}$ and $\bar{s}$, respectively. Note that ($\bar{\text{L}}$2) coincides with (L′2); the same holds true for ($\bar{\text{G}}$2) and (G′2).

4.2. Third Main Theorem. *If $\bar{\varphi}$ is order preserving, then Σ admits a ϱ-separating extending order if and only if T^* admits an $\overline{\text{LG}}$-decomposition.*

In this case, there is a one-to-one correspondence between the set of $\overline{L}\overline{G}$-decompositions of T^* and the set of ϱ-separating extending orders on Σ such that if $[X, Y]$ and $(<)$ correspond, then $[X, Y]$ is that decomposition of T^* given in Lemma 1.3 and $(<)$ is given by (4.1).

Proof. Let $\bar{\varphi}$ be order preserving and let $[X, Y]$ be an $\overline{L}\overline{G}$-decomposition of T^*. Let $(<)$ be defined by (4.1). To show that $(<)$ is transitive we consider two cases:

(i) $s_1 < s_2$ and $s_2 (<) t$. If $\bar{s}_1 < \bar{s}_2$ or $\bar{s}_2 < t\bar{\varphi}$, then $s_1 (<) t$. If $\bar{s}_2 = t\bar{\varphi}$ then $t \in Y$ by definition of $(<)$. If $\bar{s}_1 = \bar{s}_2$, also, then $s_1 (<) t$.

(ii) $t_1 < t_2$ and $t_2 (<) s$. If $t_2\bar{\varphi} < \bar{s}$, then $t_1 (<) s$. By definition of $(<)$, if $t_2\bar{\varphi} = \bar{s}$, then $t_2 \in X$. Applying Theorem 3.5, we see that $[X, Y]$ determines an extending order on $\bar{\Sigma}$. In this situation Lemma 2.11 implies $t_1\bar{\varphi} < t_2\bar{\varphi}$ or $t_1 \in X$. In either case $t_1 (<) s$. To prove that $(<)$ is right monotone we follow Lemma 2.8. (Clearly, $\overline{L}\overline{G}$ – implies L'G' –.) (i) Let $s (<) t$ and $tt' \in X$. Then $t \in X$, so that $\bar{s} < t\bar{\varphi}$. By $(\overline{L}1)$, $\bar{s}t'\bar{\varphi} < t\bar{\varphi}t'\bar{\varphi}$; thus $s(t'\varphi) (<) tt'$. (ii) is dual to (i). We conclude that $(<)$ is an extending order on Σ.

Conversely, let $(<)$ be ϱ-separating with null decomposition $[X, Y]$. Then X satisfies $(\overline{L}2) = (L'2)$. If $tt' \in X$ and $\bar{s} < t\bar{\varphi}$, then $s (<) t (<) t\varphi$. It follows that $s(t'\varphi) (<) tt' (<) t\varphi t'\varphi$. Since $(<)$ is ϱ-separating, the latter implies $s(t'\bar{\varphi}) < t\bar{\varphi}t'\varphi$. The case $t't \in X$ is similar to the above. Dually, Y satisfies Condition $\overline{G}$. The remainder of the theorem is immediate.

Two extending orders $(<)$ and $[<]$ on Σ and $\bar{\Sigma}$, respectively, are *ϱ-equivalent* if $t (<) s \Leftrightarrow t [<] \bar{s}$.

4.3 Corollary. *There is a one-to-one correspondence between the set of ϱ-separating extending orders on Σ and the set of $\bar{\varphi}$-admissible extending orders on $\bar{\Sigma}$ such that $(<)$ on Σ and $[<]$ on $\bar{\Sigma}$ correspond if and only if they are ϱ-equivalent.*

Proof. This is an immediate consequence of Theorem 4.2 and Theorem 3.5 applied to the extension $\bar{\Sigma}$.

By part (c) of Lemma 2.3, if $M = \emptyset$, then every extending order on Σ is ϱ-separating. Thus Corollary 4.4 holds.

4.4 Corollary. *If T is without annihilators, then every extending order on Σ is ϱ-separating. In this case, Σ admits an extending order if and only if T^* admits an $\overline{L}\overline{G}$-decomposition; hence Σ admits at most four such orders.*

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