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SEVERAL NEW CHARACTERIZATIONS OF THE SPHERE

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1. Preliminaries. Let M be an ovaloid of class C^4 in the 3-dimensional Euclidean space E^3. Let $\{U_\alpha\}$ be its covering by domains such that in each U_α there is a field of orthonormal frames $(M; v_1, v_2, v_3)$ with $v_1, v_2 \in T(M)$. The orientation of v_3 be chosen in such a way that the principal curvatures are positive. Then

$$\begin{aligned} &(1) \qquad \mathrm{d}M = \omega^1 v_1 + \omega^2 v_2 , \quad \mathrm{d}v_1 = \omega_1^2 v_2 + \omega_1^3 v_3 , \\ &\mathrm{d}v_2 = -\omega_1^2 v_1 + \omega_2^3 v_3 , \quad \mathrm{d}v_3 = -\omega_1^3 v_1 - \omega_2^3 v_2 ; \\ &\mathrm{d}\omega^i = \omega^j \wedge \omega_j^i , \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j ; \quad \omega_i^j + \omega_j^i = 0 , \quad \omega^3 = 0 . \end{aligned}$$

Using the well known prolongation procedure, we get the existence of functions (in each U_α) $a, b, c; \alpha, \ldots, \delta; A, \ldots, E$ such that

$$(2) \qquad \omega_1^3 = a\omega^1 + b\omega^2 , \quad \omega_2^3 = b\omega^1 + c\omega^2 ;$$

$$\begin{aligned} &(3) \qquad \mathrm{d}a - 2b\omega_1^2 = \alpha\omega^1 + \beta\omega^2 , \\ &\mathrm{d}b + (a - c)\,\omega_1^2 = \beta\omega^1 + \gamma\omega^2 , \\ &\mathrm{d}c + 2b\omega_1^2 = \gamma\omega^1 + \delta\omega^2 ; \end{aligned}$$

$$\begin{aligned} &(4) \qquad \mathrm{d}\alpha - 3\beta\omega_1^2 = A\omega^1 + (B - bK)\,\omega^2 , \\ &\mathrm{d}\beta + (\alpha - 2\gamma)\,\omega_1^2 = (B + bK)\,\omega^1 + (C + aK)\,\omega^2 , \\ &\mathrm{d}\gamma + (2\beta - \delta)\,\omega_1^2 = (C + cK)\,\omega^1 + (D + bK)\,\omega^2 , \\ &\mathrm{d}\delta + 3\gamma\omega_1^2 = (D - bK)\,\omega^1 + E\omega^2 . \end{aligned}$$

As always,

$$(5) \qquad K = ac - b^2 , \quad 2H = a + c$$

denote the Gauss and mean curvature respectively.

Let $f : M \to \mathcal{R}$ be a function. The derivatives f_i, f_{ij} of its restriction to U_α with respect to the frames $(M; v_1, v_2, v_3)$ are introduced by means of the formulas

$$\text{(6)} \qquad \mathrm{d}f = f_1\omega^1 + f_2\omega^2 ;$$
$$\mathrm{d}f_1 - f_2\omega_1^2 = f_{11}\omega^1 + f_{12}\omega^2 , \quad \mathrm{d}f_2 + f_1\omega_1^2 = f_{12}\omega^1 + f_{22}\omega^2 .$$

It is easy to see that the functions

$$\text{(7)} \qquad f_1^2 + f_2^2 , \quad (a - c)(f_{11} - f_{22}) + 4bf_{12}$$

do not depend on the choice of the moving frames. The domain U_α be covered by coordinates (u, v) such that

$$\text{(8)} \qquad \omega^1 = r(u, v)\,\mathrm{d}u , \quad \omega^2 = s(u, v)\,\mathrm{d}v ; \quad rs \neq 0 ;$$

such coordinates do exist. We get

$$\text{(9)} \qquad \omega_1^2 = -s^{-1}r_v\,\mathrm{d}u + r^{-1}s_u\,\mathrm{d}v$$

and

$$\text{(10)} \qquad \begin{aligned} f_1 &= r^{-1}f_u , \quad f_2 = s^{-1}f_v ; \\ f_{11} &= r^2 f_{uu} - r^{-3}r_u f_u + r^{-1}s^{-2}r_v f_v , \\ f_{12} &= r^{-1}s^{-1}f_{uv} - r^{-2}s^{-1}r_v f_u - r^{-1}s^{-2}s_u f_v , \\ f_{22} &= s^{-2}f_{vv} + r^{-2}s^{-1}s_u f_u - s^{-3}s_v f_v . \end{aligned}$$

Thus we have the following version of the (weak) maximum principle: *Let f satisfy, in U_α, the equation*

$$\text{(11)} \qquad a_{11}f_{11} + 2a_{12}f_{12} + a_{22}f_{22} + a_1f_1 + a_2f_2 + a_0f = \varphi$$

with $a_{ij}x^ix^j$ positive, $a_0 \leqq 0$ and $\varphi \geqq 0$, then f cannot have a positive maximum inside of U_α without being constant; see [1], p. 109.

2. Two generalized H-theorems. On M, consider the function

$$\text{(12)} \qquad f = 2(H^2 - K) = \tfrac{1}{2}(a - c)^2 + 2b^2 ;$$

of course, $f \geqq 0$, and we have $f = 0$ in the umbilical points. We get

$$\text{(13)} \qquad f_1 = (a - c)(\alpha - \gamma) + 4b\beta , \quad f_2 = (a - c)(\beta - \delta) + 4b\gamma ;$$

$$f_{11} = (c^2 - ac + 4b^2) K + (\alpha - \gamma)^2 + 4\beta^2 + (a - c)(A - C) + 4bB, \tag{14}$$
$$f_{12} = 2b(a + c) K + (\alpha - \gamma)(\beta - \delta) + 4\beta\gamma + (a - c)(B - D) + 4bC,$$
$$f_{22} = (a^2 - ac + 4b^2) K + (\beta - \delta)^2 + 4\gamma^2 + (a - c)(C - E) + 4bD.$$

The covariant derivatives of the mean curvature H are given by

$$2H_1 = \alpha + \gamma, \; 2H_2 = \beta + \delta; \tag{15}$$

$$2H_{11} = cK + A + C, \quad 2H_{12} = B + D, \quad 2H_{22} = aK + C + E. \tag{16}$$

Multiplying the equations (14), (16) successively by $1, 0, 1, c - a, -4b, a - c$ and adding them, we get

$$f_{11} + f_{22} - 4Kf = 2\{(a - c)(H_{11} - H_{22}) + 4bH_{12} + H_1^2 + H_2^2\} + \tag{17}$$
$$+ 2\{(2\gamma - H_1)^2 + (2\beta - H_2)^2\}.$$

Thus we obtain

Theorem 1. *Let $M \subset E^3$ be an ovaloid, and let its mean curvature satisfy*

$$(a - c)(H_{11} - H_{22}) + 4bH_{12} + H_1^2 + H_2^2 \geqq 0. \tag{18}$$

Then M is a sphere.

Next, we are going to prove the following

Theorem 2. *Let $M \subset E^3$ be a part of an ovaloid such that:* (i) *M has a net of lines of curvature,* (ii) *∂M consists of umbilical points,* (iii) *v_1 and v_2 being the unit tangent vector fields of the lines of curvature and $S : M \to \mathscr{R}$ a function satisfying*

$$S^2 \leqq 57 - 40\sqrt{2}, \tag{19}$$

we have

$$v_1 v_1 H - v_2 v_2 H + S[v_1 v_2] H = 0. \tag{20}$$

Then M is a part of a sphere.

The lines of curvature being given by

$$b(\omega^1)^2 + (c - a)\,\omega^1\omega^2 - b(\omega^2)^2 = 0, \tag{21}$$

we have to suppose

$$b = 0; \tag{22}$$

a and c are then the principal curvatures respectively. From (3) and (4),

(23)
$$v_1 a = \alpha\,, \quad v_2 a = \beta\,, \quad v_1 c = \gamma\,, \quad v_2 c = \delta\,;$$

(24)
$$\begin{aligned}
(a - c)\, v_1 v_1 a &= 3\beta^2 + (a - c)\, A\,,\\
(a - c)\, v_1 v_2 a &= (2\gamma - \alpha)\, \beta + (a - c) B\,,\\
(a - c)\, v_2 v_1 a &= 3\beta\gamma + (a - c)\, B\,,\\
(a - c)\, v_2 v_2 a &= (2\gamma - \alpha)\, \gamma + (a - c)\,(C + aK)\,,\\
(a - c)\, v_1 v_1 c &= (\delta - 2\beta)\, \beta + (a - c)\,(C + cK)\,,\\
(a - c)\, v_1 v_2 c &= -3\beta\gamma + (a - c)\, D\,,\\
(a - c)\, v_2 v_1 c &= (\delta - 2\beta)\, \gamma + (a - c)\, D\,,\\
(a - c)\, v_2 v_2 c &= -3\gamma^2 + (a - c)\, E\,.
\end{aligned}$$

The equation (20) implies

(25)
$$\begin{aligned}
&\beta^2 + \gamma^2 + \alpha\gamma + \beta\delta - (a - c)^2 K + \\
&+ (a - c)(A - E) - S(\alpha\beta + 2\beta\gamma + \gamma\delta) = 0\,.
\end{aligned}$$

The elimination of A, C, E from $(14_{1,3})$ and (25) yields

(26)
$$\begin{aligned}
f_{11} + f_{22} - 4Kf &= (\alpha - \tfrac{3}{2}\gamma + \tfrac{1}{2}S\beta)^2 + (\delta - \tfrac{3}{2}\beta + \tfrac{1}{2}S\gamma)^2 + \\
&+ \tfrac{1}{4}\{(7 - S^2)\, \beta^2 + 20S\beta\gamma + (7 - S^2)\, \gamma^2\}\,.
\end{aligned}$$

The last term is non-negative for each β and γ because of (19). This concludes the proof of our theorem.

Let us add the following remark. Let M satisfy (i) and (ii) of Theorem 2. On M, consider the second order operators of the form

(27)
$$P = S_1 v_1 v_1 + S_2 v_2 v_2 + S_3 v_1 v_2 + S_4 v_2 v_1\,,$$

$S_1, \ldots, S_4 : M \to \mathcal{R}$ being functions. Now, our problem is to determine the class of operators (27) with the following property: The surface M satisfying $PH = 0$ (and knowing nothing more about it), we are able to prove by means of the maximum principle using the function $H^2 - K$ that it is a part of a sphere. It is not difficult to see that the just defined class of operators is given by (iii) of Theorem 2.

3. A generalization of the K-theorem. We are going to prove a somewhat modified version of Theorem 1.

Theorem 3. *Let $M \subset E^3$ be an ovaloid satisfying, at each of its points, the following conditions*: (i)

$$(a - c)(K_{11} - K_{22}) + 4bK_{12} \geqq 0\,, \tag{28}$$

(ii) *$k_1 \leqq k_2$ being the principal curvatures, we have $k_2 \leqq 3k_1$. Then M is a sphere.*

We have

$$K_1 = a\gamma + c\alpha - 2b\beta\,, \quad K_2 = a\delta + c\beta - 2b\gamma\,; \tag{29}$$

$$\begin{aligned} K_{11} &= (ac - 2b^2)K + 2\alpha\gamma - 2\beta^2 + cA - 2bB + aC\,, \\ K_{12} &= -b(a + c)K + \alpha\delta - \beta\gamma + cB - 2bC + aD\,, \\ K_{22} &= (ac - 2b^2)K + 2\beta\delta - 2\gamma^2 + cC - 2bD + aE\,. \end{aligned} \tag{30}$$

Multiplying the equations (14), (30) successively by c, $-2b$, a, $c - a$, $-4b$, $a - c$ and adding them together, we get

$$cf_{11} - 2bf_{12} + af_{22} - 4HKf = (a - c)(K_{11} - K_{22}) + 4bK_{12} + T \tag{31}$$

with

$$\begin{aligned} T = a(3\beta^2 + \delta^2 + 2\gamma^2 - 2\alpha\gamma) - \\ - 2b(\alpha + \gamma)(\beta + \delta) + c(\alpha^2 + 3\gamma^2 + 2\beta^2 - 2\beta\delta)\,. \end{aligned} \tag{32}$$

We have to prove $T \geqq 0$. The term T being an invariant of the surface (which is easy to see), it is sufficient to prove the inequality in a generic point $m \in M$ using a convenable field of moving frames around m. Let us choose this field in such a way that $b(m) = 0$. Then, at m,

$$\begin{aligned} T = c^{-1}(c\alpha - a\gamma)^2 + a^{-1}(a\delta - c\beta)^2 + \\ + 2H\{a^{-1}(3a - c)\beta^2 + c^{-1}(3c - a)\gamma^2\}\,; \end{aligned} \tag{33}$$

$a(m)$ and $c(m)$ being the principal curvatures at m, we are done.

The classical K-theorem follows easily. Our starting point is the equation (30), we have to prove that $K = \text{const.}$ implies $T \geqq 0$. Around a generic point $m \in M$, choose the moving frames as above. We have

$$K_1 = a\gamma + c\alpha = 0\,, \quad K_2 = a\delta + c\beta = 0 \quad \text{at} \quad m\,,$$

and we get the existence of numbers ϱ, σ such that

$$\alpha = \varrho a\,, \quad \gamma = -\varrho c\,, \quad \beta = \sigma a\,, \quad \delta = -\sigma c \quad \text{at} \quad m\,.$$

Then

$$T(m) = (3a^2 + 2ac + 3c^2)(a\sigma^2 + c\varrho^2) \geqq 0 .$$

4. Another generalization of the H-theorem. Consider the invariant 1-form

$$(34) \qquad \tau = (a\beta + b\gamma - b\alpha - c\beta)\,\omega^1 + (a\gamma + b\delta - b\beta - c\gamma)\,\omega^2$$

on M; it is easy to see that

$$(35) \qquad d\tau = -2\{\beta^2 + \gamma^2 - \alpha\gamma - \beta\delta + 2(H^2 - K)K\}\,\omega^1 \wedge \omega^2 .$$

The following assertion follows: Let M be a surface with ∂M umbilical, then

$$(36) \qquad \int_M \{\beta^2 + \gamma^2 - \alpha\gamma - \beta\delta + 2(H^2 - K)K\}\,\omega^1 \wedge \omega^2 = 0 .$$

Now, we are in the position to prove

Theorem 4. *Let M be a surface with ∂M umbilical, and let*

$$(37) \qquad 4(H^2 - K)K \geqq H_1^2 + H_2^2 .$$

The point $m \in M$ being non-umbilical, there exists its neighbourhood $U \subset M$ such that $K = 0$ in U. (I do not suppose M to be an ovaloid!)

Because of (15), (36) may be rewritten as

$$(38) \qquad \int_M \{(2\beta - H_2)^2 + (2\gamma - H_1)^2 - H_1^2 - H_2^2 + 4(H^2 - K)K\}\,\omega^1 \wedge \omega^2 = 0 .$$

From (37), we obtain $2\beta = H_2$, $2\gamma = H_1$, i.e.,

$$(39) \qquad \alpha = 3\gamma , \quad \delta = 3\beta .$$

In a suitable neighborhood U of the point m, we may choose the frames in such a way that $b = 0$, $a \neq c$; the equations (3) reduce to

$$(40) \qquad da = 3\gamma\omega^1 + \beta\omega^2 , \quad (a - c)\,\omega_1^2 = \beta\omega^1 + \gamma\omega^2 , \quad dc = \gamma\omega^1 + 3\beta\omega^2 .$$

By exterior differentiation, taking regard of (40_2),

$$(41) \qquad \begin{aligned} &3(d\gamma - \beta\omega_1^2) \wedge \omega^1 + (d\beta + \gamma\omega_1^2) \wedge \omega^2 = 0 , \\ &(d\beta + \gamma\omega_1^2) \wedge \omega^1 + (d\gamma - \beta\omega_1^2) \wedge \omega^2 = (c - a)\,ac\omega^1 \wedge \omega^2 , \\ &(d\gamma - \beta\omega_1^2) \wedge \omega^1 + 3(d\beta + \gamma\omega_1^2) \wedge \omega^2 = 0 . \end{aligned}$$

Thus there is a function ϱ such that

$$(42) \qquad d\beta + \gamma\omega_1^2 = (\varrho - ac^2)\,\omega^2\,, \quad d\gamma - \beta\omega_1^2 = (\varrho - a^2c)\,\omega^1\,.$$

By exterior differentiation,

$$(43) \qquad (d\varrho - 3a^2\beta\omega^2) \wedge \omega^1 = 0\,, \quad (d\varrho - 3c^2\gamma\omega^1) \wedge \omega^2 = 0\,,$$

i.e.,

$$(44) \qquad d\varrho = 3c^2\gamma\omega^1 + 3a^2\beta\omega^2\,.$$

A further exterior differentiation yields

$$(45) \qquad (a - c)\,\beta\gamma = 0\,.$$

Suppose $\beta = 0 \neq \gamma$, the case $\beta \neq 0 = \gamma$ being symmetric. We have

$$(46) \qquad da = 3\gamma\omega^1\,, \quad dc = \gamma\omega^1\,,$$

and the equation $H_1^2 + H_2^2 = 4(H^2 - K)\,K$ turns out to be

$$(47) \qquad a^3c - 2a^2c^2 + ac^3 = 4\gamma^2\,.$$

We get (taking regard of $\gamma \neq 0$!)

$$(48) \qquad \begin{gathered} a^3 + 13a^2c - 9ac^2 + 3c^3 = 8\varrho\,, \\ 11a^2 + 30ac - 21c^2 = 0\,, \quad 2a + c = 0\,, \quad 7\gamma = 0 \end{gathered}$$

by a series of successive differentiations; hence a contradiction. Thus we have $\beta = \gamma = 0$, i.e., $\omega_1^2 = 0$ because of (40_2) and $a \neq c$. But this means $ac\omega^1 \wedge \omega^2 = 0$, i.e., $K = 0$.

5. General Weingarten surfaces. We are going to prove

Theorem 5. *Let M be an ovaloid or a part of it bounded by umbilical points. Let $H, K : M \to \mathcal{R}$ be its mean and Gauss curvature respectively. On a domain $\mathcal{D}$ of $\mathcal{R}^2$ containing $H(M) \times K(M)$, be given a function $f(x, y)$ satisfying*

$$(49) \qquad F(x, y) := f_x^2 + 4xf_xf_y + 4yf_y^2 > 0$$

on $\mathcal{D}$. If $f(H, K) = 0$, M is a sphere or a part of it.

We have

$$(50)\qquad 2\,\mathrm{d}H = (\alpha + \gamma)\,\omega^1 + (\beta + \delta)\,\omega^2\,,$$
$$\mathrm{d}K = (a\gamma + c\alpha - 2b\beta)\,\omega^1 + (a\delta + c\beta - 2b\gamma)\,\omega^2\,.$$

Let $m \in M$ be a generic point. Around m, choose the moving frames in such a way that $b(m) = 0$. The equation $f(H, K) = 0$ implies

$$(51)\qquad (f_H + 2cf_K)\,\alpha + (f_H + 2af_K)\,\gamma = 0\,,$$
$$(f_H + 2cf_K)\,\beta + (f_H + 2af_K)\,\delta = 0 \quad \text{at} \quad m\,.$$

Because of

$$(f_H + 2cf_K)\,(f_H + 2cf_K) = F(H, K) > 0\,,$$

α and γ as well as β and δ have opposite signs at m. Thus $\beta^2 + \gamma^2 - \alpha\gamma - \beta\delta \geqq 0$ at m; applying now (36), we are done.

Bibliography

[1] *H. Huck - R. Roitzsch - U. Simon - W. Vortisch - R. Walden - B. Wegner - W. Wendland:* Beweismethoden der Differentialgeometrie im Grossen. Springer-Verlag 1973; Lecture Notes in Mathematics, Vol. 335.

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