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SUBLATTICES WITH SATURATED CHAINS[1])

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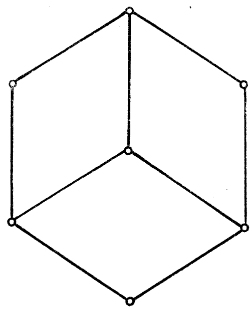
Let L be a lattice. A chain $R \subseteqq L$ will be called saturated in L if $r_1, r_2 \in R$, $x \in L$, $r_1 < x < r_2$ implies $x \in R$ whenever x is comparable with each element of R. A sublattice L_1 of L is said to be a c-sublattice of L if each chain R of L_1 which is saturated in L_1 is also saturated in L. Let K be a lattice of locally finite length. A sublattice S of K is a c-sublattice of K if and only if, whenever $s_1, s_2 \in S$ and s_1 covers s_2 in S, then s_1 covers s_2 in K.

Some classes of lattices of locally finite length can be characterized by the requirement of not containing c-sublattices of certain types.



Let B be the lattice in Fig. 1 and let B' be the lattice dual to B. Let C be the non-distributive modular lattice with five elements. For positive integers $m \geqq 3$, $n \geqq 3$ we denote by $L(m, n)$ the lattice consisting of elements $x_1 < x_2 < \ldots < x_m$, $y_1 < < y_2 < \ldots < y_n$ such that $x_1 = y_1$, $x_m = y_n$ and x_i is incomparable with y_j whenever $1 < i < m$, $1 < j < n$.

The following results are known:

(A) (GRÄTZER [3], p. 151.) *Let L be a finite modular lattice. Then L is non-distributive if and only if it contains a c-sublattice isomorphic to C.*

(B) (Cf. [4].) *Let L be a lattice of locally finite length. Then L is distributive if and only if it does not contain a c-sublattice isomorphic to some of the following lattices:* B, B', $L(m, n)$ $(m \geqq 3,\ n \geqq 4)$, C.

(C) (VILHELM [5], Thm. 2.2.) *Let L be a lattice of locally finite length. Then L is semimodular if and only if it does not contain a c-sublattice isomorphic to* B' *or* $L(m, n)$ $(m \geqq 3,\ n \geqq 4)$.

In view of these results the following question seems to be quite natural: which classes of lattices can be defined by the requirement of not containing c-sublattices of certain types? Let us formulate the corresponding notions more precisely.

Let K be a class of lattices and let K_1 be a subclass of K. The class K_1 will be said to be characterizable in K by c-sublattices if there exists a set C of lattices belonging to $K \setminus K_1$ such that, for each lattice $L \in K$, the following conditions are equivalent:

(i) L belongs to K_1;

(ii) if L_1 is a c-sublattice of L, then no lattice $L' \in C$ is isomorphic to L_1.

Let K_0 be the class of all lattices. Further let K_d and K_m be the class of all distributive lattices and the class of all modular lattices, respectively. A variety K of lattices will be called nontrivial if there exists a lattice $L \in K$ with card $L > 1$.

We have the following negative result:

Theorem. *Let* K *be a nontrivial variety of lattices such that* $K \subseteqq K_m$. *Then* K *is not characterizable in* K_0 *by c-sublattices.*

Proof. Suppose that K is characterizable in K_0 by c-sublattices. Then there exists a set C of lattices of the class $K_0 \setminus K$ such that a lattice L belongs to K if and only if it does not contain any c-sublattice isomorphic to some of the lattices of C. Let $C = \{L_j\}$ $(j \in J)$, $\alpha = \sup\{\text{card } L_j\}$ and let β be a cardinal with $\beta > \alpha$. Let ω_β be the first ordinal with the property that the power of the set of all lower ordinals equals β. Let A and B be dually isomorphic to ω_β, $C_1 = A \times B$. Let x, y, u be three distinct elements not belonging to C_1, $L = C_1 \cup \{x, y, u\}$.

We define in L a partial order $\leqq$ as follows. If $c_1, c_2 \in C_1$, then $c_1 \leqq c_2$ in L has the same meaning as $c_1 \leqq c_2$ in C_1. The element u is the least element in L. Further we put $x < y$. For $c \in C_1$, $z \in \{x, y\}$ we set $z < c$ if and only if $c = (a, 0)$, where a is any element of A and 0 is the greatest element of B; if c is not of this form, then z and c are incomparable.

It is well-known that the variety K_d of all distributive lattices is the unique atom in the lattice of all varieties of lattices (cf., e.g., [2], p. 182). Hence $K_d \subseteqq K$.

The lattice L is not modular, hence it does not belong to K. Thus according to the assumption there exists a c-sublattice L_0 of L isomorphic to a lattice of the class C.

[1]) The results of this paper have been presented at the Conference on lattice theory held at Szeged (August, 1974).

Put $L_0 \cap C_1 = C_0$. If each $c \in C_0$ has the form $(a, 0)$ with $a \in A$, then L_0 is a chain and hence $L_0 \in K_d \subseteqq K$, which is a contradiction. If $\{x, y\} \cap L_0 = \emptyset$, then L_0 is distributive, thus $L_0 \in K$, which is impossible. Hence there is $z \in \{x, y\}$ and $c = (a, b)$, $b \neq 0$, such that $\{z, c\} \subseteqq L_0$. Then

$$c_1 = c \vee z = (a, 0) \in L_0 .$$

Let A_1 be a saturated chain in L_0 having the least element z and the greatest element c_1. Then A_1 is a saturated chain in L between z and c_1. Thus the set $A_2 = \{(a_2, 0) : a_2 \in A,\ a_2 \leqq a\}$ is a subset of A_1. Obviously card $A_2 = \beta$ and hence card $L_0 \geqq \beta$. But card $L_j \leqq \alpha < \beta$ for each $L_j \in C$ and hence L_0 cannot be isomorphic to L_j.

The following questions remain open:

(a) Is the class K_d characterizable in K_m by c-sublattices?

(b) Does there exist a nontrivial variety $K \neq K_0$ that is characterizable in K_0 by c-sublattices?

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