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A STRONG COMPLEMENT PROPERTY OF DEDEKIND DOMAINS

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Let R be a commutative ring with identity. R is called completable provided that $1 \in (a_1, \ldots, a_n)$, $a_i \in R$, $i = 2, \ldots, n$, implies there is an $n \times n$ matrix A over R with first row $a_1, \ldots, a_n$ and $\det A = 1$. Similarly, R is called strongly completable if $d \in (a_1, \ldots, a_n)$, $a_i \in R$, $i = 2, \ldots, n$, then there is an $n \times n$ matrix B over R with first row $a_1, \ldots, a_n$ and $\det B = d$. I. REINER has shown that any Dedekind domain is completable. In this paper it is shown that Dedekind domains are in fact strongly completable, a fortiori, completable.

The assertion clearly holds for $n = 2$; for if $d \in (a_1, a_2)$, $d = a_1x + a_2y$, then $-y, x$ works as a second row. Suppose that the assertion is true for $k < n$, $n \geqq 3$, and let $d \in (a_1, \ldots, a_n)$. If $J = (a_1, \ldots, a_{n-2}) = (0)$, then $d \in (a_{n-1}, a_n)$, $d = a_{n-1}u + a_nv$, $u, v \in R$. In this case, let

$$B = \begin{pmatrix} a_{n-1} & a_n & a_1 & \ldots & a_{n-2} \\ -v & u & 0 & \ldots & 0 \\ & O & & I^{n-2} & \end{pmatrix}.$$

Hence $\det B = d$ and the assertion follows.

If $J = (a_1, \ldots, a_{n-2}) \neq (0)$, and if $I = (a_1, \ldots, a_n)$, then let $I = \prod_{i=1}^{t} M_i^{\alpha_i}$ and $J = \prod_{i=1}^{t} M_i^{\beta_i}$ be the representations of the ideals I and J as products of powers of distinct maximal ideals. One may order the M_i so that $0 \leqq \alpha_i < \beta_i$ for $1 \leqq i \leqq r$, and $\alpha_i = \beta_i$ for $r + 1 \leqq i \leqq t$. If $1 \leqq k \leqq r$, it follows that either a_{n-1} or a_n does not belong to $M_k^{\alpha_k+1}$. The Chinese Remainder Theorem guarantees $b \in R$ such that $b \equiv 0 \pmod{M_k^{\alpha_k+1}}$ if $a_{n-1} \notin M_k^{\alpha_k+1}$, $b \equiv 1 \pmod{M_k^{\alpha_k+1}}$ if $a_{n-1} \in M_k^{\alpha_k+1}$, for $k = 1, 2, \ldots, r$. Hence $a_{n-1} + ba_n \notin M_k^{\alpha_k+1}$, $k = 1, \ldots, r$, and if $(a_1, \ldots, a_{n-2}, a_{n-1} + ba_n) = \prod_{i=1}^{t} M_i^{\mu_i}$, it follows that $\mu_i = \alpha_i$, $i = 1, \ldots, t$ and $(a_1, \ldots, a_{n-1} + ba_n) = I$.

(For details, see [1], Lemma 3.3.) Since $d \in I$, the induction hypothesis assures us of an $(n-1) \times (n-1)$ matrix D whose first row is $a_1, \ldots, a_{n-1} + ba_n$ and such that $\det D = d$. If we let

$$B = \begin{pmatrix} & & a_n \\ D & & 0 \\ \cdots & \cdots & \cdots \\ 0 & \ldots 0 & 1 \end{pmatrix} \begin{pmatrix} I^{n-2} & & O \\ \cdots & \cdots & \cdots \\ & & 1 \quad 0 \\ O & & -b \quad 1 \end{pmatrix},$$

then B is the desired matrix.

References

[1] *M. Moore* and *A. Steger,* Some results on completability in commutative rings, Pacific J. Math., *37* No. 2 (1971), 453—460.

[2] *I. Reiner,* Unimodular complements, Amer. Math. Monthly, *63* (1956), 246—247.

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