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GEODETIC GRAPHS OF DIAMETER TWO

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Geodetic graphs were defined by O. ORE [1] as graphs in which to any pair of vertices there exists a unique path of minimal length joining them. For example, an arbitrary tree is a geodetic graph. Planar geodetic graphs were studied by J. G. STEMPLE and M. E. WATKINS [2]. Here we shall give some results concerning geodetic graphs of diameter two.

If a graph is geodetic of diameter two, then it does not contain multiple edges and any pair of its distinct vertices either is joined by an edge, or is connected by a unique path of the length two.

Theorem 1. *Let G be a geodetic graph of diameter two and of vertex connectivity degree one. Then G contains exactly one cut-vertex and each block of G is a clique.*

Proof. As G has vertex connectivity degree equal to one, it contains at least one cut-vertex. Suppose that it has two distinct cut-vertices a_1 and a_2. Let G' be the union of all simple paths joining a_1 and a_2 in G; the graph G' is a connected subgraph of G consisting of one or more blocks of G. Let G'' be the graph obtained from G by deleting all edges of G' and all vertices of G' except a_1 and a_2. Evidently G'' is disconnected and the vertices a_1, a_2 are in different connected components of G''. As they are cut-vertices in G, they cannot be isolated in G''. Thus let b_1 or b_2 be a vertex joined with a_1 or a_2 respectively by an edge in G''. Then any path in G joining b_1 and b_2 must contain both a_1 and a_2, therefore its length is at least three, which is a contradiction with the assumption that G has diameter two. Therefore G has exactly one cut-vertex; denote it by a. Let u, v be two vertices lying in distinct blocks of G and both distinct from a. Any path joining u and v must contain a. As G has diameter two, there exists a path joining u and v of length two. This path contains only the vertices u, a, v, therefore there exist edges au, av. As u and v were chosen arbitrarily, we have proved that each vertex of G distinct from a must be joined by an edge with a. Now let u_1, u_2 be two distinct vertices of the same block of G, $u_1 \neq a$, $u_2 \neq a$. Suppose that they are not joined by an edge. Then their distance is two; there exists a path P_0 of length two joining them which has the edges au_1, au_2. As G is geodetic, no other path of length two joining u_1 and u_2 may exist. However, as u_1 and u_2 lie in the same block, there exists at least one simple path joining u_1 and u_2 and having no vertex in common with P_0 except u_1 and u_2. Let P be such a path of minimal length, let this length be l; obviously $l \geqq 3$. Let the vertices of P be $u_1 = w_0, w_1, \ldots, w_l = u_2$ and the edges $w_i w_{i+1}$ for $i = 0, 1, \ldots, l-1$. The vertices $u_1 = w_0$ and w_2 are not joined by an edge; otherwise by deleting the vertex w_1 and the edges w_0w_1, w_1w_2 and by adding the edge w_0w_2 we should obtain a path of length $l - 1$ joining u_1 and u_2, which would be a contradiction with the minimality of P. Therefore the distance of w_0 and w_2 is two. But they are joined by two different paths of the length two; one of them contains the edges w_0w_1, w_1w_2, the other contains aw_0, aw_2. We have obtained a contradiction. Thus we have proved that any two vertices of the same block of G are joined by an edge and each block of G is a clique.

Fig. 1 shows examples of such graphs.

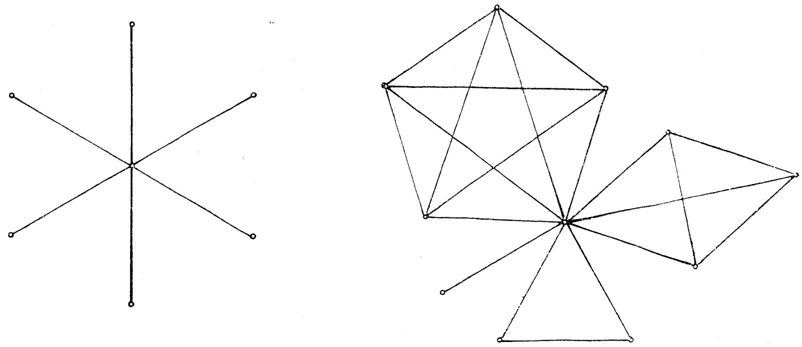

Fig. 1.

Theorem 2. *Let G be a geodetic graph of diameter two and of vertex connectivity degree at least two. Let G contain a clique K with at least two vertices. Then G contains an induced subgraph L described in the following way: L contains K as as a subgraph and, moreover, it contains the vertices $f(u)$ for each vertex u of K, the vertex w and the edges $u\,f(u), f(u)\,w$ for all vertices u of K. The vertices $f(u)$ for all u of K and w are pairwise distinct and do not belong to K.*

Proof. First suppose that K is a maximal clique of G, i.e., that it is not a proper subgraph of another clique. The clique K must be a proper subgraph of G; otherwise G would have diameter one. As G is connected, there exists at least one vertex of G not belonging to K and joined by an edge with a vertex of K; if the latter is u_1, then the former will be denoted by $f(u_1)$. As the vertex connectivity degree of G is at least two, there exists a path P connecting $f(u_1)$ with a vertex of K which does not contain u_1. If we go along P from $f(u_1)$, let u_2 be the first vertex of K which we meet. Let the vertex of P preceding u_2 be $f(u_2)$. Suppose $f(u_2) = f(u_1)$. If the clique K consists only of two vertices u_1, u_2 then the vertices $u_1, u_2, f(u_1)$ form a clique containing K as a proper subgraph, which is a contradiction with the maximality of K. If K has more than two vertices, let v be a vertex of K distinct from u_1 and u_2. There exist two paths of length two between $f(u_1)$ and v; one of them has the edges $f(u_1)\, u_1$, $u_1 v$, the other $f(u_1)\, u_2$, $u_2 v$. Therefore $f(u_1)$ and v cannot have distance two, they must be joined by an edge. As v was chosen arbitrarily, $f(u_1)$ must be joined by edges with all vertices of K and K is a proper subgraph of the clique induced by all vertices of K and $f(u_1)$. We have proved $f(u_1) \neq f(u_2)$. Now suppose that $f(u_1)$ and $f(u_2)$ are joined by an edge. Then the vertices $u_1, f(u_2)$ are joined by two paths of length two. One has the edges $u_1 f(u_1), f(u_1) f(u_2)$, the other has the edges $u_1 u_2$, $u_2 f(u_2)$. This means again that u_1 and $f(u_2)$ must be joined by an edge. Analogously u_2 and $f(u_1)$ must be joined by an edge. If K contains only two vertices, the vertices u_1, u_2, $f(u_1), f(u_2)$ induce a clique properly containing K. If K contains a vertex v distinct from u_1 and u_2, then v is connected with $f(u_1)$ by two paths of length two; one contains the edges vu_1, $u_1 f(u_1)$, another the edges vu_2, $u_2 f(u_1)$. Therefore also v is joined with $f(u_1)$. Analogously we prove that v is joined with $f(u_2)$. Therefore all vertices of K are joined with both $f(u_1)$ and $f(u_2)$ and the vertices of K together with $f(u_1)$ and $f(u_2)$ induce a clique properly containing K. We have proved that $f(u_1)$, $f(u_2)$ are not joined by an edge. They must be connected by a path of length two; let its inner vertex be w. Suppose that w belongs to K. We have either $w \neq u_1$ or $w \neq u_2$; without a loss of generality let $w \neq u_1$. Then $f(u_1)$ is joined by edges with two vertices of K, namely, u_1, w. Analogously as in the case $f(u_1) = f(u_2)$ we prove that $f(u_1)$ is joined with all vertices of K and we have again a clique properly containing K. Thus w is not in K. Evidently also $w \neq f(u_1)$, $w \neq f(u_2)$. Suppose that w is joined by an edge with a vertex v of K. Without a loss of generality let again $v \neq u_1$. Then the vertices $v, f(u_1)$ are connected by two paths of length two; one contains the edges vu_1, $u_1 f(u_1)$, the other the edges vw, $wf(u_1)$. Therefore v and $f(u_1)$ must be joined by an edge, which is not possible as proved in the case when w was supposed to be in K. The vertex w has distance two from all vertices of K. Let x be a vertex of K, $x \neq u_1$, let $f(x)$ be the inner vertex of the path of length two connecting w and x. The vertex $f(x)$ is not in K, because otherwise w would be joined by an edge with a vertex of K, which was proved to be impossible. If $f(x) = f(u_1)$, then this vertex would be joined by edges with both u_1 and x, which is also impossible; the proof is analogous to that of the inequality $f(u_1) \neq f(u_2)$. In the same way we prove that $f(x) \neq f(u_2)$. Analogously to the above proofs we can prove that for any x and y of K (not excluding u_1 and u_2), $x \neq y$, we have $f(x) \neq f(y)$ and these vertices are not joined by an edge. We can prove also that for $x \neq y$ the vertices $f(x)$, y are not joined by an edge; this is analogous to the proof that $f(u_1)$ is not joined with u_2. We have obtained the induced subgraph L of G. It remains to prove the assertion in the case when K is not a maximal clique. Then there exists a maximal clique K_0 containing K

as a subgraph. We construct the graph L_0 for K_0 analogously as L for K in the previous case. The subgraph of L_0 induced by the set of vertices of K, by the vertices $f(u)$ for u from K and by w is the required subgraph L.

Note that any L is itself a geodetic graph of diameter two and vertex connectivity degree at least two. For any $n \geqq 5$, finite or infinite, we can construct L with n vertices. We conclude

Corollary. *A geodetic graph of diameter two and of vertex connectivity degree at least two can have an arbitrary number of vertices greater than or equal to five.*

Some graphs L are in Fig. 2.

Nevertheless, there are also geodetic graphs of diameter two and of vertex connectivity degree at least two which have not this form. The well-known Petersen graph in Fig. 3 is such a graph.

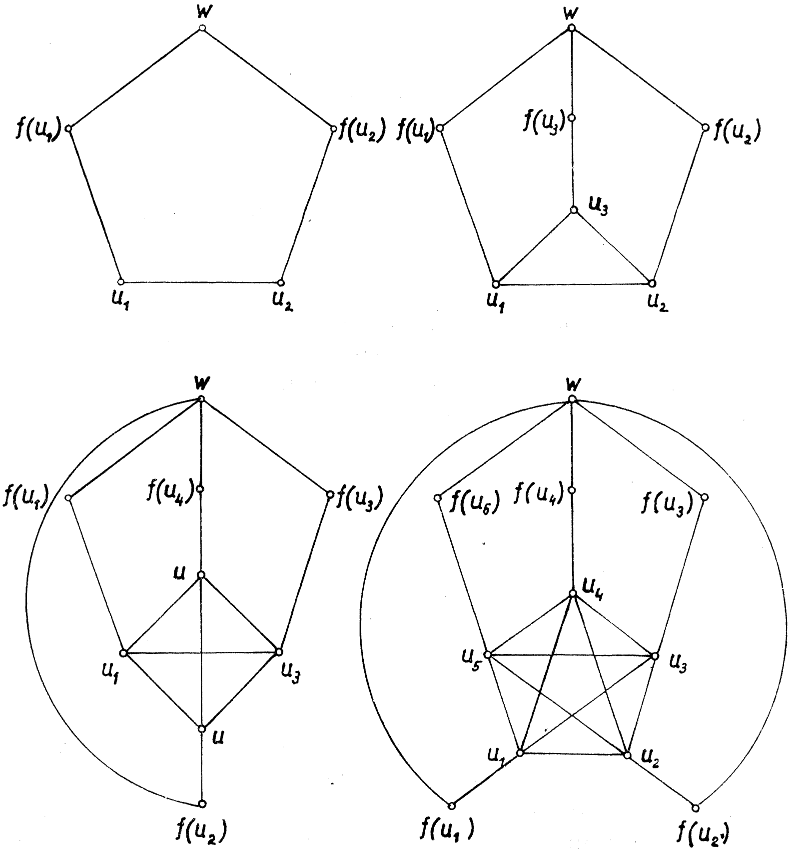

Fig. 2.

Theorem 3. *Let C be a circuit of length five of a geodetic graph G of diameter two. Then either C has no diagonal edges or the set of vertices of C induces a clique of G.*

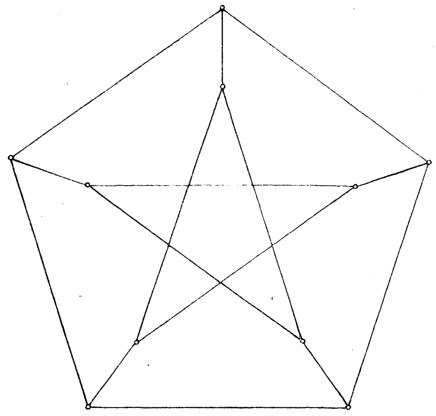

Fig. 3.

Proof. Let the vertices of C be u_1, u_2, u_3, u_4, u_5 and the edges u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1. Suppose that C has a diagonal edge; without a loss of generality we may suppose that this edge is u_1u_3. Then u_1 and u_4 are connected by two paths of length two; one contains the edges u_1u_3, u_3u_4, the other contains u_1u_5, u_4u_5. Therefore u_1 and u_4 must be joined by an edge and the vertex u_1 is joined by edges with all vertices of C. But analogously, from the existence of the diagonal edge u_1u_3 or u_1u_4 we can prove that also u_3 or u_4 respectively is joined with all other vertices of C. Now we have edges u_2u_4, u_3u_5 and their existence implies that also u_2 and u_5 are joined with all vertices of C (except itself). The vertex set of C induces a clique of G.

Theorem 4. *Let G be a geodetic graph of diameter two and of vertex connectivity degree at least two. Then to any two distinct vertices of G there exists a circuit of length five without diagonal edges containing both of them.*

Proof. If these two vertices are joined by an edge, they induce a clique K and according to Theorem 2 there exists an induced subgraph L of G which contains K and is a circuit of length five (the first graph in Fig. 2). Now let u, v be two vertices of G not joined by an edge. There exists a path P_0 of length two joining them; let its inner vertex be w. As G has the vertex connectivity degree at least two, there exists at least one path joining u and v and not containing w. Let P be such a path of the minimal length l; evidently $l \geqq 3$. If $l = 3$, we have obtained a circuit of length five which is the union of P_0 and P. If $l > 3$, then let the vertices of P be $u = x_0$,

$x_1, \ldots, x_l = v$ and let the edges of P be $x_i x_{i+1}$ for $i = 0, 1, \ldots, l - 1$. The vertices x_0, x_3 must have distance one or two; therefore there exists a path P_1 of length one or two joining x_0 and x_3. The union of P_1 and of the subpath of P joining x_3 and x_l is a path of length $l - 1$ or $l - 2$ joining u and v, which is a contradiction with the minimality of l. Therefore $l = 3$ and we have a circuit C of length five which is the union of P_0 and P. It remains to prove that C has no diagonal edge. If C had some diagonal edge, then according to Theorem 3 the vertex set of C would induce a clique of G and u and v would be joined by an edge, which would be a contradiction.

References

[1] *O. Ore:* Theory of Graphs. Providence 1962.

[2] *J. G. Stemple* and *M. E. Watkins:* On planar geodetic graphs. J. Comb. Theory *2* (1968), 101–117.

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