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# ADJOINT DOMAINS AND GENERALIZED SPLINES*)

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**1. Introduction.** In this paper we are concerned with minimizing the differential expression

$$l(y) = \sum_{i=0}^{n} A_i y^{(n-i)} \tag{1.1}$$

in a suitable Hilbert space norm subject to the constraints

$$\lambda_i(y) = \sum_{j=1}^{n} \int_a^b \mathrm{d}w_{ij}(t)\, y^{(n-j)} = r_i\,, \quad i = 1, \ldots, q\,. \tag{1.2}$$

$l(y)$ is understood to be a *regular* vector differential expression; i.e., the $A_i$ are $m \times m$ matrix valued (m.v.) functions with complex components in class $C^{(n-i)}$ with $\det A_0 \neq 0$ on a compact interval $[a, b]$. The functionals $\lambda_i$ are represented by $p \times m$ m.v. measures $w_{ij}$ also with complex components; we assume further that the $w_{ij}$ are singular (supported by sets of zero Lebesgue measure) and are of bounded total variation. The $r_i$ are arbitrary $p$-dimensional vectors over the complex field $\mathscr{C}$.

Following JEROME and SCHUMAKER [15] we call the minimizing function $f$, if it exists, an *Lg-spline interpolating the data $r_i$ with respect to the functionals $\lambda_i$.*

In the last fifteen years several authors, notably GOLOMB and WEINBERGER [11], DE BOOR [8], GREVILLE [12], [13], SCHOENBERG [23], [24], [25], ATTEIA [4], [5], DE BOOR and LYNCH [9], AHLBERG and NILSON [1], SARD [22], SCHULTZ and VARGA [26], ANSELONE and LAURENT [2], JEROME and SCHUMAKER [15], JEROME [14], and LUCAS [19], [20] among others have considered problems of this nature (for a fairly complete survey of the literature up to 1969 see [16]).

$l(y)$ usually has been regarded as a real scalar differential operator and the $\lambda_i$ taken as scalar functionals depending on at most finitely many points of $[a, b]$.

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The recent paper of Jerome and Schumaker [15] for example discussed the possibility of functionals represented by general measures, but considered in detail only $\lambda_i$ of "extended Hermite-Birkhoff type"

$$(1.3)\qquad \lambda_i(f) = \sum_{j=1}^{n} \alpha_{ij} f^{(n-j)}(x_i)\,, \quad i = 1, \ldots, k\,, \quad a \leqq x_i < \ldots < x_k \leqq b\,.$$

In this paper we extend the minimization problem to the setting of vector valued differential operators and to the functionals (1.2).

Although Hilbert space methods have been employed (cf. [2], [4], [5], [22]) for very general minimization problems, the approach of this paper seems new. We regard the differential expression (1.1) and homogenous equations $\lambda_i(y) = 0$ together as generating a closed densely defined unbounded operator $L$ on the Hilbert space of square integrable vector valued functions. Applying the orthogonal projection theorem, the determination of the $Lg$-spline then becomes equivalent to the determination of the $L^2$ adjoint of $L$, a problem solved by the author in an earlier paper [7]. In this way analytic properties of various generalized splines can easily be derived as special cases (see Section 5). We believe that this point of view is a natural one and that it is capable of further development.

**2. Preliminaries and Notation.** Before proceeding further we observe that the system (1.1), (1.2) is equivalent to the system

$$(2,1)\qquad l(y) = \sum_{i=0}^{n} A_i y^{(n-i)}\,,$$

$$\lambda(y) = \sum_{i=1}^{n} \int_a^b \mathrm{d}v_i(t)\, y^{(n-i)} = r\,,$$

where $v_i$ is the $s \times m$ m.v. measure

$$(2,2)\qquad \begin{pmatrix} w_{1i} \\ \vdots \\ w_{qi} \end{pmatrix},$$

where $s = pq$ and $r$ is the vector $(r_1, \ldots, r_q)^t$ in $\mathscr{C}^s$.

Let $\mathscr{L}_m^2$ be the Hilbert space of $m \times 1$ vector valued functions $y$ on $[a, b]$ under the norm

$$(2,3)\qquad \|y\| = \left(\int_a^b y^* y\, \mathrm{d}t\right)^{1/2} = \left(\int_a^b \sum_{i=1}^{m} |y_i|^2\, \mathrm{d}t\right)^{1/2}.$$

We define

$$(2,4)\qquad W_m^{2,n} = \{y : y^{(n-1)} \text{ is absolutely continuous; } y^{(n)} \in \mathscr{L}_m^2\}\,.$$

Furthermore let

$$W_m^{2,n}(r) = \{y : y \in W_m^{2,n};\ \lambda(y) = r\}\,. \tag{2,5}$$

We assume implicitly that $W_m^{2,n}(r)$ is non-empty. With this notation (1.1), (1.2) can be viewed as generating the non-linear operator $L_r : W_m^{2,n}(r) \to \mathscr{L}_m^2$ given by the differential expression (1.1) on the domain $W_m^{2,n}(r)$.

Our problem then will be to minimize $l(y)$ in the norm of $\mathscr{L}_m^2$ over the functions in $W_m^{2,n}(r)$.

To this end a few notational remarks are in order. If $L$ is a linear operator, $D(L)$, $R(L)$, $N(L)$ will stand for its domain, range and null space respectively. Moreover, $L^+$ will denote the adjoint of $L$. In the case of a vector or matrix, we will use the notation $M^*$ to stand for the conjugate transpose of $M$ (the mere transpose will be denoted by $M^t$). We will represent the identity operator on the space $X$ by the symbol $I_X$ and the $n \times n$ identity matrix by $I_n$. $\mathscr{R}^n$, $\mathscr{C}^n$ will denote $n$-dimensional space under the standard Euclidean norm over the real or complex fields $\mathscr{R}$ or $\mathscr{C}$ respectively. The $i^{th}$ unit basis vector of $\mathscr{R}^n$ or $\mathscr{C}^n$ (considered as a column vector) will be denoted by $\bar{e}_i$. The notation $\lambda[E]$ will stand for the characteristic function of the set $E$. If $P$ is a point, $\mu_{(P)}$ will signify "point mass" measure; that is, $\mu_{(P)}[E] = 1$ if $P \in E$ and $\mu_{(P)}[E] = 0$ otherwise. It is evident that with this notation we can write an "atomic" m.v. measure $v$ supported at the points $P_1, \ldots, P_r$ as

$$v = \sum_{i=1}^{r} v[P_i]\,\mu_{(P_i)}\,. \tag{2,6}$$

Finally, where necessary we will assume familiarity with the basic theory of linear unbounded operators, particularly with the notions of the *adjoint* and closure of an operator.

**3. The Adjoint $L^+$.** It is clear that $L_r$ can be viewed as an "$r$ translate" of the linear homogeneous operator $L$ given by

$$l(y) = \sum_{i=0}^{n} A_i y^{(n-i)}\,, \tag{3,1}$$

$$\lambda(y) = \sum_{i=1}^{n} \int_a^b dv_i(t)\, y^{(n-i)} = 0\,.$$

In a recent paper [7] we have extensively studied $L$ and its adjoint $L^+$ as operators in an $\mathscr{L}^p$ space, $1 \leqq p < \infty$. We state our results in the case of $\mathscr{L}_m^2$ in the following theorems and corollaries. (Where proofs are omitted, see [7].)

**Theorem 3.1** *Let* $\alpha_{jr}$, $0 \leqq r \leqq j \leqq n$, *be the* $m \times m$ *matrices*

$$\alpha_{jr} = \sum_{i=0}^{j-r} (-1)^{j-i} B_r^{j-i} A_i^{*(j-i-r)} \tag{3,2}$$

*where* $B_r^{j-i}$ *is the binomial coefficient* $\binom{j-i}{r}$; *and let* $\beta$ *be the* $mn \times mn$ *matrix*

$$(3,3) \qquad \begin{pmatrix} \alpha_{00} & 0 & \dots & 0 \\ \alpha_{10} & \alpha_{11} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \dots & \alpha_{n-1,n-1} \end{pmatrix}.$$

*Then the* $\mathscr{L}_m^2$ *adjoint* $L^+$ *of* $L$ *exists and is given by the formal adjoint*

$$(3,4) \qquad l^+(z) = \sum_{i=0}^{n} (-1)^{n-i} (A_i^* z)^{(n-i)}$$

*on the domain*

$$(3,5) \qquad \mathscr{D}^* = \bigcup_{\varphi \in \mathscr{C}^s} \{z : \hat{z} + \beta^{-1}\tilde{v}^*[a, t]\,\phi \text{ is absolutely continuous;}$$
$$(\hat{z})' \in \mathscr{L}_{nm}^2 \text{ a.e., } \hat{z}(a^+) = -\beta^{-1}\tilde{v}^*[a]\,\phi;\ \hat{z}(b^-) = \beta^{-1}\tilde{v}^*[b]\,\phi\},$$

*where* $\tilde{v}^*$ *denotes the* $mn \times s$ *measure* $(v_1^*, \dots, v_n^*)^t$, $\hat{z}$ *the vector* $(z, z^{(1)}, \dots, z^{(n-1)})^t$, *and* $\mathscr{L}_{mn}^2$ *the space of* $mn \times 1$ $\mathscr{L}^2$ *integrable vector valued functions under a norm similar to that given by* (2.3)

**Corollary 3.1** *Let* $l_j^+$, $j = 0, 1, \dots, n$ *be the* $n + 1$ *"partial" adjoint expressions*

$$(3,6) \qquad \begin{aligned} l_0^+(z) &= A_0^* z, \\ &\dots\dots\dots\dots \\ l_j^+(z) &= \sum_{i=0}^{j} (-1)^{j-i} (A_i^* z)^{(j-i)}, \\ &\dots\dots\dots\dots \\ l_n^+(z) &= \sum_{i=0}^{n} (-1)^{n-i} (A_i^* z)^{(n-i)}. \end{aligned}$$

*Then* $\mathscr{D}^*$ *consists of all functions* $z$ *in* $W_m^{2,n}$ *for which* $l_j^+(z) + v_{j+1}^*[a, t]\,\phi$ *is absolutely continuous and the end point conditions*

$$(3,7) \qquad \begin{aligned} l_j^+(z)\,[a^+] &= -v_{j+1}^*[a]\,\phi, \\ l_j^+(z)\,[b^-] &= \phantom{-}v_{j+1}^*[b]\,\phi \end{aligned}$$

*are satisfied, where* $\phi$ *is an arbitrary vector in* $\mathscr{C}^s$.

Proof. This follows because the rows of the matrix $\beta$ are simply the coefficients of ascending powers of $z$ in the expressions $l_j^+$.

**Corollary 3.2.** $\mathscr{D}^*$ *is non-empty and dense in* $\mathscr{L}_m^2$.

Proof. This follows by taking $\phi = 0$. Then $\mathscr{D}^*$ contains the functions $z$ in $W_m^{2,n}$ such that $z^{(i)}$, $i = 1, \dots, n-1$, vanishes at the points $a$, $b$. This class is well known to be dense.

Suppose the measures $\nu_1, \ldots, \nu_n$ have support only at finitely many points $a = x_0 < x_i < \ldots < x_k = b$. Then it can be shown [7] that the parameter $\phi$ can be eliminated and $\mathscr{D}^*$ can be described by the $mn(k+1)$-dimensional system

$$\text{(3,8)} \qquad \begin{pmatrix} \hat{z}(x_0^+) \\ \cdots\cdots\cdots\cdots \\ (\hat{z}(x_i^+) - \hat{z}(x_i^-)) \\ \cdots\cdots\cdots\cdots \\ \hat{z}(x_k^-) \end{pmatrix} = P \begin{pmatrix} \hat{z}(x_0^+) \\ \cdots\cdots\cdots\cdots \\ \hat{z}(x_i^+) - \hat{z}(x_i^-) \\ \cdots\cdots\cdots\cdots \\ \hat{z}(x_k^-) \end{pmatrix},$$

where $P$ is a certain constant matrix depending on the measures $\nu_i$ and the points $x_i$. Thus (3.8) is a generalization of the boundary form representation obtainable for the adjoint domain for an ordinary two point problem. For further details see [7].

The following corollary will be essential for the determination of the properties of the $Lg$-spline.

**Corollary 3.3** *$z$ in $\mathscr{D}^*$ is in class $C^{(j)}[a, b]$ if and only if the measures $\nu_1, \ldots, \nu_{j+1}$ are continuous (non-atomic). If, however, $\mathcal{O}$ is an open set in the complement of the supporting sets of $\nu_1, \ldots, \nu_k$, and $z \in N(L^+)$, then $z$ is in class $C^{(n)}$ on $\mathcal{O}$. In any event $z$ and its derivatives have at most countably many discontinuities.*

Proof. The last statement is true because $\tilde{\nu}^*$ is a measure of bounded variation. The first statement is trivial for $j = 0$ by Corollary 3.1. Let us assume it is true for $j = 0, \ldots, j-1$. Then from (3.6) and (3.7) and for $t \in (a, b)$

$$\text{(3,9)} \qquad l_j^+(z)[t^+] - l_j^+(z)[t^-] = (-1)^j A_0^*(z^{(j)}(t^+) - z^{(j)}(t^-)) + \\ + \sum_{r=0}^{j-1} \alpha_{jr}(z^{(r)}(t^+) - z^{(r)}(t^-)) = \nu_{j+1}^*[t].$$

By hypothesis $\sum_{r=0}^{j-1} \alpha_{jr}(z^{(r)}(t^+) - z^{(r)}(t^-))$ vanishes if and only if $\nu_1, \ldots, \nu_j$ are non-atomic. Hence because of the non-singularity of $A_0^*$, $z^{(j)}(t^+) - z^{(j)}(t^-)$ depends only on $\nu_{j+1}^*[t]$. To prove the second statement, let $z \in N(L^+)$ and $t \in \mathcal{O}$. Then

$$\text{(3,10)} \qquad l_j^+(z)[t^+] - l_j^+(z)[t^-] = \nu_{j+1}^*[t] = 0.$$

The equations (3.10) are equivalent to the system

$$\text{(3,11)} \qquad \beta(t)(\hat{z}(t^+) - \hat{z}(t^-)) = 0.$$

Since $\beta(t)$ is invertible, $z$ is in $C^{(n-1)}$ on $\mathcal{O}$. Because $z \in N(L^+)$, $l^+(z) \equiv 0$ on $[a, b]$. Therefore for $t$ in $\mathcal{O}$,

$$\text{(3,12)} \qquad 0 = (-1)^n A_0^*(z^{(n)}(t^+) - z^{(n)}(t^-)) + \sum_{r=0}^{n-1} \alpha_{jr}(z^{(r)}(t^+) - z^{(r)}(t^-)) = \\ = (-1)^n A_0^*(z^{(n)}(t^+) - z^{(n)}(t^-)).$$

As before the non-singularity of $A_0^*$ implies that $z^{(n)}$ is continuous on $\mathcal{O}$. This completes the proof of the corollary.

The next theorem will provide the key to the proof (Theorem 4.1) of the existence of the $Lg$-spline.

**Theorem 3.2.** *$L$ and its adjoint $L^+$ are unbounded closed operators in $\mathscr{L}_m^2$ with closed ranges and closed (finite dimensional) null spaces.*

*Therefore,* (see Goldberg [10] p. 95, Theorem IV. 1.2):

$$\begin{aligned} &\text{(i)} \quad R(L) = N(L^+)^{\perp}\,; \\ &\text{(ii)} \quad R(L^+) = N(L)^{\perp}\,; \\ &\text{(iii)} \quad N(L) = R(L^+)^{\perp}\,; \\ &\text{(iv)} \quad N(L^+) = R(L)^{\perp}\,. \end{aligned} \tag{3,13}$$

**4. The $Lg$-Spline.** We first prove two lemmas concerning the range of the non-linear operator $L_r$.

**Lemma 4.1.** *Suppose $\lambda(Y_1, \ldots, Y_n)$ is the $s \times mn$ matrix*

$$\left(\sum_{j=1}^{n} \int_a^b \mathrm{d}v_j Y_1^{(n-j)}, \ldots, \sum_{j=1}^{n} \int_a^b \mathrm{d}v_j Y_n^{(n-1)}\right), \tag{4,1}$$

*where $Y_1, \ldots, Y_n$ are the $m \times m$ fundamental operator solutions for the system $l(y)$. If $r$ is in the range of $\lambda(Y_1, \ldots, Y_n)$ — or equivalently if $r$ annihilates the null space $\lambda^*(Y_1, \ldots, Y_n)$ — then:*

$$\begin{aligned} &\text{(i)} \quad R(L_r) = R(L)\,; \\ &\text{(ii)} \quad W_m^{2,n}(r) = D(L)\,. \end{aligned} \tag{4,2}$$

*If on the other hand $r$ is not in the range of $\lambda(Y_1, \ldots, Y_n)$ — equivalently if $r$ does not annihilate the null space of $\lambda^*(Y_1, \ldots, Y_n)$ — then*

$$\begin{aligned} &\text{(iii)} \quad R(L_r) \cap R(L) \quad \textit{is empty}\,; \\ &\text{(iv)} \quad W_m^{2,n}(r) \cap D(L) \quad \textit{is empty}\,; \\ &\text{(v)} \quad R(L_r) = R(L) + \xi\,, \quad \textit{where } \xi \textit{ is any function in } R(L_r)\,; \\ &\text{(vi)} \quad W_m^{2,n}(r) = D(L) + \xi'\,, \quad \textit{where } \xi' \textit{ is any function in } W_m^{2,n}(r)\,. \end{aligned} \tag{4,3}$$

Proof. Since (i) $\Leftrightarrow$ (ii), (iii) $\Leftrightarrow$ (iv), and (v) $\Leftrightarrow$ (vi), we need only deal with (i), (iii), and (v). Define

$$N(L_r) = \{y : y \in N(l);\ \lambda(y) = r\}\,. \tag{4,4}$$

Any function $y$ in $N(l)$ can be written

$$y = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_a^b \mathrm{d}v_j \, Y_i^{(n-j)} C_i \,, \tag{4,5}$$

where $C_1, \ldots, C_n$ are arbitrary vectors in $\mathscr{C}^m$. Consequently $r$ is in the range of the matrix (4.1) if and only if $N(L_r)$ is non-empty. Suppose $N(L_r)$ is non-empty. Let $y$ be some function in the preimage of $L$ and $w$ a function in $N(L_r)$. Then $w + y \in W_m^{2,n}(r)$, and $l(w + y) \in R(L_r)$. Moreover $l(w + y) = l(y)$. Thus $R(L) \subseteq R(L_r)$. To show the reverse inclusion, let $z$ be in $W_m^{2,n}(r)$ and $\varphi$ in $N(L_r)$. Then $z - \varphi$ is in $D(L)$ and $l(z - \varphi) = l(z)$. Therefore $R(L_r) \subset R(L)$; and the two ranges are identical. This proves (i). Suppose on the other hand $r$ is not in the range of the matrix (4.1). Then $N(L_r)$ is empty. If (iii) is false, there exists $w \in R(L_r) \cap R(L)$. Then there exists $u$ in $D(L)$ and $v$ in $W_m^{2,n}(r)$ such that $L_r(v) = w = L(u)$. Evidently $u - v \in N(L_r)$ which is impossible. Finally statement (vi) and consequently (v) follows from the fact that the difference of any two members of $W_m^{2,n}(r)$ is in $D(L)$.

**Lemma 4.2.** *$R(L_r)$ is a closed variety in $\mathscr{L}_m^2$.*

Proof. By Theorem 3.2 of the previous section, $R(L)$ is closed in $\mathscr{L}_m^2$. From (v) of Lemma 4.1 $R(L_r)$ is closed.

The next theorem and its corollaries are the main results of this paper.

**Theorem 4.1.** *An Lg-spline $f$ exists for $l(y)$ in $W_m^{2,n}(r)$. Moreover $f$ in $W_m^{2,n}(r)$ is an Lg-spline if and only if $l(f) \in N(L^+)$, that is, — to use equivalent notation, if and only if $L^+ L_r(f) = 0$.*

Proof. Because $R(L_r)$ is closed (Lemma 4.2) there exists a *unique* element $\tau$ of $R(L_r)$ of minimum norm. By the definition of $R(L_r)$ there exists $f$ in $W_m^{2,n}(r)$ such that $l(f) = \tau$. Since $R(L_r)$ is a translate of $R(L)$, $\tau$ is orthogonal to $R(L)$. By Theorem 3.2 (iv) $\tau \in N(L^+)$; that is, $l(f) \in N(L^+)$, or equivalently $L^+ L_r(f) = 0$. This shows the "only if" part of the second statement. Conversely, suppose $l(f)$ is in $N(L^+)$ with $f$ in $W_m^{2,n}(r)$ (equivalently $L^+ L_r(f) = 0$). Let $g$ be any other function in $W_m^{2,n}(r)$. Then

$$l(g) = l(g - f) + l(f) \,. \tag{4,6}$$

Since $l(g - f) \in R(L)$, we have the "first integral relation":

$$\|l(g)\|^2 = \|l(g - f)\|^2 + \|l(f)\|^2 \,; \tag{4,7}$$

and thus

$$\|l(g)\| \geqq \|l(f)\| \,. \tag{4.8}$$

Therefore, $f$ is an $Lg$-spline. This completes the proof of the theorem.

**Corollary 4.1.** *If $N(L)$ is $\{0\}$, $f$ is unique; otherwise $f$ is determined modulo $N(L)$. Moreover, if $N(L_r)$ is non-empty, $f$ can be any function in $N(L_r)$ and zero is the minimum of $\|l(f)\|$.*

Proof. The second statement is obvious from the definition of $N(L_r)$. Suppose $g$ is another $Lg$-spline. Since $g - f \in W_m^{2,n}(0)$, $l(g - f) \in R(L)$. Since $g - f \in N(L^+) = R(L)^{\perp}$, $l(g - f) = 0$. This shows that $f$ is determined modulo $N(L)$ and is unique if $N(L) = \{0\}$.

The next corollary accounts for the remarkable fact that the spline is often *much* smoother than most of the functions in $W_m^{2,n}(r)$.

**Corollary 4.2.** *The Lg-spline $f$ possesses the following degrees of smoothness.*

(i) *$f \in C^{(n-1)}[a, b]$ always;*

(ii) *$f \in C^{(n-1+j)}[a, b]$ if and only if $\nu_1, \ldots, \nu_j$ are continuous;*

(iii) *$f \in C^{(2n)}$ in every open set $\mathcal{O}$ in the complement of the supports of the measures $\nu_1, \ldots, \nu_n$.*

Proof. Since $f$ is in $W_m^{2,n}(r)$, it is at least in $C^{(n-1)}[a, b]$. This proves (i). By the previous theorem, $l(f) \in N(L^+) \subset \mathscr{D}^*$. Therefore by Corollary 3.3, $l(f) \in C^{(j-1)}[a, b]$ if and only if $\nu_1, \ldots, \nu_j$ are continuous. Also $l(f)$ is in $C^{(n)}$ on $\mathcal{O}$. The first statement implies that $f \in C^{(n-1+j)}[a, b]$; and the second one means that $f$ is in $C^{(2n)}$ on $\mathcal{O}$. This proves (ii) and (iii).

We remark here that if $\lambda$ only involves the measure $\nu_n$, then $f \in C^{(2n-1)}[a, b]$ if $\nu_n$ is continuous; otherwise $f \in C^{(2n-2)}[a, b]$.

The next corollary concerns the behavior of the spline at the ends of the interval $[a, b]$.

**Corollary 4.3.** *Suppose* $\inf\{\operatorname{supp} \tilde{\nu}\} = c > a$ *and* $\sup\{\operatorname{supp} \tilde{\nu}\} = d < b$. *Then $l(f)$ vanishes on the intervals $[a, c)$, $(d, b]$.*

Proof. Since $l(f) \in N(L^+)$, $l(f)$ satisfies the end point conditions (3.7) and $l^+ \, l(f) = 0$ on $[a, b]$. By assumption $\tilde{\nu}^*[a]$ and $\tilde{\nu}^*[b]$ are zero. Thus $l(f)\,[a^+] = 0$ and $l(f)\,[b^-] = 0$. Hence the uniqueness theorem for linear systems tells us that $l(f)$ vanishes on intervals $[a, \varepsilon) \subset [a, c)$ and $(\sigma, b] \subset (d, b]$. Since $l(f)$ is continuous on $[a, b]$, repeating the argument shows that $l(f)$ vanishes on the *whole* intervals $[a, c)$ and $(d, b]$.

**5. Examples.** In this section we apply the theory of the last section to some concrete problems. All our examples will involve only scalar real operators and data.

Example 1. It has been known for some time (see [8], [9], [12], [28]) that a unique function $f$ exists minimizing

$$\int_a^b (f^{(k)}(x))^2 \, dx \tag{5,1}$$

and interpolating the data

$$\lambda_i(x_i) = f(x_i) = r_i \quad i = 1, \ldots, n\,, \tag{5,2}$$

where $a < x_1 < \ldots < x_n < b$ and $k \leqq n$. $f$ moreover is a natural spline of degree $2k - 1$ with knots at the points $x_i$. We proceed to derive this result using our methods.

Here $\nu_i = 0$, $i = 1, \ldots, k - 1$, and $\nu_k$ is the $n \times 1$ m.v. measure

$$\nu_k = \sum_{i=1}^{n} \bar{e}_i \mu_{(x_i)}\,. \tag{5,3}$$

Then the data (5.2) can be written

$$\int_a^b \mathrm{d}\nu_k f = r\,, \tag{5,4}$$

where $r = (r_1, \ldots, r_n)^t$. The homogeneous operator $L$ is

$$l(y) = y^{(k)}\,, \tag{5,5}$$

$$\lambda(y) = \int_a^b \mathrm{d}\nu_k y = 0\,.$$

Since the null space of $l(y)$ consists of polynomials of degree $k - 1$ and $n \geqq k$, it is clear that $N(L) = \{0\}$, so that (Corollary 4.1) the $Lg$-spline $f$ is unique. $L^+$ is easily seen from Theorem 3.1 and Corollary 3.3 to be

$$l^+(z) = (-1)^k\, z^{(k)}\,, \tag{5,6}$$

$$\mathscr{D}^* = \{z : z \in C^{(k-2)}[a, b];\; z^{(k)} \in \mathscr{L}^2;\; \hat{z}(a^+) = \hat{z}(b^-) = 0\}\,.$$

The null space of $L^+$ is particularly easy to determine. It is given by polynomials of degree $k - 1$ in the intervals $(x_i, x_{i+1})$ and vanishes in $[a, x_1)$ and $(x_n, b]$. In other words $N(L^+)$ consists of splines of degree $k - 1$ with knots at the points $x_i$, and which vanish in the intervals $[a, x_1)$, $(x_n, b]$. Now such a spline can be given the canonical representation (see [12])

$$s(x) = \sum_{i=1}^{n} c_i (x - x_i)_+^{k-1}\,, \tag{5,7}$$

where "+" stands for the notation

$$x_+^m = \begin{cases} 0 & \text{if } x \leqq 0 \\ x^m & \text{if } x > 0\,. \end{cases} \tag{5,8}$$

$s(x)$ vanishes in $[a, x_1)$. The condition that it vanishes in $(x_n, b]$ (after some simple algebra) imposes the conditions

$$\sum_{j=1}^{n} c_j x_j^r = 0\,, \quad r = 0, \ldots, k - 1\,. \tag{5,9}$$

Since (Theorem 4.1),

$$(5{,}10)\qquad l(f) = f^{(k)}\,,\quad = s(x)\,,$$

$f$ must be a spline of degree $2k - 1$ given by

$$(5{,}11)\qquad f = \sum_{i=1}^{n} d_i(x - x_i)_+^{2k-1} + p(x)\,,$$

where

$$(5{,}12)\qquad d_i = \frac{(k-1)!}{(2k-1)!}\, c_i\,,$$

and $p(x)$ is a polynomial of degree $k - 1$. Because of (5.9) the $d_i$ also satisfy the condition

$$(5{,}13)\qquad \sum_{j=1}^{n} d_j x^r = 0\,, \quad r = 0, \ldots, k-1\,.$$

Thus $f$ is a natural spline of degree $2k - 1$.

In the remaining examples we put aside the problem of calculating $N(L^+)$ and solving for the spline $f$; instead we simply show that the splines satisfy certain well-known smoothness properties.

Example 2. A natural generalization of the previous example is the notion of a "generalized spline" due to GREVILLE [12]. A generalized spline is defined as a function $f$ minimizing in $\mathscr{L}^2$ the norm of the real scalar operator

$$(5{,}14)\qquad l(f) = \sum_{i=0}^{k} a_i(x) f^{(k-i)}\,; \quad a_0(x) > 0 \quad \text{on} \quad [a, b]$$

and interpolating the data (5.2). Again we assume $n \geqq k$.

As before, $v_i = 0$, $i \neq k$ and

$$(5{,}15)\qquad v_k = \sum_{i=1}^{n} \bar{e}_i \mu(x_i)\,,$$

so that $\lambda(y)$ can be represented in the form (5.4). The homogenous operator $L$ is given by

$$(5{,}16)\qquad l(y) = \sum_{i=0}^{k} a_i(x)\, y^{(k-i)}\,,$$

$$\lambda(y) = \int_a^b \mathrm{d}v_k y = 0\,.$$

If the $k$ fundamental solutions of $l(y)$ are a Chebyshev system (i.e. have the property that no linear combination has more than $k - 1$ zeros), it is clear that only the zero function can be in $N(L)$. Thus (Corollary 4.1) the $Lg$-spline $f$ is unique. By the remarks following Corollary 4.2, $f$ is in $C^{(2n-2)}[a, b]$, and by Corollary 4.3 $l(f) \equiv 0$ on $[a, x_1)$, $(x_n, b]$.

Example 3. Again a finite set of points $a \leqq x_1 < \ldots < x_n \leqq b$ is prescribed in the interval $[a, b]$. Let $E = \{(\alpha, \beta)\}$ be a set of ordered pairs such that $\alpha$ assumes each of the values $1, 2, \ldots, n$ at least once, with $\beta \in \{0, 1, \ldots, k-1\}$ ($\beta = k - 1$ for some pair $(\alpha, \beta)$ in $E$). Ahlberg and Nilson [1] and later Schoenberg [25] explored the problem of minimizing (5.1) while simultaneously interpolating the "Hermite-Birkhoff" functionals

$$\lambda_{\alpha\beta}(y) = y^{(\beta)}(x_\alpha) = r_{\alpha\beta}\,, \quad (\alpha, \beta) \in E\,. \tag{5,17}$$

The solution of this minimization problem is called a $g$-spline.

Here the $\lambda_{\alpha\beta}$ are represented by the vector functional

$$\lambda(y) = \sum_{j=1}^{k} \int_a^b \mathrm{d}v_j y^{(k-j)} = r\,, \tag{5,18}$$

where $v_j$ is the $kn \times 1$ m.v. measure

$$v_j = \sum_{(\alpha,\beta)\in E} \bar{e}_\sigma \delta(k - j, \beta)\, \mu_{(x_\alpha)}\,, \quad \sigma = (\alpha - 1)\, k + \beta + 1\,, \tag{5,19}$$

where $r = \sum_{(\alpha,\beta)\in E} \bar{e}_\sigma r_{\alpha\beta}$ and $\delta$ is the Kronecker delta function. With this notation $L$ is

$$\begin{aligned} l(y) &= y^{(k)}\,, \\ \lambda(y) &= 0\,. \end{aligned} \tag{5,20}$$

If $y^{(k)} = 0$ and $\lambda(y) = 0$ imply $y = 0$, i.e., $N(L) = \{0\}$, then as before the spline $f$ is unique. By Corollary 3.3, $f^{(k)} \in C^{(k)}(x_i, x_{i+1})$. Since $l(f) \in N(L^+)$, $ll^+(f)\,[x] = 0$ for $x \neq x_i$; i.e., $f^{(2k)}(x) = 0$ if $x \neq x_i$. From Corollary 4.3, $f^{(k)} \equiv 0$, on $[a, x_1)$, $(x_n, b]$. Also from Corollary 3.1 and the definition of the measures $v_j$

$$l^+_{k-\beta-1}\, l(f)\, [x_\alpha^+] - l^+_{k-\beta-1}\, l(f)\, [x_\alpha^-] = -v^*_{k-\beta}[x_\alpha]\, \phi = 0 \tag{5,21}$$

if and only if $(\alpha, \beta) \notin E$. On computing the expressions $l^+_{k-\beta-1}\, l(f)$, we see from (5.21) that $f^{(2k-\beta-1)}$ is continuous at $x_\alpha$ if and only if the derivative $y^{(\beta)}$ is not specified there. *Hence in particular, if $\beta'$ is the order of the highest derivative specified at $x_\alpha$, the g-spline $f$ is locally in $C^{(2k-\beta'-1)}$ around $x_\alpha$.* Finally from the endpoint conditions (3.7), we have

$$\begin{aligned} f^{(2k-j-1)}(a^+) &= 0 \\ f^{(2k-j-1)}(b^-) &= 0\,, \end{aligned} \tag{5,22}$$

if and only if $y^{(j)}$ is not specified at $x_1 = a$ or $x_n = b$.

Example 4. For our last example we discuss the minimization problem associated with the operator (5.14) and the functionals

$$\lambda_l(y) = \sum_{i=1}^{n} \Big( \sum_{j=1}^{k} \alpha_{ij}^{(l)}\, y^{(k-j)}(x_i) \Big), \quad l = 1, \ldots, p, \quad p < nk. \tag{5,23}$$

Clearly the interpolating "extended Hermite-Birkhoff functionals" (1.3) considered by Jerome and Schumaker [15] are a special case of (5.23).

Here

$$v_j = \sum_{i=1}^{n} \bar{\alpha}_{ij} \mu_{(x_i)}, \tag{5,24}$$

where $\bar{\alpha}_{ij} = (\alpha_{ij}^{(l)}, \ldots, \alpha_{ij}^{(p)})^t$. Since this example is done exactly the same way as the previous one, we will list the properties of the $Lg$-spline $f$ and leave their derivation to the reader.

(i) If $\lambda_i(y) = 0$; $i = 1, \ldots, p$ and $l(y) = 0$ imply $y = 0$, the $Lg$-spline is unique;

(ii) $l^+ \, l(f)\,[x] = 0$ for all $x \neq x_i$;

(iii) $l(f) = 0$ on $[a, x_1)$, $(x_n, b]$;

(iv) if the derivative $y^{(j)}$ is not specified at the knot $x_i$,

$$l_{k-j-1}^+ \, l(f)\,[x_i^+] - l_{k-j-1}^+ \, l(f)\,[x_i^-] = 0.$$

Note $(i)-(\text{iv})$ remain true even if the $\{x_i\}$ are not a finite ordered set of points *provided the measures $v_j$ are of bounded variation*. It is easily seen that this is equivalent to requiring $\sum_{i=1}^{\infty} |\alpha_{ij}^{(l)}| < \infty$ for every fixed $l, j$.

**6. Conclusion.** In this section we wish to indicate some possible extensions of the minimization problem considered in this paper.

Specifically, the functionals $\lambda_i$ can be generalized in many ways. We may assume for example that the measures $w_{ij}$ have absolutely continuous parts. In this case (see [6] for a discussion of the adjoint of a *first* order differential operator with a Stieltjes boundary condition with respect to a possibly non-sigular measure) the adjoint of the homogeneous operator $L$ may not be well defined. We must then have recourse to the theory of adjoint *relations* in Hilbert space developed by Arens [3]. Furthermore additional smoothness conditions must be imposed on the absolutely continuous parts of the $w_{ij}$ to avoid distribution theory. A second possibility is that the domain $W_m^{2,n}(r)$ might contain discontinuous functions. For instance the $\lambda_i$ might be of the form

$$\lambda_i(y) = \sum_{l=1}^{\infty} \sum_{j=1}^{n} \big(A_{lj} y^{(n-j)}(t_l^+) + B_{lj} y^{(n-j)}(t_l^-)\big) + \sum_{j=1}^{n} \int_a^b \mathrm{d}w_{ij} y^{(n-j)}, \tag{6,1}$$

where $\{t_i\}$ might in general be an infinite set. A particular case of this interpolation problem has been already considered by Jerome [14] who pointed out that it is useful in the theory of thin vibrating elastic beams. The adjoint of the homogeneous operator determined by such a problem has not been determined in general. Special cases have been successfully considered, however, by STALLARD [27], KRALL [17,] [18], and ZETTL [29]. A third possibility is the case of minimally interpolating infinitely many functionals. The question of the adjoint in this case remains an open problem.

In a different direction it would be quite interesting to weaken the smoothness conditions on the coefficients of $l(y)$. The notion of quasiderivatives (as in NAIMARK [21]) might be introduced with profit here.

Finally, of course, it is also necessary to calculate the interpolating spline as well as examine its analytic properties. It should be evident that such a calculation essentially depends on the computation of the null space of $L^+$. Variation of parameters then reduces the problem to the solution of a certain linear algebraic system. We have already in [6] examined $N(L^+)$ for first order operators. The same methods should extend to the higher order case. It would be especially interesting to compare them with the "reproducing kernel" approach of de Boor and Lynch [9]. But we will pursue these matters in more detail elsewhere.

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