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Miroslav Dont

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ON A HEAT POTENTIAL

MIROSLAV DONT, Praha

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INTRODUCTORY REMARKS

In the whole of the present paper we shall deal with the Euclidean plane R^2.

Let Γ be the well-known kernel in R^2 (the fundamental solution of the heat equation in R^2) defined by

$$\Gamma(x, t) = \begin{cases} \frac{1}{2}(\pi t)^{-1/2} \exp\left(-\dfrac{x^2}{4t}\right), & t > 0 \\ 0, & t \leqq 0 \end{cases} . \tag{0.1}$$

This kernel is used in the definition of the heat potentials in the plane. For our purpose we shall consider a special heat potential which is defined in the following. The author wishes to acknowledge his gratitude to JOSEF KRÁL who directed his attention to the heat potential, especially to that mentioned above, and to the relevant problems.

Let φ be a fixed continuous function of bounded variation on $\langle a, b\rangle$ (where $a < b \in R^1$ are fixed) and put

$$K = \{[\varphi(t), t];\ t \in \langle a, b\rangle\} . \tag{0.2}$$

With each bounded Baire function f on $\langle a, b\rangle$ we shall associate a function Tf on $R^2 \setminus K$ defined by

$$Tf(x, t) = -\int_a^b f(\tau)\, \partial_1 \Gamma(x - \varphi(\tau), t - \tau)\, d\tau - \\ - \int_a^b f(\tau)\, \Gamma(x - \varphi(\tau), t - \tau)\, d\varphi(\tau) , \tag{0.3}$$

where $\partial_1 \Gamma$ stands for the partial derivative of Γ with respect to the first variable. It will be shown that if the function φ satisfies a certain complementary condition (namely, a geometric condition on the set K), we can evaluate the "right" and "left" limits of Tf on K at the points of continuity of the function f (we may consider the function f on $\langle a, b\rangle$ as a function on K).

The function Tf is continuous on $R^2 \setminus K$, satisfies the heat equation (we also say that it is caloric) and exhibits a boundary discontinuity on K like that of the classical double-layer heat potential. We will investigate the potential Tf in a manner similar to that used by J. Král in his papers [6], [7], [8] for the logarithmic potential. By analogy to the so-called cyclic variation from these papers we shall introduce and investigate the so-called parabolic variation which will assume the principal role in our study.

Now let us introduce some fundamental notations, notions and assertions which will be used later.

Let $^*R^1$ stand for the real line including the points $+\infty$ and $-\infty$. The term function on a set M stands for a mapping from M to $^*R^1$; the real function on M is a mapping from M to R^1 and if we talk about a continuous function we always mean a real function.

If f is a real function defined on an interval $J \subset R^1$ let us define the variation of the function f on an open set $G \subset J$ as follows. Put $\operatorname{var}[f; \emptyset] = 0$ and if $G \neq \emptyset$ define $\operatorname{var}[f; G]$ as the least upper bound of all sums of the form

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| ,$$

where $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ are non-overlapping compact intervals contained in J. For any set $M \subset J$ put

$$\operatorname{var}[f; M] = \inf_G \operatorname{var}[f; G] ,$$

where G ranges over all open sets in J for which $M \subset G$. It is known that $\operatorname{var}[f; \cdot]$ is an outer measure and its restriction on the $\operatorname{var}[f; \cdot]$-measurable sets is a measure. The integral of a function $F : M \to {}^*R^1$ (where $M \subset J$ is a $\operatorname{var}[f; \cdot]$-measurable set) with respect to this measure will be denoted by

$$\int_M F \,\mathrm{d} \operatorname{var} f , \quad \int_M F(t) \,\mathrm{d} \operatorname{var} f(t) , \quad \text{etc.}$$

We say that the function f has locally finite variation if $\operatorname{var}[f; I] < \infty$ for every compact interval $I \subset J$. If the function f has locally finite variation on J then the integral

$$\int_M F \,\mathrm{d}f$$

stands for the Lebesgue-Stieltjes integral of the function F. Let us note that if the functions F, f are continuous on $\langle a, b\rangle \subset R^1$ and f is of bounded variation on $\langle a, b\rangle$ then

$$\int_{\langle a,b\rangle} F \, \mathrm{d}f = \int_a^b F \, \mathrm{d}f ,$$

where the Lebesgue-Stieltjes integral is on the left hand side and the Stieltjes integral on the right hand side.

Lemma 0.1. *Let f, g be continuous functions with locally finite variation on an interval $J \subset R^1$ and let F be a continuous function on J. Then*

$$\int_J F(t) \, \mathrm{d}f(t) \, g(t) = \int_J Ff \, \mathrm{d}g + \int_J Fg \, \mathrm{d}f$$

whenever at least two of those integrals converge.

Lemma 0.2. *Let f be a continuous function of bounded variation on $\langle a, b\rangle$, Φ a function on $f(\langle a, b\rangle)$. Then*

$$\int_{f(a)}^{f(b)} \Phi(x) \, \mathrm{d}x = \int_a^b \Phi(f(t)) \, \mathrm{d}f(t)$$

whenever there is a (finite or infinite) Lebesgue integral on the right hand side.

Lemma 0.3. *Let f be a continuous function with locally finite variation on an interval $J \subset R^1$, let p be a function with continuous first derivative on $f(J)$. Putting $h = p * f$, the function h has locally finite variation on the interval J and*

$$\int_J F \, \mathrm{d} \operatorname{var} h = \int_J |p'(f(t))| \, F(t) \, \mathrm{d} \operatorname{var} f(t) \tag{0.4}$$

for any lower-semicontinuous function $F \geqq 0$ on the interval J. If, in addition, the integrals in (0.4) are finite, then

$$\int_J F \, \mathrm{d}h = \int_J p'(f(t)) \, F(t) \, \mathrm{d}f(t) .$$

Lemma 0.4. *Suppose that f, g are continuous functions with locally finite variation on an interval $J \subset R^1$. Then*

$$\int_J F \, \mathrm{d} \operatorname{var} (fg) \leqq \int_J F|f| \, \mathrm{d} \operatorname{var} g + \int_J F|g| \, \mathrm{d} \operatorname{var} f .$$

In particular, the following estimate holds:

$$\operatorname{var}[fg; J] \leqq \operatorname{var}[g; J] \sup_{t \in J} |f(t)| + \operatorname{var}[f; J] \sup_{t \in J} |g(t)| .$$

Lemma 0.5. *Let f be a continuous function of bounded variation on an interval $J \subset R^1$. Suppose that $F \geqq 0$ is a lower-semicontinuous function on J. For each $x \in R^1$ put*

$$\theta(x, F) = \sum_{\substack{t \in J \\ f(t) = x}} F(t)$$

(we put $\theta(x, F) = +\infty$ whenever $F(t) > 0$ for uncountably many $t \in J$ with $f(t) = x$). Then $\theta(x, F)$ is a Lebesgue measurable function of the variable $x \in R^1$ and

$$\int_{-\infty}^{+\infty} \theta(x, F) \, \mathrm{d}x = \int_J F \, \mathrm{d} \operatorname{var} f .$$

These assertions can be found for example in [6]; Lemma 0.2 is established in [13].

1. PARABOLIC VARIATION

In this part let $a < b$ be fixed numbers in R^1 and let φ be a continuous function of bounded variation on the interval $\langle a, b \rangle$. Let K be the set defined by (0.2). K is a compact set in R^2.

For each $[x, t] \in R^2$ with $t > a$ we define a function $\alpha_{x,t}$ on the interval $\langle a, \min . \{t, b\})$ by the equality

(1.1) $$\alpha_{x,t}(\tau) = \frac{x - \varphi(\tau)}{2 \sqrt{(t - \tau)}} .$$

The function $\alpha_{x,t}$ has locally finite variation on the interval $\langle a, \min \{t, b\})$. This follows immediately from Lemma 0.4 for it holds

$$\operatorname{var}_\tau \left[\frac{x - \varphi(\tau)}{2 \sqrt{(t - \tau)}} ; \langle a, t' \rangle \right] \leqq \frac{1}{2 \sqrt{(t - t')}} (\operatorname{var} [\varphi; \langle a, b \rangle] + \sup_{\tau \in \langle a, b \rangle} |x - \varphi(\tau)|)$$

for each $t' \in \langle a, \min \{t, b\})$.

We fix a bounded lower-semicontinuous function $Q \geqq 0$ defined on $\langle a, b \rangle$.

Let $[x, t] \in R^2$. For $\alpha, r > 0$, $\alpha < +\infty$ we put

(1.2) $$n^Q_{x,t}(r, \alpha) = \sum_\tau Q(\tau) ;$$

in the sum on the right hand side we consider every $\tau \in \langle a, b \rangle$ for which $0 < t - \tau < < r$ and

$$t - \tau = \left(\frac{x - \varphi(\tau)}{2\alpha} \right)^2 .$$

Further, write

$$n^Q_{x,t}(\infty, \alpha) = n^Q_{x,t}(\alpha) ; \quad n^1_{x,t}(r, \alpha) = n_{x,t}(r, \alpha) ;$$
$$n_{x,t}(\infty, \alpha) = n_{x,t}(\alpha) .$$

Consequently, $n_{x,t}(\alpha)$ stands for the number of all points of the set

$$K \cap \left\{ [\xi, \tau] \in R^2 ; \quad t - \tau = \left(\frac{\xi - x}{2\alpha} \right)^2 , \quad [\xi, \tau] \neq [x, t] \right\} .$$

Lemma 1.1. *Given* $[x, t] \in R^2$, $r > 0$, *then the function* $n^Q_{x,t}(r, \alpha)$ *is a measurable function of the variable* $\alpha \in (0, \infty)$ *and*

$$(1.3) \qquad \int_0^\infty \exp(-\alpha^2)\, n^Q_{x,t}(r, \alpha)\, d\alpha = \int_{\max\{a, t-r\}}^{\min\{t, b\}} Q(\tau) \exp(-\alpha^2_{x,t}(\tau))\, d \operatorname{var} \alpha_{x,t}(\tau)$$

whenever $\max\{a, t - r\} < \min\{t, b\}$; *otherwise*

$$(1.4) \qquad \int_0^\infty \exp(-\alpha^2)\, n^Q_{x,t}(r, \alpha)\, d\alpha = 0 .$$

Proof. If $\max\{a, t - r\} \geqq \min\{t, b\}$, then either $t \leqq a$ or $t \geqq b + r$. The function $n^Q_{x,t}(r, \cdot)$ is the zero-function on $(0, \infty)$ in each of these two cases, which follows immediately from the definition of this function.

Suppose now that $a < t < b + r$, $x \in R^1$.

Putting $I = (\max\{a, t - r\}, \min\{t, b\})$,

$$F(\tau) = Q(\tau) \exp(-\alpha^2_{x,t}(\tau)) \quad (\tau \in I) ,$$

the function F is a nonnegative lower-semicontinuous function on the interval I.

Let us define a function m on R^1 by

$$m(\beta) = \sum_\tau F(\tau) ,$$

where τ in the sum runs over those $\tau \in I$ for which

$$\frac{x - \varphi(\tau)}{2 \sqrt{(t - \tau)}} = \beta .$$

It follows from Lemma 0.5 that m is a measurable function on R^1 and

$$(1.5) \qquad \int_{-\infty}^{+\infty} m(\beta)\,\mathrm{d}\beta = \int_I F(\tau)\,\mathrm{d}\,\mathrm{var}_\tau \left[\frac{x - \varphi(\tau)}{2\sqrt{(t - \tau)}}\right] = \\ = \int_I Q(\tau)\exp\left(-\alpha_{x,t}^2(\tau)\right)\mathrm{d}\,\mathrm{var}\,\alpha_{x,t}(\tau)\,.$$

However,

$$(1.6) \qquad m(\alpha) + m(-\alpha) = \sum_M F(\tau) = \sum_M Q(\tau)\exp\left(-\alpha^2\right) = \exp\left(-\alpha^2\right) n_{x,t}^Q(r, \alpha)$$

where $M = \{\tau \in I;\ \pm\alpha = \alpha_{x,t}(\tau)\}$ and thus

$$(1.7) \qquad \int_{-\infty}^{\infty} m(\beta)\,\mathrm{d}\beta = \int_0^{\infty} \exp\left(-\alpha^2\right) n_{x,t}^Q(r, \alpha)\,\mathrm{d}\alpha\,.$$

It is seen from (1.6) that $n_{x,t}(r, \cdot)$ is a measurable function on $(0, \infty)$. The relation (1.3) follows from (1.7) and from (1.5).

Now we are justified to define the parabolic variation.

Definition 1.1. Let $r > 0$. The function $V_K^Q(r; \cdot, \cdot)$ defined by equality

$$(1.8) \qquad V_K^Q(r; x, t) = \int_0^{\infty} \exp\left(-\alpha^2\right) n_{x,t}^Q(r, \alpha)\,\mathrm{d}\alpha$$

is called *the parabolic variation with the radius r and the weight Q of the curve φ (or of the set K)*. Furthermore we write

$$V_K^Q(\infty; x, t) = V_K^Q(x, t)\,, \quad V_K^1(r; x, t) = V_K(r; x, t)\,,$$
$$V_K^1(x, t) = V_K(x, t) \quad \text{for} \quad [x, t] \in R^2\,.$$

The function $V_K(\cdot, \cdot)$ is called *the parabolic variation (of the curve φ)*.

Let us note that (by Lemma 1.1)

$$(1.9) \qquad V_K^Q(r; x, t) = \int_{\max\{a, t-r\}}^{\min\{t, b\}} Q(\tau)\exp\left(-\alpha_{x,t}^2(\tau)\right)\mathrm{d}\,\mathrm{var}\,\alpha_{x,t}(\tau)$$

for any $r > 0$, $x \in R^1$, $a < t < b + r$.

If either $t \leqq a$ or $t \geqq b + r$ then

$$V_K^Q(r; x, t) = 0\,.$$

Especially,

$$(1.10) \qquad V_K(x, t) = \int_a^{\min\{t, b\}} \exp\left(-\alpha_{x,t}^2(\tau)\right)\mathrm{d}\,\mathrm{var}\,\alpha_{x,t}(\tau)$$

for any $x \in R^1$, $t > a$.

Lemma 1.2. *The function $V_K^Q(r; \cdot)$ is a nonnegative lower-semicontinuous function on R^2 and is finite on $R^2 - K$.*

Proof. $V_K^Q(r; \cdot)$ is nonnegative since Q and also $n_{x,t}^Q(r; \cdot)$ are.

$V_K^Q(r; \cdot)$ is undoubtedly continuous at every point of the set

$$\{[x, t] \in R^2;\ t \leqq a\} \cap \{[x, t] \in R^2;\ t \geqq b + r\}$$

since $V_K^Q(r; \cdot) = 0$ on that set.

Fix $x \in R^1$, $t \in (a, b + r)$ and suppose that

$$c < V_K^Q(r; x, t)$$

for a fixed $c \in R^1$. Taking into account that Q is a lower-semicontinuous on $\langle a, b\rangle$ and putting $I = (\max\{a, t - r\}, \min\{t, b\})$ we obtain from (1.9) that there is a continuous function q on $\langle a, b\rangle$ for which $0 \leqq q \leqq Q$ on $\langle a, b\rangle$, such that

$$\int_I q(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) > c\,.$$

It is known that if $F \geqq 0$ is a continuous function on I and f is a continuous function with locally finite variation on I then

$$\int_I F \,\mathrm{d} \operatorname{var} f = \sup_L \int_L F \,\mathrm{d} \operatorname{var} f\,,$$

where L runs over all compact intervals which are contained in I. Now it is seen from the definition of the Stieltjes integral that there are points $\tau_1, \tau_2, \ldots, \tau_n$ with

$$\max\{a, t - r\} < \tau_1 < \tau_2 < \ldots < \tau_n < \min\{t, b\}$$

such that

$$\theta(x, t) = \sum_{i=1}^{n-1} q(\tau_i) \exp\left(-\alpha_{x,t}^2(\tau_i)\right) \left|\alpha_{x,t}(\tau_i) - \alpha_{x,t}(\tau_{i+1})\right| > c\,.$$

If we fix such points $\tau_1, \tau_2, \ldots, \tau_n$, the expression θ may be considered a function on the set $M = \{[x', t'] \in R^2;\ t' > \tau_n\}$. The function θ is surely continuous on M and thus there is $\delta > 0$ with

$$\delta < \min\{\tau_1 - \max\{a, t - r\},\ \min\{t, b\} - \tau_n\}$$

such that

$$\theta(x', t') > c$$

for each $[x', t'] \in R^2$ with $|x - x'| < \delta$, $|t - t'| < \delta$.

For $t' \in R^1$ with $|t - t'| < \delta$ it holds

$$\langle \tau_1, \tau_n \rangle \subset (\max\{a, t' - r\},\ \min\{t', b\})$$

and thus (if, besides that, $|x - x'| < \delta$) the following inequality is valid:

$$V_K^Q(r; x', t') = \int_{\max\{a,t'-r\}}^{\min\{t',b\}} Q(\tau) \exp(-\alpha_{x',t'}^2(\tau))\, \mathrm{d} \operatorname{var} \alpha_{x',t'}(\tau) =$$

$$= \int_{\tau_1}^{\tau_n} q(\tau) \exp(-\alpha_{x',t'}^2(\tau))\, \mathrm{d} \operatorname{var} \alpha_{x',t'}(\tau) \geqq \theta(x', t') > c\,.$$

As we choose $c < V_K(r; x, t)$ arbitrary, the lower-semicontinuity of $V_K^Q(r; \cdot)$ at the point $[x, t]$ follows.

Now it remains to show that for any $[x, t] \in R^2 \setminus K$ it holds $V_K^Q(r; x, t) < \infty$. If $t \leqq a$ then $V_K^Q(r; x, t) = 0$. Q is bounded by the assumption. It may be supposed for instance that $Q \leqq 1$ on $\langle a, b \rangle$.

If $t > b$ then

$$V_K^Q(r; x, t) \leqq V_K^Q(x, t) \leqq V_K(x, t) =$$

$$= \int_a^b \exp(-\alpha_{x,t}^2(\tau))\, \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) \leqq \operatorname{var}[\alpha_{x,t}; \langle a, b \rangle] \leqq$$

$$\leqq \sup_{\tau \in \langle a,b \rangle} |x - \varphi(\tau)| \left(\frac{1}{2\sqrt{(t-b)}} - \frac{1}{2\sqrt{(t-a)}} \right) + \frac{1}{2\sqrt{(t-b)}} \operatorname{var}[\varphi; \langle a, b \rangle] < \infty$$

according to Lemma 0.4.

Let $t \in (a, b\rangle$, $x \in R^1$, $x \neq \varphi(t)$. Put $|x - \varphi(t)| = r$ (it holds $r > 0$). There is a $\delta > 0$ such that $|\varphi(\tau) - \varphi(t)| < \frac{1}{2}r$ for each $\tau \in (t - \delta, t)$ (since φ is a continuous function on $\langle a, b \rangle$).

Let us further put

$$t_n = t - \frac{\delta}{n^2}$$

for $n = 1, 2, \ldots$

Then (assuming $Q \leqq 1$)

$$V_K^Q(r; x, t) \leqq V_K(x, t) = \int_a^{t_1} \exp(-\alpha_{x,t}^2(\tau))\, \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) =$$

$$= \int_a^{t_1} \exp(-\alpha_{x,t}^2(\tau))\, \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) + \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \exp(-\alpha_{x,t}^2(\tau))\, \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) \leqq$$

$$\leqq \operatorname{var}[\alpha_{x,t}; \langle a, t_1 \rangle] + \sum_{n=1}^{\infty} \sup_{\tau \in \langle t_n, t_{n+1} \rangle} \exp(-\alpha_{x,t}^2(\tau)) \operatorname{var}[\alpha_{x,t}; \langle t_n, t_{n+1} \rangle]\,.$$

Obviously it holds var $[\alpha_{x,t}; \langle a, t_1\rangle] < \infty$. Denoting $r/4\sqrt{\delta} = R$ the following estimate is valid on the interval $\langle t_n, t_{n+1}\rangle$ $(n = 1, 2, \ldots)$:

$$\exp(-\alpha_{x,t}^2(\tau)) \leqq e^{-n^2R^2}$$

(since $|x - \varphi(\tau)| \geqq \frac{1}{2}r$ on the interval $(t - \delta, t)$
and

$$|\alpha_{x,t}(\tau)| \geqq \frac{r}{2}\frac{1}{2\sqrt{(t-\tau)}} \geqq \frac{r}{4}\frac{1}{\sqrt{(t-t_n)}} = nR$$

on the interval $\langle t_n, t_{n+1}\rangle$).

Furthermore,

$$\operatorname{var}[\alpha_{x,t}; \langle t_n, t_{n+1}\rangle] \leqq$$

$$\leqq \sup_{\tau\in\langle t_n,t_{n+1}\rangle} |x - \varphi(\tau)| \left(\frac{1}{2\sqrt{(t-t_{n+1})}} - \frac{1}{2\sqrt{(t-t_n)}}\right) +$$

$$+ \frac{1}{2\sqrt{(t-t_{n+1})}} \operatorname{var}[\varphi; \langle t_n, t_{n+1}\rangle] \leqq \frac{3r}{2}\frac{1}{2\sqrt{\delta}} + \frac{n+1}{2\sqrt{\delta}} \operatorname{var}[\varphi; \langle t_1, t\rangle].$$

Hence

$$V_K(x, t) \leqq \operatorname{var}[\alpha_{x,t}; \langle a, t_1\rangle] + \sum_{n=1}^{\infty} 3R\, e^{-n^2R^2} +$$

$$+ \frac{\operatorname{var}[\varphi; \langle t_1, t\rangle]}{2\sqrt{\delta}} \sum_{n=1}^{\infty} (n+1)\, e^{-n^2R^2} < \infty.$$

The assertion is proved.

The following theorem is the main result of this part. We shall use this theorem later.

Theorem 1.1. *Let $t_0 \in \langle a, b\rangle$. If there is $\delta > 0$ such that*

$$\sup\{V_K^Q(\varphi(t), t);\ t \in \langle a, b\rangle,\ |t - t_0| < \delta\} < \infty \tag{1.11}$$

then there is a neighbourhood U of $[\varphi(t_0), t_0]$ in R^2 for which

$$\sup_{[x,t]\in U} V_K^Q(x, t) < \infty. \tag{1.12}$$

If

$$\sup_{t\in\langle a,b\rangle} V_K^Q(\varphi(t), t) = c < \infty \tag{1.13}$$

then V_K^Q is bounded on the whole R^2.

Proof. First we prove the second part of the theorem. Let $\mathcal{D}$ be a fixed finite system of disjoint intervals $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ with $\langle a_i, b_i\rangle \subset \langle a, b\rangle$ $(i = 1, 2, \ldots, n)$. Put

$$c_i = \inf\{Q(\tau);\ \tau \in \langle a_i, b_i\rangle\}\,, \quad K_i = \{[\varphi(t), t];\ t \in \langle a_i, b_i\rangle\}\,.$$

Choose $s_i \in R^1$, $|s_i| \leqq c_i$ $(i = 1, 2, \ldots, n)$. Let us define a function h on the set $R^2 \setminus \bigcup_{i=1}^{n} K_i$ by

$$h(x, t) = \sum_j s_j \int_{a_j}^{\min\{t, b_j\}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau)\,. \tag{1.14}$$

We replace the j-th integral in the sum (1.14) by zero whenever $t \leqq a_j$ (in particular $h(x, t) = 0$ if $t \leqq a$).

Fix $[x, t] \in R^2$. It is seen from the well known properties of the Stieltjes integral that $V_K^Q(x, t)$ is equal to the least upper bound of all sums of the form (1.14) where $\mathcal{D}$ runs over all finite disjoint systems of intervals contained in $\langle a, b\rangle$ and s_j over all real numbers such that $|s_j| \leqq c_j$. On that account, the boundedness of the function V_K^Q on R^2 will be evident if we show that the function h is bounded on R^2 by a constant which is independent of the choice of the system $\mathcal{D}$ and the numbers s_j $(|s_j| \leqq c_j)$.

Let us define a function F_i on $R^2 \setminus K_i$ $(i = 1, 2, \ldots, n)$ by

$$F_i(x, t) = \begin{cases} 0 & t \leqq a_i \\ \displaystyle\int_{a_i}^{\min\{t, b_i\}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) & t > a_i \end{cases}.$$

We define a function G on $^*R^1$ in the following way:

$$G(t) = \begin{cases} 0 & t = -\infty \\ \displaystyle\int_{-\infty}^{t} \exp\left(-x^2\right) \mathrm{d}x & t > -\infty \end{cases} \tag{1.15}$$

(we shall use this notation also in the sequel). Let us note that

$$0 \leqq G(t) \leqq \sqrt{\pi}$$

for each $t \in {}^*R^1$ and that G is an increasing function on $^*R^1$.

Suppose that $t \in (a_i, b_i\rangle$, $x \in R^1$, $x \neq \varphi(t)$, let $t' \in (a_i, t)$. Then

$$\int_{a_i}^{t'} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) = \int_{\alpha_{x,t}(a_i)}^{\alpha_{x,t}(t')} \exp\left(-z^2\right) \mathrm{d}z = G(\alpha_{x,t}(t')) - G(\alpha_{x,t}(a_i))\,.$$

(This follows from Lemma 0.2).

If we let t' tend to t, we obtain (considering the cases $x > \varphi(t)$ and $x < \varphi(t)$ separately)

$$(1.16) \qquad F_i(x, t) = \begin{cases} \sqrt{\pi} - G\left(\dfrac{x - \varphi(a_i)}{2\sqrt{(t - a_i)}}\right) & x > \varphi(t) \\ - G\left(\dfrac{x - \varphi(a_i)}{2\sqrt{(t - a_i)}}\right) & x < \varphi(t) \end{cases}$$

for any $t \in (a_i, b\rangle$. Similarly one comes to the equality

$$(1.17) \qquad F_i(x, t) = G\left(\frac{x - \varphi(b_i)}{2\sqrt{(t - b_i)}}\right) - G\left(\frac{x - \varphi(a_i)}{2\sqrt{(t - a_i)}}\right)$$

which holds for $[x, t] \in R^2$, $t > b_i$. It is readily verified that the function F_i satisfies the heat equation on the set $R^2 \setminus K_i$. Hence the function h satisfies the heat equation on the set $R^2 \setminus \bigcup_{i=1}^{n} K_i$.

Assuming $[x_1, t_1] \in K$ we distinguish the following two cases.

a) $[x_1, t_1] \notin \bigcup_{i=1}^{n} K_i$. In this case the function h is continuous at the point $[x_1, t_1]$ and thus

$$(1.18) \qquad \lim_{[x,t]\to[x_1,t_1]} h(x, t) = h(x_1, t_1) = \sum_j s_j \int_{a_j}^{\min\{t_1,b_j\}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) \leqq$$
$$\leqq \sum_j c_j \int_{a_j}^{\min\{t_1,b_j\}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x_1,t_1}(\tau) \leqq$$
$$\leqq \sum_j \int_{a_j}^{\min\{t_1,b_j\}} Q(\tau) \exp\left(-\alpha_{x_1,t_1}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x_1,t_1}(\tau) \leqq$$
$$\leqq \int_a^{t_1} Q(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x_1,t_1}(\tau) = V_K^Q(x_1, t_1) \leqq c$$

(everywhere in those sums we consider only such members for which $a_j < t_1$).

b) $[x_1, t_1] \in \bigcup_{i=1}^{n} K_i$. We can assume, for instance, that $t_1 \in \langle a_n, b_n\rangle$. In the same way as in the case a) one may show that

$$(1.19) \qquad \lim_{[x,t]\to[x_1,t_1]} \sum_{j \neq n} s_j \int_{a_j}^{\min\{t,b_j\}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) \leqq c$$

(in the sum we again consider only such terms for which $a_j < t$).

We may suppose that $Q \leqq 1$ on $\langle a, b\rangle$. Then $c_j \leqq 1$ (and thus $|s_j| \leqq 1$). Consider a point $[x, t] \in R^2 \setminus K$ for which $t > a_n$. In the case $t \leqq b_n$ we put $\alpha_{x,t}(t) = +\infty$ if $x > \varphi(t)$; if $x < \varphi(t)$ we put $\alpha_{x,t}(t) = -\infty$ (so that

$$\alpha_{x,t}(t) = \lim_{\tau \to t-} \alpha_{x,t}(\tau)) .$$

Then

$$(1.20) \qquad s_n \int_{a_n}^{\min\{t,b_n\}} \exp(-\alpha_{x,t}^2(\tau)) \, d\alpha_{x,t}(\tau) =$$

$$= s_n \int_{\alpha_{x,t}(a_n)}^{\alpha_{x,t}(\min\{t,b_n\})} \exp(-z^2) \, dz = s_n(G(\alpha_{x,t}(\min\{t, b_n\})) - G(\alpha_{x,t}(a_n))) \leqq \sqrt{\pi} .$$

Since we replace the n-th member in the definition of the value $h(x, t)$ by zero in the case $t \leqq a_n$, it is seen from (1.19), (1.20) and (1.18) that

$$(1.21) \qquad \lim_{\substack{[x,t] \to [x_1,t_1] \\ [x,t] \notin K}} \sup h(x, t) \leqq c + \sqrt{\pi}$$

for any $[x_1, t_1] \in K$ (in the case $Q \leqq 1$ on $\langle a, b \rangle$).

Further, let us show that

$$(1.22) \qquad \lim_{|x| + |t| \to +\infty} \int_u^{\min\{t,v\}} \exp(-\alpha_{x,t}^2(\tau)) \, d\alpha_{x,t}(\tau) = 0$$

for each interval $\langle u, v \rangle \subset \langle a, b \rangle$.

First we show that for any $[x, t] \in R^2 \setminus K$ such that $t > 0$,

$$(1.23) \qquad \int_a^{\min\{t,v\}} \exp(-\alpha_{x,t}^2(\tau)) \, d\alpha_{x,t}(\tau) = \frac{1}{4} \int_u^{\min\{t,v\}} \frac{x - \varphi(\tau)}{(t - \tau)^{3/2}} \exp(-\alpha_{x,t}^2(\tau)) \, d\tau -$$

$$- \frac{1}{2} \int_u^{\min\{t,v\}} \frac{1}{\sqrt{(t - \tau)}} \exp(-\alpha_{x,t}^2(\tau)) \, d\varphi(\tau) .$$

According to Lemma 0.1 and Lemma 0.3 the equality (1.23) holds whenever at least two of those three integrals converge. However, the integral on the left hand side of (1.23) must converge since the parabolic variation V_K^Q is finite on $R^2 \setminus K$ (see Lemma 1.2).

If $t > v$ then the functions

$$\frac{x - \varphi(\tau)}{(t - \tau)^{3/2}}, \quad \frac{1}{\sqrt{(t - \tau)}}$$

are bounded on the interval $\langle u, v \rangle$ so that the integrals on the right hand side of

(1.23) converge. If $t \in (u, v\rangle$ then $x \neq \varphi(t)$ by assumption and it is seen from the definition of the function $\alpha_{x,t}$ that there are δ, $c_1 > 0$ such that

$$\exp\left(-\alpha_{x,t}^2(\tau)\right) \leqq \exp\left(-\frac{c_1}{t-\tau}\right)$$

for each $\tau \in (t - \delta, t)$. Hence it follows that the functions

$$\frac{x - \varphi(\tau)}{(t-\tau)^{3/2}} \exp\left(-\alpha_{x,t}^2(\tau)\right), \quad \frac{1}{\sqrt{(t-\tau)}} \exp\left(-\alpha_{x,t}^2(\tau)\right)$$

are bounded on the interval $\langle u, t)$ and so the integrals on the right hand side of (1.23) converge in this case, too. The equality (1.23) is proved.

From (1.23) it follows that to verify the equality (1.22), it is sufficient to show analogous equalities for the following integrals:

$$I(x, t) = \int_u^{\min\{t,v\}} \frac{x - \varphi(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right) \mathrm{d}\tau,$$

$$II(x, t) = \int_u^{\min\{t,v\}} \frac{1}{\sqrt{(t-\tau)}} \exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right) \mathrm{d}\varphi(\tau).$$

In each estimate we may assume $t > u$ (in the other case $I(x, t) = II(x, t) = 0$). It holds

$$|I(x, t)| \leqq \int_u^{\min\{t,v\}} \frac{|x - \varphi(\tau)|}{(t-\tau)^{3/2}} \exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right) \mathrm{d}\tau \leqq$$

$$\leqq (v-u) \sup\left\{\frac{|x - \varphi(\tau)|}{(t-\tau)^{3/2}} \exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right);\ \tau \in \langle u, \min\{t, v\})\right\}.$$

Let k be a given number such that $k > \max\{|a|, |b|, \sup\{|\varphi(\tau)|; \tau \in \langle a, b\rangle\}\}$ and suppose $|x| + |t| > 2k$. Then either

$$\text{a) } |t| > k \quad \text{or} \quad \text{b) } |x| > k.$$

It is sufficient to consider only $t > k$ in the case a) (otherwise $I(x, t) = II(x, t) = 0$).

Then it holds on the interval $\langle a, b\rangle$

$$\frac{|x - \varphi(\tau)|}{(t-\tau)^{3/2}} \exp\left(-\frac{(x-\varphi(\tau))^2}{4(t-\tau)}\right) = \frac{2}{t-\tau} |\alpha_{x,t}(\tau)| \exp\left(-\alpha_{x,t}^2(\tau)\right) \leqq$$

$$\leqq \frac{2}{k-b} \sup_{z \geqq 0}\{z \exp(-z^2)\} \to 0 \quad \text{for} \quad k \to \infty.$$

Putting $A = \sup_{\tau \in \langle a,b \rangle} |\varphi(\tau)|$ we have

$$\frac{|x - \varphi(\tau)|}{(t - \tau)^{3/2}} \exp\left(-\frac{(x - \varphi(\tau))^2}{4(t - \tau)}\right) \leqq \frac{k + A}{(t - \tau)^{3/2}} \exp\left(-\frac{(k - A)^2}{4(t - \tau)}\right) \leqq$$

$$\leqq \sup_{z>0} \frac{k + A}{z^{3/2}} \exp\left(-\frac{(k - A)^2}{4z}\right) = 6^{3/2} \frac{k + A}{(k - A)^3} e^{-3/2} \to 0 \quad \text{for} \quad k \to \infty$$

in the case b). Hence, in fact,

$$\lim_{|x|+|t|\to\infty} I(x, t) = 0 . \tag{1.24}$$

One may estimate the integral II similarly.

Obviously it holds

$$|II(x, t)| \leqq \text{var}\,[\varphi; \langle a, b \rangle] \sup \left\{\frac{1}{\sqrt{(t - \tau)}} \exp\left(-\alpha_{x,t}^2(\tau)\right);\ \tau \in \langle u, \min\{t, v\}\rangle\right\}$$

(if $t > u$). In the case $t > k$ it is

$$\frac{1}{\sqrt{(t - \tau)}} \exp\left(-\frac{(x - \varphi(\tau))^2}{4(t - \tau)}\right) \leqq \frac{1}{\sqrt{(k - b)}} \to 0 \quad \text{for} \quad k \to \infty$$

on the interval $\langle a, b \rangle$ and if $|x| > k$, then

$$\frac{1}{\sqrt{(t - \tau)}} \exp\left(-\frac{(x - \varphi(\tau))^2}{4(t - \tau)}\right) \leqq \frac{1}{\sqrt{(t - \tau)}} \exp\left(-\frac{(k - A)^2}{4(t - \tau)}\right) \leqq$$

$$\leqq \sup_{z>0} \frac{1}{\sqrt{z}} \exp\left(-\frac{(k - A)^2}{4z}\right) = \frac{\sqrt{2}}{k - A} e^{-1/2} \to 0 \quad \text{for} \quad k \to \infty$$

on the interval $\langle a, \min\{t, b\})$ (if $t > a$). Hence

$$\lim_{|x|+|t|\to\infty} II(x, t) = 0 . \tag{1.25}$$

The relations (1.24), (1.25) and (1.21) result in (1.22) and hence (see the definition of the function h)

$$\lim_{|x|+|t|\to\infty} h(x, t) = 0 .$$

Thus there is $B > \max\{|a|, |b|, \sup_{\tau \in \langle a,b \rangle} |\varphi(\tau)|\}$ such that $h(x, t) \leqq c + \sqrt{\pi}$ for each point $[x, t] \in R^2$ for which $|x| + |t| \geqq B$. Now it is seen (for (1.21)) that the following estimate holds on the boundary of the region $D = [(-B, B) \times (-B, B)] \setminus K$:

$$\limsup_{\substack{[x,t]\to[x_1,t_1] \\ [x,t]\in D}} h(x, t) \leqq c + \sqrt{\pi} \quad ([x_1, t_1] \in \partial D) .$$

As it was mentioned, the function h solves the heat equation on the set $R^2 \setminus K$ and thus for any point $[x, t] \in D$ it is

$$h(x, t) \leqq c + \sqrt{\pi}$$

in accordance with the maximum principle for caloric functions (see, for example, [15]). On that account $h \leqq c + \sqrt{\pi}$ on the whole $R^2 \setminus K$. Hence it follows

$$V_K^Q(x, t) \leqq c + \sqrt{\pi}$$

for any point $[x, t] \in R^2$ (if we suppose $Q \leqq 1$ on $\langle a, b\rangle$; but it follows from (1.9) that on R^2

$$V_K^{pQ}(x, t) = p\, V_K^Q(x, t)$$

for any $p \in R^1$, $p > 0$). The second part of Theorem 1.1 is proved.

Let us prove the first part of the theorem.

Suppose that there is $\delta > 0$ such that (1.11) is valid. Put

$$K_\delta = \{[\varphi(t), t];\ t \in \langle t_0 - \tfrac{1}{2}\delta,\ t_0 + \tfrac{1}{2}\delta\rangle\}\,.$$

One may define functions $V_{K_\delta}^Q$ and $V_{K-K_\delta}^Q$ in the same manner as the function V_K^Q was defined. Then it holds

$$V_K^Q(x, t) = V_{K_\delta}^Q(x, t) + V_{K-K_\delta}^Q(x, t) \quad ([x, t] \in R^2) \tag{1.26}$$

(one puts $V_{K_\delta}^Q(x, t) = 0$ if $t \leqq t_0 - \frac{1}{2}\delta$, otherwise

$$V_{K_\delta}^Q(x, t) = \int_{t_0-\delta/2}^{\min\{t, t_0+\delta/2\}} Q(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau)\,;$$

the function $V_{K-K_\delta}^Q$ may be defined by means of the equality (1.26)). By the same argument as in the first part of the proof applied to the function $V_{K_\delta}^Q$, we obtain that the function $V_{K_\delta}^Q$ is bounded on R^2, since it is bounded on K_δ.

Let us show the boundedness of $V_{K-K_\delta}^Q$ on a neighborhood of the point $[\varphi(t_0), t_0]$ in R^2. Put $x_0 = \varphi(t_0)$,

$$U = (x_0 - \tfrac{1}{4}\delta,\ x_0 + \tfrac{1}{4}\delta) \times (t_0 - \tfrac{1}{4}\delta,\ t_0 + \tfrac{1}{4}\delta)$$

and consider $[x, t] \in U$. Assuming $Q \leqq c_0$ on $\langle a, b\rangle$, we get

$$V_{K-K_\delta}^Q(x, t) = \int_a^{t_0-\delta/2} Q(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) \leqq$$

$$\leqq c_0 \operatorname{var}[\alpha_{x,t}; \langle a, t_0 - \tfrac{1}{2}\delta\rangle] \leqq \frac{c_0}{2\sqrt{\frac{1}{4}\delta}} \sup_{\tau \in \langle a, t_0-\delta/2\rangle} |x - \varphi(\tau)| +$$

$$+ \frac{c_0}{2\sqrt{\frac{1}{4}\delta}} \operatorname{var}[\varphi; \langle a, t_0 - \tfrac{1}{4}\delta\rangle] \leqq c_1$$

where c_1 is a (finite) constant which is independent of $[x, t] \in U$. Hence it is seen that the function $V^Q_{K-K_\delta}$ is bounded on U and so (taking into account (1.26)) the function V^Q_K is bounded, too.

The theorem is proved.

2. LIMITS OF THE OPERATOR T

Similarly as in the preceding part we suppose that $\langle a, b \rangle$ is a compact interval in R^1 $(a < b)$ and φ is a continuous function of bounded variation on $\langle a, b \rangle$.

$\mathscr{C}(\langle a, b \rangle)$ is defined to be the space of all continuous functions endowed with the supremum norm topology ($\|f\| = \|f\|_{\mathscr{C}} = \sup \{|f(t)|; t \in \langle a, b \rangle\}$). It is well known that the space $\mathscr{C}(\langle a, b \rangle)$ is a Banach space.

Let Q be a nonnegative lower-semicontinuous and bounded function and let $V^Q_K(r; x, t)$, $V^Q_K(x, t)$, ... mean the same as above.

Analogously to [9] we define the space $\mathscr{C}_Q(\langle a, b \rangle) = \mathscr{C}_Q$ as the space of all functions $f \in \mathscr{C}(\langle a, b \rangle)$ for which there is a real constant c (dependent on the function f) such that

$$|f| \leqq cQ \tag{2.1}$$

on $\langle a, b \rangle$ and with the property that

$$|f(t_0) - f(t)| = o(Q(t)) \quad \text{as} \quad t \to t_0\,, \quad t \in \langle a, b \rangle \tag{2.2}$$

for each point $t_0 \in \langle a, b \rangle$. Note that (2.2) holds for a function $f \in \mathscr{C}(\langle a, b \rangle)$ and a point $t_0 \in \langle a, b \rangle$ if and only if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(t_0) - f(t)| \leqq \varepsilon\, Q(t)$$

for any $t \in \langle a, b \rangle \cap (t_0 - \delta, t_0 + \delta)$. It is clear that if

$$\liminf_{\substack{t \to t_0 \\ t \in \langle a, b \rangle}} Q(t) > 0$$

then (2.2) is valid for each $f \in \mathscr{C}(\langle a, b \rangle)$.

We define a norm on the space $\mathscr{C}_Q(\langle a, b \rangle)$ by

$$\|f\|_Q = \inf \{c \in R^1;\ |f| \leqq cQ \text{ on } \langle a, b \rangle\}\,.$$

The space $\mathscr{C}_Q(\langle a, b \rangle)$ endowed with this norm is a Banach space. It is sufficient to show the completness of this space.

Let $\{f_n\}_{n=1}^\infty \subset \mathscr{C}_Q(\langle a, b \rangle)$ be a Cauchy sequence (in the norm $\|\ldots\|_Q$). Since Q is bounded by the assumption, the sequence $\{f_n\}_{n=1}^\infty$ is Cauchy in the norm $\|\ldots\|_{\mathscr{C}}$

too and thus there is a function $f \in \mathscr{C}(\langle a, b\rangle)$ such that $f_n \to f$ uniformly on $\langle a, b\rangle$. It is readily verified that

$$c = \sup \{\|f_n\|_Q ;\quad n = 1, 2, \ldots\} < \infty$$

and that $|f| \leqq cQ$ on $\langle a, b\rangle$. Given $t_0 \in \langle a, b\rangle$, let us prove that the condition (2.2) is satisfied. If

$$\liminf_{\substack{t\to t_0\\ t\in\langle a,b\rangle}} Q(t) > 0$$

then this is evident. In the other case $Q(t_0) = 0$ and thus also $f(t_0) = 0$. If we put $c_n = \sup\limits_{m>n} \|f_m - f_n\|$ we have

$$(2.3)\qquad |f(t) - f_n(t)| \leqq \sup_{m>n} |f_m(t) - f_n(t)| \leqq \\ \leqq Q(t) \sup_{m>n} \|f_m - f_n\|_Q = c_n\, Q(t)$$

for each point $t \in \langle a, b\rangle$ and any positive integer n. Since the sequence $\{f_n\}$ is Cauchy in $\mathscr{C}_Q$ it holds that $c_n \to 0$ as $n \to \infty$. Given $\varepsilon > 0$ there is a positive integer n_0 such that $c_{n_0} < \frac{1}{2}\varepsilon$. Since $f_{n_0} \in \mathscr{C}_Q$ there is $\delta > 0$ such that

$$|f_{n_0}(t)| \leqq \frac{\varepsilon}{2}\, Q(t)$$

for each $t \in \langle a, b\rangle \cap (t_0 - \delta, t_0 + \delta)$ (for $f_{n_0}(t_0) = 0$). But for these t we have

$$|f(t)| \leqq |f(t) - f_{n_0}(t)| + |f_{n_0}(t)| \leqq c_{n_0}\, Q(t) + \frac{\varepsilon}{2}\, Q(t) \leqq \varepsilon\, Q(t)$$

which shows that the condition (2.2) is satisfied and thus $f \in \mathscr{C}_Q(\langle a, b\rangle)$. Furthermore, it is seen from (2.3) that

$$\|f_n - f\|_Q \leqq c_n$$

which implies $f_n \to f$ in $\mathscr{C}_Q(\langle a, b\rangle)$.

Definition 2.1. Let f be a bounded Baire function on $\langle a, b\rangle$. The potential Tf is defined to be a function which is evaluated in the following way: given $[x, t] \in R^2$ we put $Tf(x, t) = 0$ if $t \leqq a$; if $t > a$ we put

$$(2.4)\qquad Tf(x, t) = \frac{2}{\sqrt{\pi}} \int_a^{\min\{t,b\}} f(\tau) \exp\left(-\frac{(x - \varphi(\tau))^2}{4(t - \tau)}\right) \mathrm{d}_\tau \frac{x - \varphi(\tau)}{2\sqrt{(t - \tau)}}$$

whenever the integral on the right hand side exists and is finite.

Remark 2.1. Let us have $[x, t] \in R^2$ and suppose that $V_K^Q(x, t) < \infty$. Let f be a bounded Baire function on $\langle a, b\rangle$ and let $|f| \leqq cQ$ on $\langle a, b\rangle$ for a suitable $c \in R^1$. Then the value $Tf(x, t)$ is defined (one may consider $t > a$ only, as in the other case $Tf(x, t)$ is defined naturally) since

$$\int_a^{\min\{t,b\}} f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) \leqq$$

$$\leqq c \int_a^{\min\{t,b\}} Q(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) = c\, V_K^Q(x, t) < \infty$$

so that the integral on the right hand side converges. In particular

$$|Tf(x, t)| \leqq \frac{2}{\sqrt{\pi}} V_K^Q(x, t) \|f\|_Q$$

whenever $V_K^Q(x, t) < \infty$ and $f \in \mathscr{C}_Q(\langle a, b\rangle)$.

If $[x, t] \in R^2 \setminus K$ then $V_K(x, t) < \infty$ in accordance with Lemma 1.2. Consequently, $Tf(x, t)$ is defined for such $[x, t]$ for any bounded Baire function on $\langle a, b\rangle$. In the same way as we proved (1.23) (see the proof of Theorem 1.1) one can show that

$$(2.5) \qquad Tf(x, t) = \frac{1}{2\sqrt{\pi}} \int_a^{\min\{t,b\}} f(\tau) \frac{x - \varphi(\tau)}{(t - \tau)^{3/2}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\tau -$$

$$- \frac{1}{\sqrt{\pi}} \int_a^{\min\{t,b\}} f(\tau) \frac{1}{\sqrt{(t - \tau)}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\varphi(\tau)$$

for each $[x, t] \in R^2 \setminus K$ (in the case $t > a$, of course). This equality also holds in the case $[x, t] \in K$, $Tf(x, t)$ is defined and at least one of the integrals on the right hand side is convergent.

Let us note that the first term on the right hand side of (2.5) is a double-layer heat potential while the other is a single-layer heat potential (cf. (0.3)). Hence, for a fix bounded Baire function f, Tf is a solution of the heat equation when considered as a function on the set $R^2 \setminus K$.

The following example points out that it may happen that $V_K(x, t) < \infty$ while the integrals on the right hand side in (2.5) either do not exist or are divergent. Set

$$\varphi(t) = \sqrt{-t} \quad t \in \langle -1, 0\rangle$$

and consider the point $[x, t] = [0, 0]$. The function φ is a continuous function of bounded variation on the interval $\langle -1, 0\rangle$ and $V_K(0, 0) < \infty$ since the function

$$\alpha_{x,t}(\tau) = \frac{-\sqrt{-\tau}}{\sqrt{-\tau}} = -1$$

is constant. Nevertheless, for instance,

$$\int_{-1}^{0} \frac{|x - \varphi(\tau)|}{(t - \tau)^{3/2}} \exp(-\alpha_{x,t}^2(\tau))\, d\tau = \int_{-1}^{0} \frac{1}{|\tau|} e^{-1/4}\, d\tau = \infty$$

and

$$\int_{-1}^{0} \frac{1}{\sqrt{(t - \tau)}} \exp(-\alpha_{x,t}^2(\tau))\, d\varphi(\tau) = -\frac{1}{2}\int_{-1}^{0} \frac{1}{|\tau|} e^{-1/4}\, d\tau = -\infty .$$

Remark 2.2. In the first part we defined for $[x, t] \in R^2$ with $t > a$ a function $\alpha_{x,t}$ on the interval $\langle a, \min\{t, b\})$ by

$$\alpha_{x,t}(\tau) = \frac{x - \varphi(\tau)}{2\sqrt{(t - \tau)}} .$$

If $t > b$ we can define the function $\alpha_{x,t}$ by this equality on the whole interval $\langle a, b\rangle$ and the function so defined is a continuous function of bounded variation on $\langle a, b\rangle$.

If $t \in (a, b\rangle$ and $x \neq \varphi(t)$ we put $\alpha_{x,t}(t) = +\infty$ if $x > \varphi(t)$ and $\alpha_{x,t}(t) = -\infty$ if $x < \varphi(t)$. The function $\alpha_{x,t}$ is continuous at the point t in the sense

$$\lim_{\tau \to t-} \alpha_{x,t}(\tau) = \alpha_{x,t}(t) .$$

Consider now $[x, t] \in K$ with $t > a$. We wish to find a condition under which there exists a limit (finite or infinite)

$$\lim_{\tau \to t-} \frac{x - \varphi(\tau)}{2\sqrt{(t - \tau)}} . \tag{2.6}$$

If this limit exists we shall denote its value by $\alpha_{x,t}(t)$ in the sequel.

Let us mention that the condition $V_K(x, t) < \infty$ is sufficient (but not necessary) for the existence of the limit (2.6). Indeed, if $V_K(x, t) < \infty$ then the integral

$$\int_a^t \exp(-\alpha_{x,t}^2(\tau))\, d\alpha_{x,t}(\tau)$$

converges. Let G be the same function as defined above, i.e.,

$$G(u) = \int_{-\infty}^{u} \exp(-z^2)\, dz .$$

Then we have (see Lemma 0.2)

$$\int_a^t \exp(-\alpha_{x,t}^2(\tau))\, d\alpha_{x,t}(\tau) = \lim_{t' \to t-} \int_a^{t'} \exp(-\alpha_{x,t}^2(\tau))\, d\alpha_{x,t}(\tau) =$$
$$= \lim_{t \to t-} (G(\alpha_{x,t}(t')) - G(\alpha_{x,t}(a)))$$

so that the limit

$$\lim_{\tau \to t-} G\left(\frac{x - \varphi(\tau)}{2\sqrt{(t - \tau)}}\right)$$

exists and since G is an increasing function the limit (2.6) exists, too.

We defined the value $Tf(x, t)$ by (2.4) if the integral on the right hand side of (2.4) converged (we consider $t > a$, of course).

Let us have a given point $[x, t] \in K$ such that $V_K(x, t) = +\infty$ (then certainly $t > a$) and suppose that the limit (2.6) exists at this point. In general we cannot define $Tf(x, t)$ for every $f \in \mathscr{C}(\langle a, b\rangle)$. Suppose that Q is a bounded lower-semicontinuous nonnegative function on $\langle a, b\rangle$ such that $V_K^Q(x, t) < \infty$. Then the integral on the right hand side in (2.4) converges whenever $f \in \mathscr{C}_Q(\langle a, b\rangle)$. Let us now consider a function $f \in \mathscr{C}(\langle a, b\rangle)$ of a form $f = k + f_1$ where k is a function which assumes a constant value on $\langle a, b\rangle$ (i.e. $k \in R^1$) and $f_1 \in \mathscr{C}_Q$. Then we can and shall define $Tf(x, t)$ by

$$Tf(x, t) = \frac{2}{\sqrt{\pi}} k(G(\alpha_{x,t}(t)) - G(\alpha_{x,t}(a))) + Tf_1(x, t) . \tag{2.7}$$

Remark 2.3. Let $[x, t] \in R^2$ and suppose $V_K^Q(x, t) < \infty$. As we have noted, every function $f \in \mathscr{C}_Q(\langle a, b\rangle)$ fulfils

$$|Tf(x, t)| \leqq \|f\|_Q \frac{2}{\sqrt{\pi}} V_K^Q(x, t) . \tag{2.8}$$

If the point $[x, t]$ is fixed we may consider $Tf(x, t)$ to be a value of a continuous linear functional. This functional we denote by $T_{x,t}$, that is, we write

$$T_{x,t}(f) = T_{x,t}f = Tf(x, t)$$

for $f \in \mathscr{C}_Q$. The norm of the functional $T_{x,t}$ we denote by $\|T_{x,t}\|_Q$, i.e., we put

$$\|T_{x,t}\|_Q = \sup \{T_{x,t}(f);\ f \in \mathscr{C}_Q,\ \|f\|_Q \leqq 1\} .$$

Hence the symbol $\|\ldots\|_Q$ will be used for a norm on the space $\mathscr{C}_Q(\langle a, b\rangle)$ as well as for a norm of functionals on the space $\mathscr{C}_Q(\langle a, b\rangle)$; no misunderstanding can occur.

It follows immediately from (2.8) and the definition of the norm of a functional that

$$\|T_{x,t}\|_Q \leqq \frac{2}{\sqrt{\pi}} V_K^Q(x, t) .$$

Since Q is lower-semicontinuous we have (if $t > a$)

$$\frac{2}{\sqrt{\pi}} V_K^Q(x, t) = \frac{2}{\sqrt{\pi}} \int_a^{\min\{t,b\}} Q(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) =$$

$$= \sup\left\{\frac{2}{\sqrt{\pi}} \int_a^{\min\{t,b\}} f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau);\ f \in \mathscr{C}(\langle a, b\rangle),\ |f| \leqq Q\right\} =$$

$$= \sup\left\{Tf(x, t);\ f \in \mathscr{C}(\langle a, b\rangle),\ |f| \leqq Q\right\}.$$

Let $f \in \mathscr{C}(\langle a, b\rangle)$ and suppose that $|f| \leqq Q$ on $\langle a,b\rangle$. It is easily verified that there are functions $f_n \in \mathscr{C}_Q(\langle a, b\rangle)$ such that $f_n \to f$ uniformly on $\langle a, b\rangle$ and $\|f_n\|_Q \leqq 1$. But from this it already follows that

$$\|T_{x,t}\|_Q = \frac{2}{\sqrt{\pi}} V_K^Q(x, t)\,. \tag{2.9}$$

Lemma 2.1. *Let $Q \geqq 0$ be a bounded lower-semicontinuous function on $\langle a, b\rangle$ and suppose that $Q(a) = 0$. Let M be a set in R^2 such that either $M \subset \{[x, t];\ t \in \langle a, b\rangle,\ x > \varphi(t)\}$ or $M \subset \{[x, t];\ t \in \langle a, b\rangle,\ x < \varphi(t)\}$ and suppose that $[x_0, t_0] \in K \cap \overline{M}$. Then a finite limit*

$$\lim_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} Tf(x, t) \tag{2.10}$$

exists for each $f \in \mathscr{C}_Q(\langle a, b\rangle)$ if and only if

$$\limsup_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} V_K^Q(x, t) < \infty\,. \tag{2.11}$$

If the condition is fulfilled then the limit (2.10) exists (and is finite) even for any bounded Baire function f on $\langle a, b\rangle$ for which

$$|f(t_0) - f(t)| = o(Q(t)) \quad \text{as} \quad t \to t_0\,, \quad t \in \langle a, b\rangle \tag{2.12}$$

and $f(t_0) = 0$ if $Q(t_0) = 0$.

Proof. a) Suppose that the limit (2.10) exists and is finite for each $f \in \mathscr{C}_Q(\langle a, b\rangle)$. Then (for the space $\mathscr{C}_Q(\langle a, b\rangle)$ is Banach) it follows by the Banach-Steinhaus theorem that

$$\lim_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} \|T_{x,t}\|_Q < \infty$$

and so, according to (2.9), the condition (2.11) is fulfilled.

b) Let the condition (2.11) be fulfilled. Let f be a bounded Baire function on $\langle a, b\rangle$

for which (2.12) holds and suppose that $f(t_0) = 0$. There are $\delta > 0$ and $c < \infty$ such that

$$V_K^Q(x, t) \leqq c$$

for each $[x, t] \in M \cap \{[x, t]; |x - x_0| < \delta, |t - t_0| < \delta\}$. Given $\varepsilon > 0$, one deduces from (2.12) that there is $\delta_1 > 0$ with $\delta_1 < \delta$ such that

$$|f(\tau)| \leqq Q(\tau) \frac{\varepsilon \sqrt{\pi}}{4c}$$

for each $\tau \in (t_0 - \delta_1, t_0 + \delta_1)$. Putting $t_1 = t_0 - \delta_1$ we have

$$\left| \frac{2}{\sqrt{\pi}} \int_{t_1}^{t} f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) \right| \leqq \tag{2.13}$$

$$\leqq \frac{\varepsilon}{2c} \int_{t_1}^{t} Q(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) \leqq \frac{\varepsilon}{2c} V_K^Q(x, t) \leqq \frac{\varepsilon}{2}$$

for each point $[x, t] \in M$ for which $|x - x_0| < \delta_1$ and $|t - t_0| < \delta_1$.

Let us consider a sequence $\{[x_n, t_n]\}_{n=1}^{\infty} \subset U = (x_0 - \frac{1}{2}\delta_1,\ x_0 + \frac{1}{2}\delta_1) \times (t_0 - \frac{1}{2}\delta_1, t_0 + \frac{1}{2}\delta_1)$ such that $[x_n, t_n] \to [x_0, t_0]$ as $n \to \infty$. Then, of course,

$$\sup\{\operatorname{var}[\alpha_{x_n,t_n}; \langle a, t_1\rangle];\quad n = 1, 2, \ldots\} < \infty$$

and

$$|f(\tau) \exp\left(-\alpha_{x,t_n}^2(\tau)\right)| \leqq \sup_{\tau \in \langle a, t_1\rangle} |f(\tau)| < \infty$$

for each $\tau \in \langle a, t_1\rangle$, $n = 1, 2, \ldots$ This implies

$$\lim_{n \to \infty} \frac{2}{\sqrt{\pi}} \int_a^{t_1} f(\tau) \exp\left(-\alpha_{x_n,t_n}^2(\tau)\right) \mathrm{d}\alpha_{x_n,t_n}(\tau) = \tag{2.14}$$

$$= \frac{2}{\sqrt{\pi}} \int_a^{t_1} f(\tau) \exp\left(-\alpha_{x_0,t_0}^2(\tau)\right) \mathrm{d}\alpha_{x_0,t_0}(\tau)\,.$$

In other words, the integral

$$\frac{2}{\sqrt{\pi}} \int_a^{t_1} f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) \tag{2.15}$$

considered as a function of the variable $[x, t]$ is continuous at the point $[x_0, t_0]$.

Let us suppose that, in addition, we have chosen δ_1 such that

$$\left| \frac{2}{\sqrt{\pi}} \int_a^{t_1} f(\tau) \exp\left(-\alpha_{x_0,t_0}^2(\tau)\right) \mathrm{d}\alpha_{x_0,t_0}(\tau) - Tf(x_0, t_0) \right| < \frac{\varepsilon}{4}$$

(this is possible for $V_K^Q(x_0, t_0) < \infty$ and $f \in \mathscr{C}_Q(\langle a, b\rangle)$).

Hence, the continuity of the integral (2.15) at the point $[x_0, t_0]$ implies that there is a neighborhood $U_1 \subset U$ such that

(2.16) $$\left|\frac{2}{\sqrt{\pi}} \int_a^{t_1} f(\tau) \exp\left(-\alpha^2_{x,t}(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) - Tf(x_0, t_0)\right| < \frac{\varepsilon}{2}$$

for each point $[x, t] \in U_1$. It is seen from (2.16) that

$$|Tf(x, t) - Tf(x_0, t_0)| < \varepsilon\,.$$

So we have proved that if $f(t_0) = 0$ (and othe other conditions are fulfilled) the limit

(2.17) $$\lim_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} Tf(x, t) = Tf(x_0, t_0)$$

exists. If $Q(t_0) = 0$ the proof is complete since by the assumption we consider in this case only functions f for which $f(t_0) = 0$.

In the case $Q(t_0) > 0$ it suffices to prove that the limit exists for the function f which assumes the constant value 1 (for $T_{x,t}$ is linear). Since V_K^Q is lower-semicontinuous (see Lemma 1.2) it follows from (2.11) that $V_K^Q(x_0, t_0) < \infty$. It is easily verified that if $Q(t_0) > 0$ and $V_K^Q(x_0, t_0) < \infty$ then $V_K(x_0, t_0) < \infty$, too. If $Q(t_0) > 0$ then, of course, $t_0 > a$ by the assumption. According to Remark 2.2 it follows from $V_K(x_0, t_0) < \infty$ that the limit

(2.18) $$\lim_{\tau\to t_0-} \frac{x_0 - \varphi(\tau)}{2\sqrt{(t - \tau)}}$$

exists.

Let us now show that if the limit (2.18) exists then even the limits

(2.19) $$\lim_{\substack{[x,t]\to[x_0,t_0]\\ x>\varphi(t)}} T1(x, t)\,,$$

(2.20) $$\lim_{\substack{[x,t]\to[x_0,t_0]\\ x<\varphi(t)}} T1(x, t)$$

(1 stands for the constant function equal to 1 on $\langle a, b\rangle$) exist without any assumption on the parabolic variation. Let $[x, t] \in R^2$, $t \in (a, b\rangle$, $x > \varphi(t)$. Then we have

$$T1(x, t) = \frac{2}{\sqrt{\pi}} \int_a^t \exp\left(-\alpha^2_{x,t}(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) =$$

$$= \lim_{t'\to t-} \frac{2}{\sqrt{\pi}} \int_a^{t'} \exp\left(-\alpha^2_{x,t}(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) = \lim_{t'\to t-} \frac{2}{\sqrt{\pi}} \left(G(\alpha_{x,t}(t')) -\right.$$

$$\left.- G(\alpha_{x,t}(a))\right) = \frac{2}{\sqrt{\pi}} \left(G(\alpha_{x,t}(t)) - G(\alpha_{x,t}(a))\right) = 2 - \frac{2}{\sqrt{\pi}}\, G\left(\frac{x - \varphi(a)}{2\sqrt{(t - a)}}\right).$$

The function $(x - \varphi(a))/2\sqrt{(t - a)}$ considered as a function of the variable $[x, t]$ is continuous on the set $\{[x, t] \in R^2; x > a\}$. As the function G is continuous, we get

$$(2.21) \qquad \lim_{\substack{[x,t]\to[x_0,t_0] \\ x>\varphi(t)}} T1(x, t) = 2 - \frac{2}{\sqrt{\pi}} G\left(\frac{x_0 - \varphi(a)}{2\sqrt{(t_0 - a)}}\right).$$

Similarly we find

$$(2.22) \qquad \lim_{\substack{[x,t]\to[x_0,t_0] \\ x<\varphi(t)}} T1(x, t) = -\frac{2}{\sqrt{\pi}} G\left(\frac{x_0 - \varphi(a)}{2\sqrt{(t_0 - a)}}\right).$$

This completes the proof.

Remark 2.4. Let $t_0 \in (a, b\rangle$, $x_0 = \varphi(t_0)$ and let either $M \subset D^+ = \{[x, t];\ t \in (a, b),\ x > \varphi(t)\}$ or $M \subset D^- = \{[x, t];\ t \in (a, b),\ x < \varphi(t)\}$. Suppose that $[x_0, t_0] \in \overline{M}$ and the condition (2.11) is fulfilled. Further suppose that the limit

$$\alpha_{x_0,t_0}(t_0) = \lim_{t\to t_0-} \frac{x_0 - \varphi(t)}{2\sqrt{(t_0 - t)}}$$

exists (finite or infinite). This is, of course, fulfilled for instance whenever $Q(t_0) > 0$, because then $V_K(x_0, t_0) < \infty$ follows from (2.11). Let us consider a function f of the form $f = k + f_1$, where $k \in R^1$ is a constant and f_1 is a bounded Baire function for which the condition (2.12) is fulfilled (and for which $f_1(t_0) = 0$ if $Q(t_0) = 0$). We can define the value $Tf(x_0, t_0)$ by (2.7). According to that equality, the linearity of $T_{x,t}$ and to the equalities (2.17), (2.21), (2.22) we now have

$$(2.23) \qquad \lim_{\substack{[x,t]\to[x_0,t_0] \\ [x,t]\in M}} Tf(x, t) = Tf(x_0, t_0) + f(t_0)\left(2 - \frac{2}{\sqrt{\pi}} G(\alpha_{x_0,t_0}(t_0))\right)$$

if $M \subset D^+$; if $M \subset D^-$ then

$$(2.24) \qquad \lim_{\substack{[x,t]\to[x_0,t_0] \\ [x,t]\in M}} Tf(x, t) = Tf(x_0, t_0) - f(t_0)\frac{2}{\sqrt{\pi}} G(\alpha_{x_0,t_0}(t_0)).$$

If $\alpha_{x_0,t_0}(t_0) = 0$ then we get equalities analogous to classical equalities for the limit of a double-layer heat potential:

$$\lim_{\substack{[x,t]\to[x_0,t_0] \\ [x,t]\in M}} Tf(x, t) = Tf(x_0 t_0) \pm f(t_0),$$

where we consider + or − when $M \subset D^+$ or $M \subset D^-$, respectively.

The following assertion is a corollary of Lemma 2.1 and Theorem 1.1 (by virtue of the lower-semicontinuity of V_K^Q).

Theorem 2.1. *Let $Q \geqq 0$ be a bounded lower-semicontinuous function on $\langle a, b\rangle$ such that $Q(a) = 0$. Let $t_0 \in \langle a, b\rangle$, $x_0 = \varphi(t_0)$. Then there exist finite limits*

$$(2.25) \qquad \lim_{\substack{[x,t]\to[x_0,t_0] \\ t\in\langle a,b\rangle, x>\varphi(t)}} Tf(x, t),$$

$$(2.26) \qquad \lim_{\substack{[x,t]\to[x_0,t_0] \\ t\in\langle a,b\rangle, x<\varphi(t)}} Tf(x, t)$$

for each $f \in \mathscr{C}_Q(\langle a, b\rangle)$ if and only if there is $\delta > 0$ such that

$$(2.27) \qquad \sup \{V_K^Q(\varphi(t), t);\ t \in (t_0 - \delta, t_0 + \delta) \cap \langle a, b\rangle\} < \infty .$$

If the condition (2.27) is fulfilled for some $\delta > 0$ then the limits (2.25), (2.26) exist and are finite for any bounded Baire function f on $\langle a, b\rangle$ such that

$$|f(t) - f(t_0)| = o(Q(t)) \quad \text{as} \quad t \to t_0\,, \quad t \in \langle a, b\rangle$$

and $f(t_0) = 0$ if $Q(t_0) = 0$.

The last assertion enable us to define operators $\tilde{T}_+$ and $\tilde{T}_-$ on $\mathscr{C}_0(\langle a, b\rangle)$ by the following equalities:

$$(2.27) \qquad \tilde{T}_+ f(t) = \lim_{\substack{[x',t']\to[\varphi(t),t] \\ t'\in\langle a,b\rangle, x'>\varphi(t)}} Tf(x', t'),$$

$$(2.28) \qquad \tilde{T}_- f(t) = \lim_{\substack{[x',t']\to[\varphi(t),t] \\ t'\in\langle a,b\rangle, x'<\varphi(t)}} Tf(x', t')$$

for $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$.

These operators are important in connection with the Fourier problem of the heat equation. The reader is referred to the article "On a boundary value problem for the heat equation" which will be published in this journal.

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Author's address: 166 27 Praha 6, Suchbátarova 2, ČSSR (FEL ČVUT).