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THE EXISTENCE OF 2-FACTORS IN SQUARES OF GRAPHS

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The *square* G^2 of a connected graph G is that graph having the same vertex set as G and such that two vertices of G^2 are adjacent if and only if the distance between these vertices in G is at most two. Figure 1 shows two graphs Y and Z and their squares.

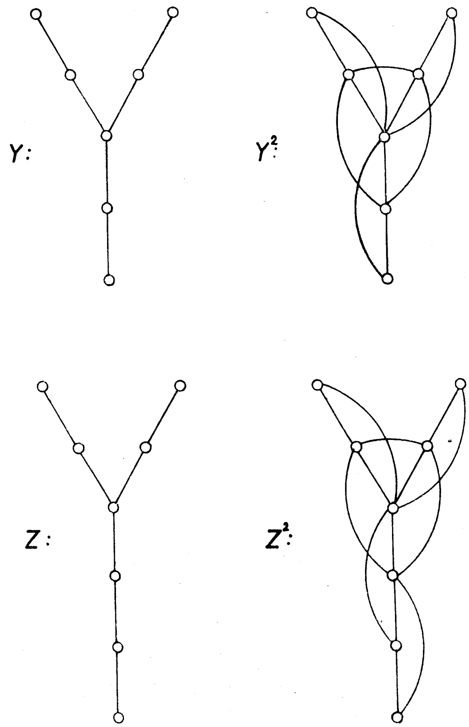

Fig. 1.

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An *n-factor* of a graph G is a spanning subgraph of G which is regular of degree n. A 2-factor of G, then, is a collection of disjoint cycles which spans G. FLEISCHNER [3] proved that the square of every cyclic block is hamiltonian and hence contains a 2-factor. In [5] NEUMAN proved that the square T^2 of a tree T with at least three vertices is hamiltonian if and only if T does not contain the graph Y (of Fig. 1) as a subgraph. HOBBS [4] proved that if every vertex of a graph G has degree at least two, then G^2 has a 2-factor. By Neuman's result, neither Y^2 nor Z^2 is hamiltonian; however, it is not difficult to show that Z^2 contains a 2-factor while Y^2 does not.

It is the object of this paper to present a necessary and sufficient condition for the square G^2 of a graph G to possess a 2-factor. Before stating this result, we give one additional definition; all other definitions not given here may be found in [1]. An *end-path* is a path in which at least one end-vertex of the path has degree one and all vertices which are not end-vertices have degree two.

The following lemma will prove convenient.

Lemma. *Let G be any cyclic block, and let v be any vertex of G. Then there exists a vertex u in G adjacent with v such that $G - v - u$ is connected.*

Proof. Since blocks contain no cut-vertices, the graph $G - v$ is connected. Suppose for every vertex u of G adjacent with v that $G - v - u$ is disconnected. This implies that every vertex adjacent with v is a cut-vertex of $G - v$. Let B be an end-block of $G - v$ (a block of $G - v$ containing exactly one cut-vertex of G), and let w be the cut-vertex of $G - v$ belonging to B. No vertex of B, except possibly w, is adjacent to v in G. Hence, w is a cut-vertex of G; this contradicts the fact that G is a block and establishes the lemma.

We now present our main result.

Theorem. *Let G be a connected graph having at least three vertices. A necessary and sufficient condition for the square G^2 of G to contain a 2-factor is that there exists in G no vertex which is the end-vertex of three end-paths of length two.*

Proof. Suppose G is a connected graph containing a vertex v which is the end-vertex of three end-paths of length two. Let the three vertices of degree one in these three end-paths be denoted by v_1, v_2, and v_3. Assume G^2 contains a 2-factor F. For each $i = 1, 2, 3$, the vertex v_i is incident with two edges in G^2, one of which is the edge v_iv. Now each vertex v_i and thus each edge v_iv belongs to F; however, this implies that v is incident with three edges in F. This is impossible since every vertex in F has degree two. Therefore, our assumption is incorrect, and G^2 does not contain a 2-factor.

For the converse, we proceed by induction on the number p of vertices of G. The result follows immediately for $p = 3$, 4, and 5. Assume that if H is a connected graph of order at least three but less than $p(\geqq 6)$ such that H contains no vertex which is the end-vertex of three end-paths of length two, then H^2 has a 2-factor. Let G be a connected graph of order p such that G contains no vertex which is the end-vertex of three end-paths of length two.

If G is a block, then by Fleischner's theorem, G^2 is hamiltonian so that G^2 has a 2-factor. Hence, we may assume G to have cut-vertices and two or more blocks. An *end-block* of G is a block of G containing exactly one cut-vertex of G. Among all end-blocks of G, we consider those end-blocks B with the property that, with at most one exception, every block containing the cut-vertex in B is an end-block. We refer to such end-blocks as *terminal end-blocks*.

Three cases are now considered, depending on the number of vertices in terminal end-blocks.

Case 1. *Suppose G contains a terminal end-block B having four or more vertices.* Let v be the cut-vertex of G belonging to B. Denote by G_1 the connected graph obtained by deleting from G all vertices of B different from v.

If G_1 contains a vertex which is the end-vertex of three end-paths of length two, then, necessarily, v is a vertex of degree one on one of these three end-paths. By Fleischner's Theorem, B^2 contains a hamiltonian cycle F_1, and by the induction hypothesis, $(G_1 - v)^2$ contains a 2-factor F_2. Thus, $F_1 \cup F_2$ is a 2-factor of G^2.

We henceforth assume that G_1 contains no vertex which is the end-vertex of three end-paths of length two. Suppose, first, that G_1 has at least three vertices. Then, by the induction hypothesis, G_1^2 contains a 2-factor F_1. In [2] it was shown that if H is a cyclic block with at least four vertices, then $H^2 - x$ is hamiltonian for every vertex x of H. By applying this result, we arrive at a hamiltonian cycle F_2 in the graph $B^2 - v$. Hence $F_1 \cup F_2$ is a 2-factor of G^2.

Next assume that G_1 has two vertices. Let u be the vertex of G_1 different from v. We investigate two subcases.

Sub-case A. *Assume $B - v$ contains a vertex which is the end-vertex of three or more end-paths of length two.* Let $v_1, v_2, \ldots, v_k$, $k \geqq 3$, be all vertices of degree one on all end-paths of length two in $B - v$. Since B has no vertices of degree one, v is adjacent to each of the vertices $v_1, v_2, \ldots, v_k$ in B. Hence $B - \{v, v_1, v_2, \ldots, v_k\}$ is connected, contains more than three vertices, and has no vertex which is the end-vertex of three end-paths of length two; thus, by the induction hypothesis, the square of $B - \{v, v_1, v_2, \ldots, v_k\}$ has a 2-factor F_1. The subgraph of G^2 induced by the vertices in the set $\{u, v, v_1, v_2, \ldots, v_k\}$ contains a hamiltonian cycle F_2. Then $F_1 \cup F_2$ is a 2-factor of G^2.

Sub-case B. *Assume $B - v$ contains no vertex which is the end-vertex of three or more end-paths of length two.* By the lemma, there exists a vertex w in B which is adjacent with v such that $B - v - w$ is connected. Suppose there exists no vertex in $B - v - w$ which is the end-vertex of three or more end-paths of length two. Since $p \geqq 6$, $B - v - w$ contains at least three vertices. Therefore, by the induction hypothesis, $(B - v - w)^2$ contains a 2-factor F_1. Furthermore, the subgraph induced by the vertices u, v, and w in G^2 is a triangle F_2, and $F_1 \cup F_2$ is a 2-factor in G^2.

Now suppose that there exists in $B - v - w$ a vertex which is the end-vertex of three or more end-paths of length two, and let $w_1, w_2, \ldots, w_k$, $k \geqq 3$, be all vertices of degree one on all end-paths of length two in $B - v - w$. Then $B - \{v, w, w_1, w_2, \ldots, w_k\}$ is a connected graph with at least three vertices which does not contain a vertex which is the end-vertex of three end-paths of length two. Hence the square of $B - \{v, w, w_1, w_2, \ldots, w_k\}$ contains a 2-factor F_1. Moreover, in G^2 the subgraph induced by $\{u, v, w, w_1, w_2, \ldots, w_k\}$ contains a 2-factor F_2 so that $F_1 \cup F_2$ is a 2-factor of G^2.

Case 2. *Suppose G contains to terminal end-block having four or more vertices but does contain terminal end-blocks with exactly three vertices.* Let B be a triangle which is a terminal end-block of G, and let v be the cut-vertex of G in B. If all blocks containing v are end-blocks, then it follows immediately that G^2 is hamiltonian and hence has a 2-factor. We therefore assume that not all blocks containing v are end-blocks.

If G has other end-blocks containing v, then define H_0 to be the graph obtained from G by deleting those vertices different from v in the end-blocks containing v. Also, define $H_1 = H_0 - v$. Necessarily, each of H_0 and H_1 is connected, and at least one of H_0 and H_1 has order at least three and contains no vertex which is the end-vertex of three end-paths of length two. Let H denote whichever of H_0 and H_1 has the above property. Then H^2 contains a 2-factor F_1, and the remaining vertices of G induce in G^2 a hamiltonian cycle F_2. Thus, $F_1 \cup F_2$ is a 2-factor of G^2.

Assume now that B is the only end-block of G containing v, and define G_1 to be the graph obtained by deleting the vertices of B from G. If G_1 contains no vertex which is the end-vertex of three end-paths of length two, then G_1^2 has a 2-factor F_1 and $F_1 \cup B$ is a 2-factor of G^2. Otherwise, let $v_1, v_2, \ldots, v_k$, $k \geqq 3$, be the vertices of degree one in all end-paths of length two in G_1. If G_0 is the graph obtained by removing the vertices of B and the vertices $v_1, v_2, \ldots, v_k$ from G, it follows, by the induction hypothesis, that G_0^2 has a 2-factor F'. In G^2 the subgraph induced by the vertices of B and $\{v_1, v_2, \ldots, v_k\}$ contains a hamiltonian cycle F''. Therefore, $F' \cup F''$ is a 2-factor in G^2.

Case 3. *Suppose that the only terminal end-blocks in G are acyclic.* Let B_1 be a terminal end-block containing the vertices v and v_1, where v is the cut-vertex. If all blocks containing v are end-blocks, then G is a star graph and G^2 is hamiltonian and thus contains a 2-factor. Hence, we assume not all blocks containing v are end-blocks.

If G contains at least three vertices of degree one adjacent with v, say $v_1, v_2, \ldots, v_k$ ($k \geqq 3$), then at least one of $H_0 = G - \{v_1, v_2, \ldots, v_k\}$ and $H_1 = H_0 - v$ is a connected graph of order at least three containing no vertex which is the end-vertex of three end-paths of length two. Such a graph H has the property that H^2 contains a 2-factor F_1 while the remaining vertices of G induce in G^2 a hamiltonian cycle F_2. Thus, $F_1 \cup F_2$ is a 2-factor of G^2.

Suppose next that the only vertices of degree one adjacent with v are v_1 and v_2. Define $G_1 = G - \{v, v_1, v_2\}$. If G_1 has no vertex which is the end-vertex of three end-paths of length two, then G_1^2 has a 2-factor F_1. The subgraph of G^2 induced by $\{v, v_1, v_2\}$ is a 2-factor F_2, and $F_1 \cup F_2$ is a 2-factor of G^2. If, on the other hand, G_1 contains a vertex which is the end-vertex of three or more end-paths of length two, we let $u_1, u_2, \ldots, u_k$, $k \geqq 3$, be all vertices of degree one on all end-paths of length two in G_1. Here the subgraph of G^2 induced by $\{v, v_1, v_2, u_1, u_2, \ldots, u_k\}$ has a hamiltonian cycle F' while the square of $G_0 = G_1 - \{u_1, u_2, \ldots, u_k\}$ has a 2-factor F''. Then $F' \cup F''$ is a 2-factor of G^2.

Finally, suppose that v_1 is the only vertex of degree one adjacent with v. Then we have a situation analogous to that considered in Case 1. The graph G^2 can be shown to have a 2-factor by essentially the same argument made in Subcases A and B.

This completes the proof.

We conclude by presenting a corollary. The *subdivision graph* $S(G)$ of a graph G is that graph in which every edge $e = uv$ is replaced by a new vertex w and two new edges uw and wv. The *total graph* $T(G)$ of G is that graph whose vertex set can be put in one-to-one correspondence with the set of vertices and edges of G in such a way that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident. It is a consequence of the definitions, that for every graph G, $T(G) = [S(G)]^2$. From this, we arrive at the following.

Corollary. *A necessary and sufficient condition for the total graph $T(G)$ of a connected graph G having at least two vertices to possess a 2-factor is that G does not contain three vertices of degree one which are adjacent with the same vertex.*

References

[1] *M. Behzad* and *G. Chartrand,* Introduction to the Theory of Graphs. Allyn and Bacon, Boston (1972).

[2] *G. Chartrand, A. M. Hobbs, H. A. Jung, S. F. Kapoor,* and *C. St. J. A. Nash-Williams,* The square of a block is Hamiltonian connected. J. Combinatorial Theory *16* (1974), 290—292.

[3] *H. Fleischner,* The square of every two-connected graph is Hamiltonian. J. Combinatorial Theory (Series B) *16* (1974), 29—34.

[4] *A. M. Hobbs,* Some hamiltonian results in powers of graphs. J. Res. Nat. Bur. Standards *77*B (1973), 1—10.

[5] *F. Neuman,* On a certain ordering of the vertices of a tree. Časopis Pěst. Mat. *89* (1964), 323—339.

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