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COMPLETELY DECOMPOSABLE ABELIAN GROUPS ANY REGULAR SUBGROUP OF WHICH IS COMPLETELY DECOMPOSABLE

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In this paper we shall give the full description of all torsion free abelian groups with the maximum condition for the types which have the property formulated in the title. Concerning the groups without the maximum condition it suffices to know all such groups G the type set of which contains an infinite increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ such that for every $\hat{\tau} \in \hat{\tau}(G)$ it is $\hat{\tau} \leqq \hat{\tau}_n$ for some n, the set $\{\hat{\tau}, \hat{\tau} \in \hat{\tau}(G), \hat{\tau} \leqq \hat{\tau}_n\}$ is inversely well-ordered and for every prime p the inequality $\tau_n(p) \neq \infty$ holds for a finite number of n's only.

We start with the formulation of our results:

Theorem 1. *Let G be a completely decomposable torsion free abelian group satisfying the maximum condition for the types. Then any regular subgroup of G is completely decomposable if and only if $G = D \dotplus H$ where D is divisible and H reduced and*

1) *for any three elements $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3$ from $T(H)$ the set $\{\hat{\tau}_1 \cap \hat{\tau}_2, \hat{\tau}_1 \cap \hat{\tau}_3, \hat{\tau}_2 \cap \hat{\tau}_3, \hat{\tau}_1 \cap \hat{\tau}_2 \cap \hat{\tau}_3\}$ is ordered and contains at most two different elements,*

2) *if the elements $\hat{\tau}_1, \hat{\tau}_2$ from $T(H)$ are incomparable, then $H(\hat{\tau}_1)$ (and $H(\hat{\tau}_2)$) is of finite rank,*

3) *for any four not necessarily different elements $\hat{\tau}_i$, $i = 1, 2, 3, 4$ from $T(H)$ with $\hat{\tau}_1 \cap \hat{\tau}_2 \parallel \hat{\tau}_3 \cap \hat{\tau}_4$ it holds $(\hat{\tau}_1 \cap \hat{\tau}_2) \vee (\hat{\tau}_3 \cap \hat{\tau}_4) = \hat{R}$.*

Let us denote by $\mathfrak{N}$ the class of all completely decomposable torsion free abelian groups G having three following properties:

1) the type set of G contains an infinite increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ such that for every $\hat{\tau} \in \hat{\tau}(G)$ it is $\hat{\tau} \leqq \hat{\tau}_n$ for some n and the set $\{\hat{\tau}, \hat{\tau} \in \hat{\tau}(G), \hat{\tau} \leqq \hat{\tau}_n\}$ is inversely well-ordered,

2) for every prime p the inequality $\tau_n(p) \neq \infty$ holds for a finite number of n's only,

3) any regular subgroup of G is completely decomposable.

Theorem 2. *Any regular subgroup of a completely decomposable torsion free abelian group G is completely decomposable if and only if either $\hat{\tau}(G)$ satisfies the maximum condition and therefore G is of the form from Theorem 1, or G is of the form $G = D \dotplus G_1 \dotplus G_2$ where D is divisible, $G_1 \in \mathfrak{N}$, $\hat{\tau}(G_2)$ is inversely well-ordered and $\hat{\tau}_1 \geqq \hat{\tau}_2$ for any $\hat{\tau}_i \in \hat{\tau}(G_i)$, $i = 1, 2$.*

1. INTRODUCTION

By a group we shall always mean an additively written torsion free abelian group. $\hat{\tau}(G)$ denotes the type set of the group G, i.e., the set of the types $\hat{\tau}^G(g)$ of all non-zero elements $g \in G$. If G is completely decomposable, $G = \dot{\sum}_{\alpha \in A} J_\alpha$ then $T(G)$ is the set of the types of all J_α, $\alpha \in A$. $\hat{R}$ is the type of the group of all rationals. For the other notation and terminology we refer to [1]. Especially, for $S \subseteq G$, $\{S\}_*^G$ denotes the pure closure of the set S in the group G and $\hat{\tau}_1 \parallel \hat{\tau}_2$ denotes the incomparability of the types $\hat{\tau}_1, \hat{\tau}_2$.

Following WANG [5] we shall call a subgroup H of G regular in G if any element of H has in H the same type as in G.

For the convenience of the reader we shall formulate some results from [2], [3], [4] and [5]. The proofs will be omitted.

1.1. Lemma. *Let J_i, $i = 1, 2, 3$ be three reduced torsion free groups of rank one and of the types $\hat{\tau}_i$, $i = 1, 2, 3$. If any pure subgroup of $G = J_1 \dotplus J_2 \dotplus J_3$ is completely decomposable then the set $\{\hat{\tau}_1 \cap \hat{\tau}_2, \hat{\tau}_1 \cap \hat{\tau}_3, \hat{\tau}_2 \cap \hat{\tau}_3, \hat{\tau}_1 \cap \hat{\tau}_2 \cap \hat{\tau}_3\}$ is ordered (in the natural order of the types) and contains at most two different elements.*

Proof. See [2], Lemma 2.

1.2. Proposition. *A completely decomposable torsion free group G the type set of which satisfies the maximum condition has the property that any one of its pure subgroups is completely decomposable if and only if $G = D \dotplus H$ where D is divisible and H reduced and*

a) *for any four not necessarily different elements $\hat{\tau}_i$, $i = 1, 2, 3, 4$ from $T(H)$ with $\hat{\tau}_1 \cap \hat{\tau}_2 \parallel \hat{\tau}_3 \cap \hat{\tau}_4$ it holds $H(\hat{\tau}_1 \cap \hat{\tau}_2) \cap H(\hat{\tau}_3 \cap \hat{\tau}_4) = 0$,*

b) *for any two incomparable elements $\hat{\tau}', \hat{\tau}''$ from $T(H)$ the set $\{\hat{\tau}_1 \cap \hat{\tau}_2, \hat{\tau}_i \in T(H), \hat{\tau}_i \geqq \hat{\tau}' \cap \hat{\tau}'', i = 1, 2\}$ satisfies the minimum condition,*

c) *for any three not necessarily different elements $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3$ from $T(H)$ with $\hat{\tau}_3 \parallel \hat{\tau}_1 \cap \hat{\tau}_2$ the subgroup $H(\hat{\tau}_1 \cap \hat{\tau}_2)$ is of finite rank,*

d) *the set of all maximal elements of $T(H)$ is at most countable.*

Proof. See [2], Theorem 1.

Let us denote by $\mathfrak{M}$ the class of all completely decomposable groups G having the following two properties:

1) the type set of G contains an infinite increasing sequence $\hat{\tau}_1 < \hat{\tau}_2 < \ldots$ such that for every $\hat{\tau} \in \hat{\tau}(G)$ it is $\hat{\tau} \leqq \hat{\tau}_n$ for some n and the set $\{\hat{\tau}, \hat{\tau} \in \hat{\tau}(G), \hat{\tau} \leqq \hat{\tau}_n\}$ is inversely well-ordered for every n,

2) any pure subgroup of G is completely decomposable.

1.3. Proposition. *A completely decomposable torsion free group G the type set of which does not satisfy the maximum condition has the property that any one of its pure subgroups is completely decomposable if and only if $G = D \dotplus G_1 \dotplus G_2$ where D is divisible, $G_1 \in \mathfrak{M}$, $T(G_2)$ is inversely well-ordered and $\hat{\tau}_1 \geqq \hat{\tau}_2$ for any $\hat{\tau}_i \in \hat{\tau}(G_i)$, $i = 1, 2$.*

Proof. See [2], Theorem 2.

1.4. Proposition. *A completely decomposable group G has the property that any one of its subgroups H with G/H bounded is isomorphic to G if and only if:*

α) For any two incomparable elements $\hat{\tau}_1, \hat{\tau}_2$ from $T(G)$ it is $\hat{\tau}_1 \vee \hat{\tau}_2 = \hat{R}$.

β) If $\{\hat{\tau}_n\}$ is an infinite increasing sequence of elements from $T(G)$ then for any prime p the inequality $\tau_n(p) \neq \infty$ holds for a finite number of n's only.

Proof. See [3], Theorem 3.

1.5. Proposition. *Let G be a completely decomposable group satisfying the condition α) from 1.4. If $T(G)$ satisfies the maximum condition and H is regular in G with G/H torsion then H is isomorphic to G.*

Proof. See [4], Theorem 2.

1.6. Proposition. *Let $G = G_1 \dotplus G_2$ be a group such that G_2 is completely decomposable with $T(G_2)$ inversely well-ordered and $\hat{\tau}_1 \geqq \hat{\tau}_2$ for all $\hat{\tau}_i \in \hat{\tau}(G_i)$, $i = 1, 2$. Then any regular subgroup H of G is a direct sum of $H \cap G_1$ and a completely decomposable group.*

Proof. See [5].

1.7. Lemma. *Let G be a completely decomposable torsion free group having the property that for any four not necessarily different elements $\hat{\tau}_i$, $i = 1, 2, 3, 4$ from $T(G)$ with $\hat{\tau}_1 \cap \hat{\tau}_2 \parallel \hat{\tau}_3 \cap \hat{\tau}_4$ it is $G(\hat{\tau}_1 \cap \hat{\tau}_2) \cap G(\hat{\tau}_3 \cap \hat{\tau}_4) = 0$. Then any finite set $\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_n$ of elements of $T(G)$ contains two elements having the same intersection as this set.*

Proof. See [2], Lemma 10.

2. AUXILIARY RESULTS

In this section we shall deal with conditions 1) – 3) of Theorem 1 and conditions a) – d) of Proposition 1.2.

2.1. Lemma. *A reduced completely decomposable group H satisfying condition* 3) *satisfies condition* a).

Proof. $H(\hat{\tau}_1 \cap \hat{\tau}_2) \cap H(\hat{\tau}_3 \cap \hat{\tau}_4) = H((\hat{\tau}_1 \cap \hat{\tau}_2) \vee (\hat{\tau}_3 \cap \hat{\tau}_4)) = H(\hat{R}) = 0$.

2.2. Lemma. *A reduced completely decomposable group H satisfying conditions* 2) *and* 3) *satisfies condition* b).

Proof. It suffices to show that $H^*(\hat{\tau}' \cap \hat{\tau}'') \subseteq \{H(\hat{\tau}'), H(\hat{\tau}'')\}$ since the last groups are of finite rank by 2). Let $\hat{\tau} > \hat{\tau}' \cap \hat{\tau}''$ be an arbitrary element of $T(H)$. Then $\hat{\tau} \parallel \hat{\tau}', \hat{\tau} \parallel \hat{\tau}''$ is impossible since in this case $\hat{\tau} = \hat{\tau} \vee (\hat{\tau}' \cap \hat{\tau}'') = (\hat{\tau} \vee \hat{\tau}') \cap (\hat{\tau} \vee \hat{\tau}'') = \hat{R}$ by 3). Similarly $\hat{\tau}' \cap \hat{\tau}'' < \hat{\tau} < \hat{\tau}'$ leads to a contradiction $\hat{\tau} = \hat{\tau} \vee (\hat{\tau}' \cap \hat{\tau}'') = (\hat{\tau} \vee \hat{\tau}') \cap (\hat{\tau} \vee \hat{\tau}'') = \hat{\tau}'$ since $\hat{\tau}$ is clearly incomparable with $\hat{\tau}''$. From this the assertion easily follows.

2.3. Lemma. *A reduced completely decomposable group H satisfying conditions* 1), 2) *and* 3) *satisfies condition* c).

Proof. For $\hat{\tau}_1 \cap \hat{\tau}_2 \in T(H)$ it suffices to use condition 2). If $\hat{\tau}_1 \cap \hat{\tau}_2 \notin T(H)$ then necessarily $\hat{\tau}_1 \parallel \hat{\tau}_2$. Since $\hat{\tau}_3 \parallel \hat{\tau}_1 \cap \hat{\tau}_2$ we have $\hat{\tau}_1 \cap \hat{\tau}_2 > \hat{\tau}_1 \cap \hat{\tau}_2 \cap \hat{\tau}_3 = \hat{\tau}_1 \cap \hat{\tau}_3 = \hat{\tau}_2 \cap \hat{\tau}_3$ by 1). Thus 3) yields $\hat{\tau}_3 = (\hat{\tau}_1 \vee \hat{\tau}_2) \cap \hat{\tau}_3 = (\hat{\tau}_1 \cap \hat{\tau}_3) \vee (\hat{\tau}_2 \cap \hat{\tau}_3) = \hat{\tau}_1 \cap \hat{\tau}_2 \cap \hat{\tau}_3$ – a contradiction completing the proof.

2.4. Lemma. *For a reduced completely decomposable group H condition* d) *follows from condition* 3).

Proof. We can suppose that the maximal elements of $T(H)$ are well-ordered in a way. Let $\{\hat{\tau}_\alpha, \alpha < \omega\}$ be the set of all maximal elements of $T(H)$. For any $\alpha < \omega$ let us denote by N_α the set of all primes p with $\tau_\alpha(p) \neq \infty$ and $M_\alpha = \bigcup_{\beta < \alpha} N_\beta$. By hypothesis, any N_α, $\alpha < \omega$ is non-empty. On the other hand, $M_\alpha \cap N_\alpha = \emptyset$ by 3). Hence the system $\{M_\alpha, \alpha < \omega\}$ of subsets of the set of all primes is increasing and therefore countable, from which the assertion easily follows.

3. THE PROOF OF THEOREM 1

a) Necessity. Condition 1): follows at once from Lemma 1.1. Condition 2): follows at once from Proposition 1.2 c). Condition 3): Taking in H the elements a, b with $\hat{\tau}^H(a) = \hat{\tau}_1 \cap \hat{\tau}_2$, $\hat{\tau}^H(b) = \hat{\tau}_3 \cap \hat{\tau}_4$ we get a completely decomposable pure subgroup $S = \{a, b\}_*^H$ and Proposition 1.4 yields condition 3).

b) Sufficiency. Let $G = D \dotplus H$ be a completely decomposable group where D is divisible and H reduced with conditions 1)–3) and let T be any regular subgroup of G. From the well-known properties of divisible groups (see e.g. [1]) it follows that we can assume T to be written in the form $T = T \cap D \dotplus T'$ where $T' \subseteq H$. Denoting by S the pure closure of T' in H we obtain the complete decomposability of S by means of Lemmas 2.1–2.4 and Proposition 1.2. By Lemmas 2.1 and 1.7 $T(S)$ satisfies condition α) of Proposition 1.4 so that Proposition 1.5 completes the proof.

4. THE PROOF OF THEOREM 2

a) Necessity. It suffices to use either Theorem 1 or Propositions 1.3 and 1.4.

b) Sufficiency. If $\hat{\tau}(G)$ satisfies the maximum condition then it suffices to use Theorem 1. In the other case let $G = D \dotplus G_1 \dotplus G_2$ be a completely decomposable group with D divisible, $G_1 \in \mathfrak{N}$, $T(G_2)$ inversely well-ordered and $\hat{\tau}_1 \geqq \hat{\tau}_2$ for any $\hat{\tau}_i \in \hat{\tau}(G_i)$, $i = 1, 2$ and let T be any regular subgroup of G. By the well-known properties of divisible groups we can assume that T is of the form $T = T \cap D \dotplus T'$ where $T' \subseteq G_1 \dotplus G_2$. Proposition 1.6 now completes the proof.

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