Czechoslovak Mathematical Journal

Milan Tvrdý

Linear boundary value type problems for functional-differential equations and their adjoints

Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 1, 37–66

Persistent URL: http://dml.cz/dmlcz/101292

Terms of use:

© Institute of Mathematics AS CR, 1975

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

LINEAR BOUNDARY VALUE TYPE PROBLEMS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS AND THEIR ADJOINTS

MILAN TVRDÝ, Praha

(Received July 27, 1973, — in revised version January 20, 1974)

0. INTRODUCTION

The paper deals with boundary value type problems for functional-differential equations

$$(0,1) \qquad \dot{x}(t) = \int_{-r}^{0} [d_\vartheta P(t, \vartheta)]\, x(t + \vartheta) + f(t) \quad \text{a.e. on} \quad [a, b]$$

or

$$(0,2) \quad \dot{x}(t) = A(t)\, x(t) + B(t)\, x(t - r) + \int_{a-r}^{b} [d_s G(t, s)]\, x(s) + f(t) \text{ a.e. on } [a, b],$$

where $-\infty < a < b < \infty$ and the functions $P(t, \vartheta)$, $G(t, s)$, $A(t)$, $B(t)$ and $f(t)$ fulfil some natural assumptions. In particular, we derive their adjoints and in some special cases prove the Fredholm alternative. (The results of A. HALANAY [5] or E. A. LIFŠIC [9] on the existence of periodic solutions to the equation (0,1) and the results of [12] on integral boundary value problems for ordinary integro-differential equations are included.) Our approach is based on the ideas of D. WEXLER [14] and ŠT. SCHWABIK [11] and differs from that of A. Halanay [6] or D. HENRY [8] (cf. also J. K. HALE [7]). The adjoint problems obtained seem to be more natural than those of D. Henry [8] and follow directly from the principles of functional analysis. (It is shown that after some artificial steps our adjoint reduces to that of D. Henry.) Initial functions are continuous on $[a - r, a]$ or of bounded variation on $[a - r, a]$. In § 4 boundary value type problems for hereditary differential equations considered in the sense of M. C. DELFOUR, S. K. MITTER [3] (with square integrable initial functions) are treated.

1. PRELIMINARIES

Let $-\infty < \alpha < \beta < +\infty$. The closed interval $\alpha \leqq t \leqq \beta$ is denoted by $[\alpha, \beta]$, its interior $\alpha < t < \beta$ by (α, β) and the corresponding half-open intervals by $[\alpha, \beta)$ and $(\alpha, \beta]$. Given a $p \times q$-matrix $M = (m_{i,j})_{i=1,\dots,p, j=1,\dots,q}$, M' denotes its transpose and

$$\|M\| = \max_{i=1,\dots,p} \sum_{j=1}^{q} |m_{i,j}| .$$

$\mathscr{R}_n$ is the space of real column n-vectors with the norm $\|x\| = \max_{i=1,\dots,p} |x_i|$. The space of real row n-vectors is $\mathscr{R}_n^*$. (Elements of $\mathscr{R}_n^*$ are denoted by x', where $x \in \mathscr{R}_n$;

$$\|x'\| = \sum_{i=1}^{n} |x_i| .)$$

$\mathscr{C}_n(\alpha, \beta)$ is the Banach space (B-space) of continuous functions $u : [\alpha, \beta] \to \mathscr{R}_n$ with the norm $\|u\|_{\mathscr{C}} = \sup_{t\in[\alpha,\beta]} \|u(t)\|$; $\mathscr{BV}_n(\alpha, \beta)$ is the B-space of functions $u : [\alpha, \beta] \to \mathscr{R}_n$ of bounded variation on $[\alpha, \beta]$ with the norm $\|u\|_{\mathscr{BV}} = \|u(\beta)\| + \mathrm{var}_\alpha^\beta u$; $\mathscr{V}_n^0(\alpha, \beta)$ is the set of functions $u' : [\alpha, \beta] \to \mathscr{R}_n^*$ of bounded variation on $[\alpha, \beta]$, right continuous on (α, β) and vanishing at β (being equipped with the norm $\|u'\|_{\mathscr{BV}}$, $\mathscr{V}_n^0(\alpha, \beta)$ becomes a B-space); $\mathscr{AC}_n(\alpha, \beta)$ is the B-space of absolutely continuous functions $u : [\alpha, \beta] \to \mathscr{R}_n$ with the norm $\|u\|_{\mathscr{AC}} = \|u\|_{\mathscr{BV}}$; $\mathscr{L}_n(\alpha, \beta)$ is the B-space of Lebesgue integrable (L-integrable) functions $u : [\alpha, \beta] \to \mathscr{R}_n$ with the norm

$$\|u\|_{\mathscr{L}} = \int_\alpha^\beta \|u(t)\| \, \mathrm{d}t ;$$

$\mathscr{L}_n^\infty(\alpha, \beta)$ is the B-space of essentially bounded functions $u' : [\alpha, \beta] \to \mathscr{R}_n^*$ with the norm $\|u'\| = \sup\,\mathrm{ess}_{t\in[\alpha,\beta]} \|u'(t)\|$.

Given a B-space $\mathscr{X}$, $\mathscr{X}^*$ denotes its dual and the value of a functional $y \in \mathscr{X}^*$ on $x \in \mathscr{X}$ is denoted by $\langle x, y \rangle_{\mathscr{X}}$. The zero functional on $\mathscr{X}$ is denoted by $o_{\mathscr{X}}$. Hereafter $\mathscr{L}_n^*(\alpha, \beta)$ and $\mathscr{C}_n^*(\alpha, \beta)$ are identified with $\mathscr{L}_n^\infty(\alpha, \beta)$ and $\mathscr{V}_n^0(\alpha, \beta)$, respectively, while

$$\langle x, y' \rangle_{\mathscr{L}} = \int_\alpha^\beta y'(t)\, x(t) \, \mathrm{d}t \quad \text{and} \quad \langle u, v' \rangle_{\mathscr{C}} = \int_\alpha^\beta [\mathrm{d}v'(t)]\, u(t)$$

for $x \in \mathscr{L}_n(\alpha, \beta)$, $y' \in \mathscr{L}_n^\infty(\alpha, \beta)$, $u \in \mathscr{C}_n(\alpha, \beta)$ and $v' \in \mathscr{V}_n^0(\alpha, \beta)$. (There are isometric isomorphisms between $\mathscr{L}_n^*(\alpha, \beta)$ and $\mathscr{L}_n^\infty(\alpha, \beta)$ and between $\mathscr{C}_n^*(\alpha, \beta)$ and $\mathscr{V}_n^0(\alpha, \beta)$, cf. e.g. [4].)

Let $\mathscr{X}$, $\mathscr{Y}$ be B-spaces. Given a linear bounded operator $T : \mathscr{X} \to \mathscr{Y}$ (defined on the whole $\mathscr{X}$), T^* denotes its adjoint ($T^* : \mathscr{Y}^* \to \mathscr{X}^*$, $\langle Tx, y \rangle_{\mathscr{Y}} = \langle x, T^*y \rangle_{\mathscr{X}}$ for all $x \in \mathscr{X}$ and $y \in \mathscr{Y}^*$), Ker (T) is the set of all $x \in \mathscr{X}$ such that $Tx = 0$ = zero element of $\mathscr{Y}$ and Im (T) is the range of T. Given two operators $T_1 : \mathscr{X}_1 \to \mathscr{Y}$, $T_2 : \mathscr{X}_2 \to \mathscr{Y}$, the homogeneous equations $T_1 x = 0$ and $T_2 z = 0$ are said to be equivalent if there is a one-to-one correspondence between $\operatorname{Ker}(T_1)$ and $\operatorname{Ker}(T_2)$.

2. GENERAL BOUNDARY VALUE TYPE PROBLEM AND ITS ADJOINT

2,1. Assumptions. We assume $-\infty < a < b < +\infty$, $r > 0$*). $A(t)$ and $B(t)$ are $n \times n$-matrix functions L-integrable on $[a, b]$, $G(t, s)$ is a Borel measurable in (t, s) on $[a, b] \times [a - r, b]$ $n \times n$-matrix function such that $\operatorname{var}_{a-r}^{b} G(t, \cdot) < \infty$ for any $t \in [a, b]$ and

$$\int_a^b (\|G(t, b)\| + \operatorname{var}_{a-r}^{b} G(t, \cdot))\,\mathrm{d}t < \infty\,,$$

$f(t) \in \mathscr{L}_n(a, b)$. Λ is an arbitrary B-space, $l \in \Lambda$ and the operators $M : \mathscr{C}_n(a - r, a) \to \Lambda$, $N : \mathscr{AC}_n(a, b) \to \Lambda$ are linear and bounded, while $\operatorname{Im}(N^*) \subset \mathscr{C}_n^*(a, b) = \mathscr{V}_n^0(a, b)$ (i.e., given $\lambda \in \Lambda^*$, there is a function $(N^*\lambda)(t) \in \mathscr{V}_n^0(a, b)$ such that

$$\langle Nx, \lambda \rangle_\Lambda = \langle x, N^*\lambda \rangle_{\mathscr{AC}} = \int_a^b [\mathrm{d}(N^*\lambda)(t)]\, x(t) \text{ for all } x \in \mathscr{AC}_n(a, b))\,.$$

Without any loss of generality we may also assume that, given $t \in [a, b]$, the function $G(t, \cdot)$ is right continuous on $(a - r, b)$, while $G(t, b) = 0$. Given $\lambda \in \Lambda^*$, let us denote by $(M^*\lambda)(t)$ the row n-vector function such that $(M^*\lambda)(t) - (N^*\lambda)(a) \in \mathscr{V}_n^0(a - r, a)$ and

$$\langle Mu, \lambda \rangle_\Lambda = \langle u, M^*\lambda \rangle_{\mathscr{C}} = \int_{a-r}^a [\mathrm{d}\{(M^*\lambda)(t) - (N^*\lambda)(a)\}]\, u(t)$$

for all $u \in \mathscr{C}_n(a - r, a)$.

We are interested in the following boundary value type problem:

2,2. Problem (P). Determine $x \in \mathscr{AC}_n(a, b)$ and $u \in \mathscr{C}_n(a - r, a)$ such that

$$(2,1) \quad \dot{x}(t) = A(t)\,x(t) + \begin{Bmatrix} B(t)\,u(t - r), & t < a + r \\ B(t)\,x(t - r), & t \geqq a + r \end{Bmatrix} + \int_{a-r}^a [\mathrm{d}_s G(t, s)]\,u(s) + \int_a^b [\mathrm{d}_s G(t, s)]\,x(s) + f(t) \quad \text{a.e. on } [a, b]\,,$$

$$(2,2) \quad x(a) = u(a)\,,$$

$$(2,3) \quad Mu + Nx = l\,,$$

where Assumptions 2,1 are fulfilled.

*) If $r = 0$, the equation (2,1) reduces to an ordinary integro-differential equation with initial data in R_n. The case of $r = 0$ will be treated separately later on (cf. Sec. 5,5).

2,3. Notation. Let us put

$$\mathscr{X} = \mathscr{AC}_n(a, b) \times \mathscr{C}_n(a - r, a), \quad \mathscr{Y} = \mathscr{L}_n(a, b) \times \Lambda \times \mathscr{R}_n$$

and

$$(2,4) \qquad U : \begin{pmatrix} x \\ u \end{pmatrix} \in \mathscr{X} \to \begin{pmatrix} Dx - Ax - B_1 x - B_2 u - G_1 x - G_2 u \\ Mu + Nx \\ u(a) - x(a) \end{pmatrix} \in \mathscr{Y},$$

where

$$D : x \in \mathscr{AC}_n(a, b) \quad \to \dot{x}(t) \in \mathscr{L}_n(a, b),$$

$$A : x \in \mathscr{AC}_n(a, b) \quad \to A(t)\, x(t) \in \mathscr{L}_n(a, b),$$

$$B_1 : x \in \mathscr{AC}_n(a, b) \quad \to \begin{cases} 0, & t < a + r \\ B(t)\, x(t - r), & t \geqq a + r \end{cases} \in \mathscr{L}_n(a, b),$$

$$B_2 : u \in \mathscr{C}_n(a - r, a) \to \begin{cases} B(t)\, u(t - r), & t < a + r \\ 0, & t \geqq a + r \end{cases} \in \mathscr{L}_n(a, b),$$

$$G_1 : x \in \mathscr{AC}_n(a, b) \quad \to \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) \in \mathscr{L}_n(a, b),$$

$$G_2 : u \in \mathscr{C}_n(a - r, a) \to \int_{a-r}^a [\mathrm{d}_s G(t, s)]\, u(s) \in \mathscr{L}_n(a, b).$$

All these operators are linear and bounded. The given problem (P) can be reformulated as the operator equation

$$U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ 0 \end{bmatrix}.$$

Clearly, $\mathscr{X}^* = \mathscr{AC}_n^*(a, b) \times \mathscr{V}_n^0(a - r, a)$, $\mathscr{Y}^* = \mathscr{L}_n^\infty(a, b) \times \Lambda^* \times \mathscr{R}_n^*$ and

$$\left\langle \begin{pmatrix} x \\ u \end{pmatrix}, (g, h') \right\rangle_{\mathscr{X}} = \langle x, g \rangle_{\mathscr{AC}} + \int_{a-r}^a [\mathrm{d}h'(t)]\, u(t),$$

$$\left\langle \begin{bmatrix} f \\ l \\ k \end{bmatrix}, (y', \lambda, \gamma') \right\rangle_{\mathscr{Y}} = \int_a^b y'(t) f(t)\, \mathrm{d}s + \langle l, \lambda \rangle_\Lambda + \gamma' k$$

for $x \in \mathscr{AC}_n(a, b)$, $u \in \mathscr{C}_n^*(a - r, a)$, $g \in \mathscr{AC}_n^*(a, b)$, $h' \in \mathscr{V}_n^0(a - r, a)$, $f \in \mathscr{L}_n(a, b)$, $l \in \Lambda$, $k \in \mathscr{R}_n$, $y' \in \mathscr{L}_n^\infty(a, b)$, $\lambda \in \Lambda^*$ and $\gamma' \in \mathscr{R}_n^*$. Let $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathscr{X}$ and $(y', \lambda, \gamma') \in \mathscr{Y}^*$, then

$$\left\langle U \begin{pmatrix} x \\ u \end{pmatrix}, (y', \lambda, \gamma') \right\rangle_{\mathscr{Y}} = \langle Dx - Ax - B_1 x - B_2 u - G_1 x - G_2 u, y' \rangle_{\mathscr{L}} +$$

$$+ \langle Mu + Nx, \lambda \rangle_\Lambda + \gamma'(u(a) - x(a)) =$$

$$= \langle x, D^*y` - A^*y` - B_1^*y` - G_1^*y` + N^*\lambda + K_1^*\gamma`\rangle_{\mathscr{AC}} +$$
$$+ \langle u, -B_2^*y` - G_2^*y` + M^*\lambda + K_2^*\gamma`\rangle_{\mathscr{C}},$$

where

$$K_1 : x \in \mathscr{AC}_n(a, b) \to -x(a) \in \mathscr{R}_n$$

and

$$K_2 : u \in \mathscr{C}_n(a - r, a) \to u(a) \in \mathscr{R}_n .$$

Consequently

$$U^* : (y`, \lambda, \gamma`) \in \mathscr{Y}^* \to \begin{bmatrix} D^*y` - A^*y` - B_1^*y` - G_1^*y` + N^*\lambda + K_1^*\gamma` \\ -B_2^*y` - G_2^*y` + M^*\lambda + K_2^*\gamma` \end{bmatrix} \in \mathscr{X}^*$$

and the adjoint to (P) is the system of equations for $(y`, \lambda, \gamma`) \in \mathscr{Y}^*$

(2,5)
$$\begin{aligned} D^*y` - A^*y` - B_1^*y` - G_1^*y` + N^*\lambda + K_1^*\gamma` &= o_{\mathscr{AC}}, \\ - B_2^*y` - G_2^*y` + M^*\lambda + K_2^*\gamma` &= o_{\mathscr{C}}. \end{aligned}$$

2,4. An analytic form of the adjoint problem. By the definition of an adjoint operator and by the unsymmetric Fubini theorem (2) it holds for all $x \in \mathscr{AC}_n(a, b)$, $u \in \mathscr{C}_n(a - r, a)$, $y` \in \mathscr{L}_n^\infty(a, b)$, $\lambda \in \Lambda^*$ and $\gamma` \in \mathscr{R}_n^*$

$$\left\langle \begin{pmatrix} x \\ u \end{pmatrix}, U^*(y`, \lambda, \gamma`) \right\rangle_{\mathscr{X}} = \left\langle U \begin{pmatrix} x \\ u \end{pmatrix}, (y`, \lambda, \gamma`) \right\rangle_{\mathscr{Y}} =$$
$$= \int_a^b y`(t)\,\dot{x}(t)\,\mathrm{d}t - \int_a^b y`(t)\,A(t)\,x(t)\,\mathrm{d}t - \int_{a+r}^b y`(t)\,B(t)\,x(t - r)\,\mathrm{d}t -$$
$$- \int_a^{a+r} y`(t)\,B(t)\,u(t - r)\,\mathrm{d}t - \int_a^b y`(t) \left(\int_a^b [\mathrm{d}_s G(t, s)]\,x(s) \right) \mathrm{d}t -$$
$$- \int_a^b y`(t) \left( \int_{a-r}^a [\mathrm{d}_s G(t, s)]\,u(s) \right) \mathrm{d}t + \langle Mu + Nx, \lambda \rangle_\Lambda + \gamma`(u(a) - x(a)) =$$
$$= \int_a^b y`(t)\,\dot{x}(t)\,\mathrm{d}t - \int_a^b [\mathrm{d}g`(t)]\,x(t) - \int_{a-r}^a [\mathrm{d}h`(t)]\,u(t),$$

where

(2,6)
$$g`(t) = - \int_t^b y`(s)\,A(s)\,\mathrm{d}s + \int_a^b y`(s)\,G(s, t)\,\mathrm{d}s - (N^*\lambda)(t) -$$
$$- \begin{Bmatrix} \int_{t+r}^b y`(s)\,B(s)\,\mathrm{d}s, & t \leqq b - r \\ 0 & , t > b - r \end{Bmatrix} - \begin{Bmatrix} \gamma`, & t = a \\ 0, & t > a \end{Bmatrix} \quad \text{for} \quad t \in [a, b],$$

$$h`(t) = - \int_{t+r}^{a+r} y`(s)\,B(s)\,\mathrm{d}s + \int_a^b y`(s)\,(G(s, t) - G(s, a))\,\mathrm{d}s + \begin{Bmatrix} \gamma`, & t < a \\ 0, & t = a \end{Bmatrix} -$$
$$- (M^*\lambda)(t) + (N^*\lambda)(a) \quad \text{for} \quad t \in [a - r, a].$$

Now, $(y`, \lambda, \gamma`) \in \operatorname{Ker}(U^*)$ iff

$$(2,7) \qquad 0 = \int_a^b y`(t)\, \dot{x}(t)\, dt - \int_a^b [dg`(t)]\, x(t) - \int_{a-r}^a [dh`(t)]\, u(t)$$

for all $x \in \mathscr{AC}_n(a, b)$ and $u \in \mathscr{C}_n(a - r, a)$. In particular, if $x(t) = 0$ on $[a, b]$, (2,7) means that

$$\int_{a-r}^a [dh`(t)]\, u(t) = 0 \quad \text{for all} \quad u \in \mathscr{C}_n(a - r, a)\,.$$

Since $h` \in \mathscr{V}_n^0(a - r, a)$, this is possible iff $h`(t) = 0$ on $[a - r, a]$. Thus

$$(2,8) \qquad \int_{t+r}^{a+r} y`(s)\, B(s)\, ds - \int_a^b y`(s)\, (G(s, t) - G(s, a))\, ds + $$
$$+ (M^*\lambda)(t) - (N^*\lambda)(a) - \gamma` = 0 \quad \text{on} \quad [a - r, a)\,.$$

The equality (2,7) now becomes (after integrating by parts)

$$(2,9) \quad \int_a^b y`(t)\, \dot{x}(t)\, dt = -g`(a)\, x(a) - \int_a^b g`(t)\, \dot{x}(t)\, dt \quad \text{for all} \quad x \in \mathscr{AC}_n(a, b)\,.$$

Since we may choose $x(t) = x(a) \neq 0$ on $[a, b]$, (2,9) implies furthermore $g`(a) = 0$ or

$$(2,10) \quad \gamma` = -\int_a^b y`(s)\, A(s)\, ds - \int_{a+r}^b y`(s)\, B(s)\, ds + \int_a^b y`(s)\, G(s, a)\, ds - (N^*\lambda)(a)\,.$$

Consequently, (2,9) reduces to

$$\int_a^b y`(t)\, \dot{x}(t)\, dt = -\int_a^b g`(t)\, \dot{x}(t)\, dt \quad \text{for all} \quad x \in \mathscr{AC}_n(a, b)$$

or

$$\int_a^b (y`(t) + g`(t))\, z(t)\, dt = 0 \quad \text{for all} \quad z \in \mathscr{L}_n(a, b)\,.$$

Hence $y`(t) = g`(t)$ a.e. on $[a, b]$, i.e.

$$(2,11) \qquad y`(t) = \int_t^b y`(s)\, A(s)\, ds + \begin{cases} \int_{t+r}^b y`(s)\, B(s)\, ds, & t \leqq b - r \\ 0 & , t > b - r \end{cases} -$$
$$- \int_a^b y`(s)\, G(s, t)\, ds + (N^*\lambda)(t) \quad \text{a.e. on} \quad [a, b]\,.$$

Let $z` \in \mathscr{L}_n^\infty(a, b)$. Then $(z`, \lambda, \gamma`) \in \operatorname{Ker}(U^*)$ iff there exists $y` \in \mathscr{L}_n^\infty(a, b)$ fulfilling (2,8) and (2,10) and such that $y(t) = z(t)$ a.e. on $[a, b]$ and (2,11) holds for all $t \in$

$\in (a, b)$. Finally, inserting (2,10) into (2,8) and taking into account that the right hand side of (2,11) is of bounded variation on $[a, b]$ and right continuous on (a, b), we complete the proof of the following

2,5. Theorem. *Let $z' \in \mathscr{L}_n^\infty(a, b)$, $\lambda \in \Lambda^*$ and $\gamma' \in \mathscr{R}_n^*$. Then $(z', \lambda, \gamma') \in \mathrm{Ker}\,(U^*)$ iff there exists $y \in \mathscr{BV}_n(a, b)$ right continuous on (a, b) (the values $y(a)$, $y(b)$ may be arbitrary) such that $y(t) = z(t)$ a.e. on $[a, b]$ and*

$$(2,12)\quad \int_a^b y'(s)\, A(s)\, \mathrm{d}s + \int_{t+r}^b y'(s)\, B(s)\, \mathrm{d}s - \int_a^b y'(s)\, G(s, t)\, \mathrm{d}s + (M^*\lambda)(t) = 0 \quad \text{for} \quad t \in [a - r, a)\,,$$

$$(2,13)\quad y'(t) = \int_t^b y'(s)\, A(s)\, \mathrm{d}s + \begin{cases} \int_{t+r}^b y'(s)\, B(s)\, \mathrm{d}s, & t \leqq b - r \\ 0, & t > b - r \end{cases} - \int_a^b y'(s)\, G(s, t)\, \mathrm{d}s + (N^*\lambda)(t) \quad \text{for} \quad t \in (a, b)\,,$$

while γ' is given by (2,10).

2,6. Definition. The problem (P*) of finding $y \in \mathscr{BV}_n(a, b)$ right continuous on (a, b) and $\lambda \in \Lambda^*$ such that (2,12) and (2.13) hold is called the *conjugate* problem to (P).

(In virtue of Theorem 2,5 the adjoint problem (2,5) to (P) and the problem (P*) conjugate to (P) are equivalent.)

2,7. Corollary. *The problem (P) has a solution only if*

$$(2,14)\quad \int_a^b y'(s) f(s)\, \mathrm{d}s + \langle l, \lambda \rangle_\Lambda = 0$$

for all solutions (y', λ) of the conjugate problem (P). If the operator U defined by (2,4) has a closed range $\mathrm{Im}\,(U)$ in $\mathscr{L}_n(a, b) \times \Lambda \times \mathscr{R}_n$, then the condition (2,14) is also sufficient for the existence of a solution to the problem (P).*

(The proof follows from Theorem 2,5 and from the fundamental "alternative" theorem concerning linear equations in B-spaces ([4], VI § 6).)

2,8. Remark. Let $\mathscr{X}$, $\mathscr{Y}$ be B-spaces and let $L : \mathscr{X} \to \mathscr{Y}$ be linear and bounded. A set $\mathscr{Y}^+ \subset \mathscr{Y}^*$ of linear continuous functionals on $\mathscr{Y}$ is said to be total in $\mathscr{Y}^*$ if $\langle y, g \rangle_{\mathscr{Y}} = 0$ for all $g \in \mathscr{Y}^+$ implies $y = 0$. Furthermore, if $L^+ : \mathscr{Y}^+ \to \mathscr{X}^*$ is a linear operator such that $\langle Lx, g \rangle_{\mathscr{Y}} = \langle x, L^+ g \rangle_{\mathscr{X}}$ for all $x \in \mathscr{X}$ and $g \in \mathscr{Y}^+$, we shall say that L^+ is a conjugate operator to L with respect to $\mathscr{Y}^+$. Clearly, L^+ is a restriction of the adjoint operator L^* to L on $\mathscr{Y}^+$. Hence $\mathrm{Ker}\,(L^+) \subset \mathrm{Ker}\,(L^*)$. (For some more details concerning conjugate operators see [11].)

Now, let $\mathscr{V}_n(a, b)$ be the space of row n-vector functions of bounded variation on $[a, b]$ and right continuous on (a, b). Then $\mathscr{V}_n(a, b)$ is a total subset in $\mathscr{L}_n^\infty(a, b)$. (In fact, let $f \in \mathscr{L}_n(a, b)$ and

$$0 = \int_a^b y'(t) f(t) \, dt \quad \text{for all} \quad y' \in \mathscr{V}_n(a, b) .$$

Then

$$(2,15) \qquad 0 = \int_a^b y'(t) \, dg(t) \quad \text{for all} \quad y' \in \mathscr{V}_n(a, b) ,$$

where $g \in \mathscr{AC}_n(a, b)$ is an indefinite integral of f on $[a, b]$. Let $g_i(t_1) \neq g_i(t_2)$ for a component g_i of the vector $g = (g_1, g_2, \ldots, g_n)'$ and for some $t_1, t_2 \in [a, b]$, $t_1 < t_2$. Analogously to the second part of the proof of Lemma 5,1 in [10] we put $y'(t) = (y_1(t), y_2(t), \ldots, y_n(t))$, where $y_j(t) = 0$ on $[a, b]$ for $j \neq i$, $y_i(t) = 0$ for $t \in [a, t_1)$, $y_i(t) = 1$ for $t \in [t_1, t_2)$ and $y_i(t) = 0$ for $t \in [t_2, b]$. Then $y' \in \mathscr{V}_n(a, b)$ and

$$\int_a^b y'(t) \, dg(t) = \sum_{j=1}^n \int_a^b y_j(t) \, dg_j(t) = \int_a^b y_i(t) \, dg_i(t) = \int_{t_1}^{t_2} dg_i(t) = g_i(t_2) - g_i(t_1) \neq 0$$

which contradicts (2,15). Hence $g(t) = \text{const.}$ on $[a, b]$ and $f(t) = 0$ a.e. on $[a, b]$.)

The operator $D : x \in \mathscr{AC}_n(a, b) \to \dot{x} \in \mathscr{L}_n(a, b)$ is linear and bounded. It is easy to verify that its conjugate operator D^+ with respect to $\mathscr{V}_n(a, b)$ is given by

$$D^+ : y' \in \mathscr{V}_n(a, b) \to \begin{Bmatrix} 0, & t = a \\ -y'(t), & t \in (a, b) \\ 0, & t = b \end{Bmatrix} \in \mathscr{V}_n^0(a, b) .$$

Let us put $\mathscr{Y}^+ = \mathscr{V}_n(a, b) \times \Lambda^* \times \mathscr{R}_n^*$. Then $\mathscr{Y}^+$ is a total subset in $\mathscr{Y}^* = \mathscr{L}_n^\infty(a, b) \times \Lambda^* \times \mathscr{R}_n^*$ and the conjugate operator U^+ to U with respect to $\mathscr{Y}^+$ is given by

$$U^+ : (y', \lambda, \gamma') \in \mathscr{Y}^+ \to (\xi'(t), \eta'(t)) \in \mathscr{V}_n^0(a, b) \times \mathscr{V}_n^0(a - r, a) ,$$

where

$$\xi'(t) = \begin{Bmatrix} 0, & t = a \\ -y'(t), & t \in (a, b) \\ 0, & t = b \end{Bmatrix} + \int_t^b y'(s) A(s) \, ds + \begin{Bmatrix} \int_{t+r}^b y'(s) B(s) \, ds, & t \leqq b - r \\ 0 & , t > b - r \end{Bmatrix} -$$

$$- \int_a^b y'(s) G(s, t) \, ds + (N^*\lambda)(t) + \begin{Bmatrix} \gamma', & t = a \\ 0, & t > a \end{Bmatrix} \quad \text{for} \quad t \in [a, b] ,$$

$$\eta'(t) = \int_{t+r}^{a+r} y'(s) B(s) \, ds - \int_a^b y'(s) (G(s, t) - G(s, a)) \, ds + (M^*\lambda)(t) - (N^*\lambda)(a) -$$

$$- \begin{Bmatrix} \gamma', & t < a \\ 0, & t = a \end{Bmatrix} \quad \text{for} \quad t \in [a - r, a] .$$

The equation $U^+(y`, \lambda, \gamma`) = 0$ is identical with the system of equations (2,8), (2,10), (2,13) and hence it is equivalent also with the problem (P*) introduced in Definition 2,6. In Section 2,4 we proved actually that $\mathrm{Ker}\,(U^*) \subset \mathscr{Y}^+$ and hence $\mathrm{Ker}\,(U^+) = \mathrm{Ker}\,(U^*)$.

2,9. Remark. The above procedure can be also applied to the case of initial functions of bounded variation on $[a - r, a]$. This means that instead of $u \in \mathscr{C}_n(a - r, a)$ we are looking for $u \in \mathscr{BV}_n(a - r, a)$. The adjoint problem is again equivalent to the system of the form (2,12), (2,13). Only we have to suppose in addition that $\mathrm{Im}\,(M^*) \subset \mathscr{V}_n^0(a - r, a)$.

2,10. Remark. Some examples of spaces Λ and operators M, N fulfilling Assumptions 2,1 are given in the following § 3. Some conditions on the closedness of $\mathrm{Im}\,(U)$ are given in § 5.

2,11. Remark. The couple $(y`, \lambda)$ being a solution to (P*), the values $y`(a)$, $y`(b)$ may be arbitrary. We can require e.g. $y`(a) = y`(b) = 0$ or $y`(a+) = y`(a)$, $y`(b-) = y`(b)$. In the latter case we add to the system (2,12), (2,13) the conditions

$$(2,16) \qquad y`(a) = -\int_a^b y`(s)\,(G(s, a+) - G(s, a-))\,\mathrm{d}s + (N^*\lambda)(a+) - (M^*\lambda)(a-)\,,$$

$$y`(b) = \int_a^b y`(s)\,G(s, b-)\,\mathrm{d}s - (N^*\lambda)(b-)\,.$$

(Indeed, by (2,12)

$$\int_a^b y`(s)\,A(s)\,\mathrm{d}s + \int_{a+r}^b y`(s)\,B(s)\,\mathrm{d}s = \int_a^b y`(s)\,G(s, a-)\,\mathrm{d}s - (M^*\lambda)(a-)\,.)$$

2,12. Remark. $\mathscr{AC}_n^*(a, b)$ is isometrically isomorphic with $\mathscr{L}_n^\infty(a, b)\, x\, \mathscr{R}_n^*$. Given $g \in \mathscr{AC}_n^*(a, b)$, there exist uniquely determined $\beta` \in \mathscr{R}_n^*$ and $y`(t) \in \mathscr{L}_n^\infty(a, b)$ such that

$$\langle x, g\rangle_{\mathscr{AC}} = \beta` x(a) + \int_a^b y`(t)\dot{x}(t)\,\mathrm{d}t$$

for all $x \in \mathscr{AC}_n(a,b)$. (See [4] IV, 13, 29.) By a similar argument as in 2,4 we could derive the analytic form of the adjoint problem also in the case that N is a general linear bounded operator $\mathscr{AC}_n(a, b) \to \Lambda$ without supposing $\mathrm{Im}(N^*) \subset \mathscr{V}_n^0(a, b)$.

If $N^* : \lambda \in \Lambda^* \to (N^*\lambda, \tilde{N}^*\lambda) \in \mathscr{L}_n^\infty(a, b) \times \mathscr{R}_n^*$ *and* $(M^*\lambda)(t) - (\tilde{N}^*\lambda) \in \mathscr{V}_n^0(a - r, a)$ *for any* $\lambda \in \Lambda^*$, *then the problem of finding* $(y`(t), \lambda) \in \mathscr{L}_n^\infty(a, b) \times \Lambda^*$ *such that* (2,12) *holds on* $[a - r, a)$ *and* (2,13) *holds a.e. on* $[a, b]$ *is equivalent to the adjoint of the given problem* (P).

Let us mention some special cases of the given problem (P) which arise by a special choice of the boundary operators M, N and of the terminal space Λ.

3,1. The case $\Lambda = \mathscr{L}_m(c, d)$. Let $\Lambda = \mathscr{L}_m(c, d)$ $(-\infty < c < d < +\infty)$ and

$$(3,1) \qquad M : u \in \mathscr{C}_n(a - r, a) \to \int_{a-r}^{a} [\mathrm{d}_s M(\alpha, s)]\, u(s) \in \Lambda\,,$$

$$(3,2) \qquad N : x \in \mathscr{AC}_n(a, b) \to \int_a^b [\mathrm{d}_s N(\alpha, s)]\, x(s) \in \Lambda\,,$$

where $M(\alpha, s)$ is a Borel measurable in $(\alpha, s) \in [c, d] \times [a - r, a]$ $m \times n$-matrix function such that $\operatorname{var}_{a-r}^a M(\alpha, \cdot) < \infty$ for any $\alpha \in [c, d]$ and

$$\int_c^d (\|M(\alpha, a)\| + \operatorname{var}_{a-r}^a M(\alpha, \cdot))\, \mathrm{d}\alpha < \infty$$

and $N(\alpha, s)$ is a Borel measurable in $(\alpha, s) \in [c, d] \times [a, b]$ $m \times n$-matrix function such that $\operatorname{var}_a^b N(\alpha, \cdot) < \infty$ for any $\alpha \in [c, d]$ and

$$\int_c^d (\|N(\alpha, b)\| + \operatorname{var}_a^b N(\alpha, \cdot))\, \mathrm{d}\alpha < \infty\,.$$

Without any loss of generality we may assume that for any $\alpha \in [c, d]$, $M(\alpha, \cdot)$ is right continuous on $(a - r, a)$, $N(\alpha, \cdot)$ is right continuous on (a, b), $M(\alpha, a) = N(\alpha, a)$ and $N(\alpha, b) = 0$.

Let $x \in AC_n(a, b)$, $u \in C_n(a - r, a)$, $\lambda' \in \mathscr{L}_m^\infty(c, d)$. Then by the unsymmetric Fubini theorem ([2])

$$\langle Mu, \lambda' \rangle_{\mathscr{L}} = \int_c^d \lambda'(\alpha) \left(\int_{a-r}^a [\mathrm{d}_s M(\alpha, s)]\, u(s) \right) \mathrm{d}\alpha =$$
$$= \int_{a-r}^a \left[\mathrm{d}_s \int_c^d \lambda'(\alpha)\, (M(\alpha, s) - M(\alpha, a))\, \mathrm{d}\alpha \right] u(s)$$

and

$$\langle Nx, \lambda' \rangle_{\mathscr{L}} = \int_c^d \lambda'(\alpha) \left(\int_a^b [\mathrm{d}_s N(\alpha, s)]\, x(s) \right) \mathrm{d}\alpha = \int_a^b \left[\mathrm{d}_s \int_c^d \lambda'(\alpha)\, N(\alpha, s)\, \mathrm{d}\alpha \right] x(s)\,,$$

where

$$(3,3) \qquad (N^* \lambda')(t) = \int_c^d \lambda'(\alpha)\, N(\alpha, t)\, \mathrm{d}\alpha \in \mathscr{V}_n^0(a, b)$$

and

$$(3,4)\qquad (M^*\lambda')(t) - (M^*\lambda')(a) = \\ = \int_c^d \lambda'(\alpha)(M(\alpha, t) - M(\alpha, a))\,d\alpha \in \mathscr{V}_n^0(a - r, a).$$

Hence in this case the adjoint problem is equivalent to the system (2,12), (2,13), where M^* and N^* have the special form (3,4) and (3,3), respectively.

3,2. The case $\Lambda = \mathscr{C}_m(c, d)$. Similarly we can treat the case of $\Lambda = \mathscr{C}_m(c, d)$ $(-\infty < c < d < +\infty)$ with the operators M, N given by (3,1), (3,2), where $M(\cdot, s)$ and $N(\cdot, \sigma)$ are continuous on $[c, d]$ for any $s \in [a - r, a]$ and $\sigma \in [a, b]$. (Let us note that in this case any linear bounded operator $M : \mathscr{C}_n(a - r, a) \to \Lambda$ can be expressed in the form (3,1), where $M(\alpha, s)$ fulfils all our assumptions.) Analogously as in 3,1 we obtain

$$M^* : \lambda' \in \mathscr{V}_m^0(c, d) \to \int_c^d [d\lambda'(\alpha)]\, M(\alpha, t) \in \mathscr{V}_n(a - r, a),$$

$$N^* : \lambda' \in \mathscr{V}_m^0(c, d) \to \int_c^d [d\lambda'(\alpha)]\, N(\alpha, t) \in \mathscr{V}_n^0(a, b).$$

3,3. Finite dimensional terminal space. Let $\Lambda = \mathscr{R}_m$ and

$$M : u \in \mathscr{C}_n(a - r, a) \to \int_{a-r}^a [dM(s)]\, u(s) \in \mathscr{R}_m,$$

$$N : x \in \mathscr{A}\mathscr{C}_n(a, b) \to \int_a^b [dN(s)]\, x(s) \in \mathscr{R}_m,$$

where $M(t)$ and $N(t)$ are $m \times n$-matrix functions of bounded variation on $[a - r, a]$ and $[a, b]$, respectively. We may assume also M right continuous on $(a - r, a)$, N right continuous on (a, b), $M(a) = N(a)$ and $N(b) = 0$.

Let $x \in \mathscr{A}\mathscr{C}_n(a, b)$, $u \in \mathscr{C}_n(a - r, a)$ and $\lambda' \in \mathscr{R}_m^*$, then

$$\langle Mu, \lambda' \rangle_{\mathscr{R}} = \lambda'(Mu) = \int_{a-r}^a [d\{\lambda'(M(s) - M(a))\}]\, u(s)$$

and

$$\langle Nx, \lambda' \rangle_{\mathscr{R}} = \lambda'(Nx) = \int_a^b [d(\lambda' N(s))]\, x(s),$$

where $(M^*\lambda')(t) - (M^*\lambda')(a) = \lambda'(M(t) - M(a)) \in \mathscr{V}_n^0(a - r, a)$ and $(N^*\lambda')(t) = \lambda' N(t) \in \mathscr{V}_n^0(a, b)$.

The adjoint problem is equivalent to the conjugate problem (P^*) given by (2,12), (2,13) with M^* and N^* defined above. Moreover, we may write it in the form more similar to the adjoint of the boundary value problem for ordinary integro-differential equation ([12]). Let us put for $t \in [a, b]$

$$\tilde{M} = N(a+) - M(a-), \quad \tilde{N} = -N(b-),$$

$$C(t) = G(t, a+) - G(t, a-), \quad D(t) = -G(t, b-),$$

$$L(s) = \begin{cases} N(a+) & \text{for } s = a, \\ N(s) & \text{for } a < s < b, \\ N(b-) & \text{for } s = b, \end{cases} \qquad G_0(t, s) = \begin{cases} G(t, a+) & \text{for } s = a, \\ G(t, s) & \text{for } a < s < b, \\ G(t, b-) & \text{for } s = b. \end{cases}$$

Then, requiring $y'(a+) = y'(a)$, $y'(b-) = y'(b)$ (cf. Remark 2,11) we obtain the conjugate problem (P*) to (P) in the following form:

$$\int_a^b y'(s)\, A(s)\, ds + \int_{t+r}^b y'(s)\, B(s)\, ds - \int_a^b y'(s)\, \dot{G}(s, t)\, ds + \lambda' M(t) = 0, \quad \text{on } [a - r, a),$$

$$y'(t) = y'(b) + \int_t^b y'(s)\, A(s)\, ds + \begin{cases} \int_{t+r}^b y'(s)\, B(s)\, ds, & t \leqq b - r \\ 0 & , t > b - r \end{cases} -$$

$$- \int_a^b y'(s)\, (G_0(s, t) - G_0(s, b))\, ds + \lambda'(L(t) - L(b)) \quad \text{on } [a, b],$$

$$y'(a) = \lambda' \tilde{M} - \int_a^b y'(s)\, C(s)\, ds, \quad y'(b) = -\lambda' \tilde{N} + \int_a^b y'(s)\, D(s)\, ds.$$

3,4. Boundary value type problems for functional-differential equations of retarded type. In this section we shall deal with boundary value problems for standard functional-differential equation

(3,5) $$\dot{x}(t) = \int_{-r}^0 [d_\vartheta P(t, \vartheta)]\, x(t + \vartheta) + f(t) \quad \text{a.e. on } [a, b],$$

(3,6) $$x(t) = u(t) \quad \text{on } [a - r, a],$$

(3,7) $$Mu + Nx = l \in \Lambda,$$

where the initial functions $u(t)$ are continuous on $[a - r, a]$ and the following assumptions are fulfilled:

$P(t, \vartheta)$ is a Borel measurable in $(t, \vartheta) \in [a, b] \times (-\infty, +\infty)$ $n \times n$-matrix function such that $P(t, \vartheta) = P(t, -r)$ for $\vartheta \leqq -r$, $P(t, \vartheta) = P(t, 0)$ for $\vartheta \geqq 0$, $\text{var}_{-r}^0 P(t, \cdot) < \infty$ for all $t \in [a, b]$ and

$$\int_a^b (\|P(t, 0)\| + \text{var}_{-r}^0 P(t, \cdot))\, dt < \infty.$$

Λ is a B-space and the operators $M : \mathscr{C}_n(a - r, a) \to \Lambda$ and $N : \mathscr{AC}_n(a, b) \to \Lambda$ are linear and bounded, while $\operatorname{Im}(N^*) \subset \mathscr{V}_n^0(a, b)$. Furthermore, $l \in \Lambda$ and $f(t) \in \mathscr{L}_n(a, b)$. We may also assume that $P(t, \cdot)$ is right continuous on $(-r, 0)$ and $P(t, 0) = 0$ for any $t \in [a, b]$.

Let us put for $t \in [a, b]$

$$B(t) = P(t, -r+) - P(t, -r), \quad G(t, s) = \begin{cases} P(t, -r+) & \text{if } s \leqq t - r, \\ P(t, s - t) & \text{if } t - r \leqq s \leqq t, \\ P(t, 0) = 0 & \text{if } s \geqq t. \end{cases}$$

Then $B(t)$ and $G(t, s)$ fulfil Assuptions 2,1. Moreover, given $t \in [a, b]$, $G(t, \cdot)$ is right continuous on $(a - r, b)$, $G(t, b) = 0$ and

$$\int_{-r}^{0} [\mathrm{d}_\vartheta P(t, \vartheta)]\, x(t + \vartheta) = \int_{t-r}^{t} [\mathrm{d}_s P(t, s - t)]\, x(s) =$$
$$= B(t)\, x(t - r) + \int_{a-r}^{b} [\mathrm{d}_s G(t, s)]\, x(s)\,.$$

The problem (3,5)–(3,7) is reduced to the problem of the type (P). Furthermore, for $t \in [a - r, a]$

$$\int_{t+r}^{b} y'(s)\, B(s)\, \mathrm{d}s - \int_{a}^{b} y'(s)\, G(s, t)\, \mathrm{d}s = \int_{t+r}^{b} y'(s)\, (P(s, -r+) - P(s, -r))\, \mathrm{d}s -$$
$$- \int_{a}^{t+r} y'(s)\, P(s, t - s)\, \mathrm{d}s - \int_{t+r}^{b} y'(s)\, P(s, -r+)\, \mathrm{d}s = - \int_{a}^{b} y'(s)\, P(s, t - s)\, \mathrm{d}s\,.$$

Analogously for $t \in (a, b - r)$

$$\int_{t+r}^{b} y'(s)\, B(s)\, \mathrm{d}s - \int_{a}^{b} y'(s)\, G(s, t)\, \mathrm{d}s = - \int_{t}^{b} y'(s)\, P(s, t - s)\, \mathrm{d}s$$

and

$$- \int_{a}^{b} y'(s)\, G(s, t)\, \mathrm{d}s = - \int_{t}^{b} y'(s)\, P(s, t - s)\, \mathrm{d}s \quad \text{for} \quad t \in [b - r, b]\,.$$

The following theorem is now a direct consequence of Theorem 2,5.

3,5. Theorem. *The problem of finding $y \in \mathscr{BV}_n(a, b)$ right continuous on (a, b) (the values $y(a)$, $y(b)$ may be arbitrary) and $\lambda \in \Lambda^*$ such that*

$$(3,8) \qquad - \int_{a}^{b} y'(s)\, P(s, t - s)\, \mathrm{d}s + (M^* \lambda)(t) = 0 \quad on \quad [a - r, a)\,,$$

$$(3,9) \qquad y'(t) = - \int_{t}^{b} y'(s)\, P(s, t - s)\, \mathrm{d}s + (N^* \lambda)(t) \quad on \quad (a, b)$$

is equivalent to the adjoint problem to the problem (3,5)–(3,7).

(*The functions* $(M^*\lambda)(t)$ *and* $(N^*\lambda)(t)$ *are again such that for any* $\lambda \in \Lambda^*$ $(M^*\lambda)(t) - (N^*\lambda)(a) \in \mathscr{V}_n^0(a - r, a)$, $(N^*\lambda)(t) \in \mathscr{V}_n^0(a, b)$ *and*

$$\langle Mu, \lambda\rangle_\Lambda = \int_{a-r}^{a} [\mathrm{d}\{(M^*\lambda)(t) - (M^*\lambda)(a)\}]\, u(t),$$

$$\langle Nx, \lambda\rangle_\Lambda = \int_a^b [\mathrm{d}(N^*\lambda)(t)]\, x(t)$$

for all $u \in \mathscr{C}_n(a - r, a)$, $x \in \mathscr{AC}_n(a, b)$ *and* $\lambda \in \Lambda^*$.)

3,6. Two-point boundary value type problem. Let us consider the "two-point" boundary value type problem given by the system (3,5), (3,6) and

(3,10) $$Mu + N_b x = l \in \Lambda,$$

where the functions $P(t, \vartheta)$, $f(t)$ and the operator M satisfy the corresponding assumptions of Section 3,4. Given $\lambda \in \Lambda^*$, let $(M^*\lambda)(t)$ denote now a function from $\mathscr{V}_n^0(a - r, a)$ such that

$$\langle Mu, \lambda\rangle_\Lambda = \int_{a-r}^{a} [\mathrm{d}(M^*\lambda)(t)]\, u(t)$$

for all $u \in \mathscr{C}_n(a - r, a)$ and $\lambda \in \Lambda^*$. The operator $N_b = NS_b : \mathscr{AC}_n(a, b) \to \Lambda$ is the composition of a linear bounded operator $N : \mathscr{C}_n(b - r, b) \to \Lambda$ and of a shift operator $S_b : x \in \mathscr{AC}_n(a, b) \to x/[b - r, b] \in \mathscr{C}_n(b - r, b)$ (which is also linear and bounded). Let $0 < r \leqq b - a$.

Let $x \in \mathscr{AC}_n(a, b)$ and $\lambda \in \Lambda^*$. Then

$$\langle N_b x, \lambda\rangle_\Lambda = \langle S_b x, N\lambda\rangle_{\mathscr{C}} = \int_{b-r}^{b} [\mathrm{d}(N^*\lambda)(t)]\, x(t)$$

where $(N^*\lambda)(t) \in \mathscr{V}_n^0(b - r, b)$, and putting

$$(\tilde{N}^*\lambda)(t) = \begin{cases} (N^*\lambda)(b - r+) & \text{for } t = b - r, \\ (N^*\lambda)(t) & \text{for } b - r < t \leqq b \end{cases}$$

and

$$(N_b^*\lambda)(t) = \left.\begin{cases} (N^*\lambda)(b - r) & \text{for } a \leqq t < b - r \\ (\tilde{N}^*\lambda)(t) & \text{for } b - r \leqq t \leqq b \end{cases}\right\} \in \mathscr{V}_n^0(a, b),$$

we get finally

$$\langle N_b x, \lambda\rangle_\Lambda = \int_a^b [\mathrm{d}(N_b^*\lambda)(t)]\, x(t).$$

Since all the assumptions of Section 3,4 are satisfied, the following assertion is an immediate consequence of Theorem 3,5.

3,7. Corollary. *The problem of finding* $y \in \mathscr{BV}_n(a, b)$ *right continuous on* (a, b) *(the values* $y(a)$, $y(b)$ *may be arbitrary) and* $\lambda \in \Lambda^*$ *such that*

$$(3,11) \qquad \int_a^b y`(s)\, P(s, t - s)\, ds - (M^*\lambda)(t) = (N^*\lambda)(b - r) \quad \text{on} \quad [a - r, a),$$

$$(3,12) \qquad y`(t) + \int_t^b y`(s)\, P(s, t - s)\, ds = (N^*\lambda)(b - r) \quad \text{on} \quad (a, b - r),$$

$$(3,13) \qquad y`(t) + \int_t^b y`(s)\, P(s, t - s)\, ds - (\tilde{N}^*\lambda)(t) = 0 \quad \text{on} \quad [b - r, b)$$

is equivalent to the adjoint problem to the two-point problem (3,5), (3,6), (3,10).

3,8. Relationship with the adjoint of D. Henry. Let us continue the investigation of the two-point boundary value type problem (3,5), (3,6), (3,10). We shall show that the adjoint problem (3,11) derived in 3,6 can be reduced to the form of D. Henry [8]. Let us put for $\vartheta \in [-r, 0]$ $P(t, \vartheta) = P(t + b - a, \vartheta)$ if $t \in [a - r, a)$. Given a function $z(t)$ defined on $[a - r, b]$ and $t \in [a, b]$, we put

$$z_t^0(\alpha) = \begin{cases} z(t + \alpha) & \text{if} \quad \alpha \in [-r, 0), \\ 0 & \text{if} \quad \alpha = 0. \end{cases}$$

Let $\mathscr{V}_n(-r, 0)$ be the space of all row n-vector functions of bounded variation on $[-r, 0]$ and right continuous on $(-r, 0)$. Let $R(\beta, \alpha)$ be the resolvent kernel for the Volterra integral equation

$$z`(\alpha) + \int_\alpha^0 z`(\beta)\, P(b + \beta, \alpha - \beta)\, d\beta = 0, \quad \alpha \in [-r, 0].$$

Gronwall's inequality applied to the „resolvent equation"

$$R(\beta, \alpha) + \int_\beta^\alpha R(\beta, \gamma)\, P(b + \gamma, \alpha - \gamma)\, d\gamma = P(b + \beta, \alpha - \beta); \quad \alpha, \beta \in [-r, 0]$$

yields analogously as in the proof of Lemma 1 in [14] that $\operatorname{var}_{-r}^0 R(\beta, \cdot) < \infty$ for any $\beta \in [-r, 0]$, while the function $r(\beta) = \operatorname{var}_{-r}^0 R(\beta, \cdot)$ is bounded on $[-r, 0]$. Hence the resolvent operator

$$R : w`(\alpha) \in V_n(-r, 0) \to \int_\alpha^0 w`(\beta)\, R(\beta, \alpha)\, d\beta \in \mathscr{V}_n(-r, 0)$$

is linear and bounded and for any $w` \in \mathscr{V}_n(-r, 0)$, the unique solution $z`(\alpha)$ on $[-r, 0]$ to

$$z`(\alpha) + \int_\alpha^0 z`(\beta)\, P(b + \beta, \alpha - \beta)\, d\beta = w`(\alpha)$$

is given by

$$z' = w' - Rw' = (I - R)\, w',$$

where I denotes the identity operator.

Now, let (y', λ) be a solution to (3,11)–(3,13). Let us extend the function $y'(t)$ on the interval $[a - r, a]$ in such a way that

$$(3{,}15) \qquad y'(t) + \int_t^a y'(s)\, P(s, t - s)\, ds = -(M^*\lambda)(t) \quad \text{for} \quad t \in [a - r, a)$$

and

$$(3{,}16) \qquad y'(a) + \int_a^b y'(s)\, P(s, a - s)\, ds = (N^*\lambda)(b - r).$$

Since

$$\int_t^a y'(s)\, P(s, t - s)\, ds = \int_{t-a}^0 y'(a + \beta)\, P(a + \beta, t - a - \beta)\, d\beta =$$

$$= \int_\alpha^0 y'(a + \beta)\, P(b + \beta, \alpha - \beta)\, d\beta, \quad \text{where} \quad \alpha = t - a,$$

(3,15) yields

$$(3{,}17) \qquad y_a'^0 = -(I - R)(M^*\lambda).$$

The last equation (3,13) in our conjugate system is obviously equivalent to the condition

$$(3{,}18) \qquad y_b'^0 = (I - R)(\tilde{N}^*\lambda).$$

Finally, owing to (3,15) and (3,16) the equations (3,11) and (3,12) can be replaced by the single equation

$$(3{,}19) \qquad y'(t) + \int_t^b y'(s)\, P(s, t - s)\, ds = (N^*\lambda)(b - r) \quad \text{on} \quad [a - r, b - r).$$

The system (3,17)–(3,19) is just the adjoint problem of D. Henry from [8]. (Only we have the expression depending on λ instead of an arbitrary constant on the right hand side of the Volterra integral equation on $[a - r, b - r)$.)

Obviously the couple (y', λ) being a solution to the system (3,17)–(3,19), it is a solution to (3,11)–(3,13).

3,9. Periodic solutions. Let $a = 0$, $b = T < \infty$ $(r \leqq T)$. Let $P(\cdot, \vartheta)$ be for any $\vartheta \in [-r, 0]$ a T-periodic function on $(-\infty, +\infty)$. Let us consider the periodic problem consisting of the equations (3,5), (3,6) and

$$(3{,}20) \qquad u(t) - x(T + t) = 0 \quad \text{for} \quad t \in [-r, 0]$$

(i.e., in (3,10) we have $A = \mathscr{C}_n(-r, 0)$, $l = 0$, $M = I$, $N_T = NS_T$, $N : z(t) \in \in \mathscr{C}_n(T - r, T) \to -z(T + s) \in \mathscr{C}_n(-r, 0)$.)

By Corollary 3,7 the adjoint problem is equivalent to the system of equations for $y^{\backprime}(t)$ of bounded variation on $[-r, T]$ and right continuous on $(-r, T - r) \cup \cup (T - r, T)$ and for $\lambda^{\backprime}(t) \in \mathscr{V}_n^0(-r, 0)$,

$$(3,21) \qquad y^{\backprime}(t) + \int_t^T y^{\backprime}(s)\, P(s, t - s)\, \mathrm{d}s = -\lambda^{\backprime}(-r) \quad \text{on} \quad [-r, T - r],$$

$$(3,22) \qquad y^{\backprime}(t) + \int_t^0 y^{\backprime}(s)\, P(s, t - s)\, \mathrm{d}s = -\lambda^{\backprime}(t) \quad \text{on} \quad [-r, 0),$$

$$(3,23) \qquad y^{\backprime}(t) + \int_t^T y^{\backprime}(s)\, P(s, t - s)\, \mathrm{d}s = -\lambda^{\backprime}(t - T) \quad \text{on} \quad [T - r, T).$$

Indeed, since actually we are looking for $y^{\backprime}(t)$ in the space $\mathscr{L}_n^\infty(-r, T)$, we may change the values of $y^{\backprime}$ on a set of measure zero in $[-r, T]$. Hence we may put

$$y^{\backprime}(0) + \int_0^T y^{\backprime}(s)\, P(s, -s)\, \mathrm{d}s = -\lambda^{\backprime}(-r)$$

and

$$y^{\backprime}(T - r) + \int_{T-r}^T y^{\backprime}(s)\, P(s, T - r - s)\, \mathrm{d}s = -\lambda^{\backprime}(-r).$$

($P(s, -s+) = P(s, -s)$ for any $s \neq r$ and thus $y^{\backprime}(0+) = y^{\backprime}(0)$.)

Furthermore, since by the periodicity assumption on $P(\cdot, \vartheta)$

$$\int_t^T y^{\backprime}(s)\, P(s, t - s)\, \mathrm{d}s = \int_{t-T}^0 y^{\backprime}(T + \beta)\, P(\beta, t - T - \beta)\, \mathrm{d}\beta \quad \text{for} \quad t \in [T - r, T],$$

the system (3,22), (3,23) is equivalent to the condition

$$(3,24) \qquad y^{\backprime}(t) = y^{\backprime}(T + t) \quad \text{for} \quad t \in [-r, 0).$$

3,10. Corollary. *The adjoint to the periodic problem* (3,5), (3,6), (3,20) *is equivalent to the problem of finding* $y(t) \in \mathscr{BV}_n(-r, T)$ *right continuous on* $(-r, T - r) \cup \cup (T - r, T)$ *which satisfies* (3,21) *and* (3,24), *where* $\lambda^{\backprime}(-r)$ *stands for an arbitrary constant n-vector.*

(*In other words, the problem of finding T-periodic solutions to the equation*

$$y^{\backprime}(t) + \int_t^T y^{\backprime}(s)\, P(s, t - s)\, \mathrm{d}s = \text{const}$$

is a well posed adjoint problem to the problem of finding T-periodic solutions to the equation (3,5).)

4. BOUNDARY VALUE TYPE PROBLEMS FOR HEREDITARY DIFFERENTIAL EQUATIONS OF THE DELFOUR-MITTER TYPE

4,1. Notation. Let $-\infty < \alpha < \beta < +\infty$. $\mathscr{L}_n^2(\alpha, \beta)$ is the Hilbert space of square integrable (column) n-vector functions on $[\alpha, \beta]$ with the inner product

$$u, v \in \mathscr{L}_n^2(\alpha, \beta) \to (u, v)_{\mathscr{L}} = \int_\alpha^\beta u^*(s)\, v(s)\, \mathrm{d}s = \int_\alpha^\beta v^*(s)\, u(s)\, \mathrm{d}s\,.$$

(The corresponding norm on $\mathscr{L}_n^2(\alpha, \beta)$ is given by

$$u \in \mathscr{L}_n^2(\alpha, \beta) \to \|u\|_{\mathscr{L}^2} = \left(\int_\alpha^\beta \|u(s)\|^2\, \mathrm{d}s\right)^{1/2}.$$

$\mathscr{W}_n^{1,2}(\alpha, \beta)$ is the Hilbert space of functions $x : [\alpha, \beta] \to \mathscr{R}_n$ which are absolutely continuous on $[\alpha, \beta]$ and whose derivatives Dx are square integrable on $[\alpha, \beta]$. The inner product and the corresponding norm are on $\mathscr{W}_n^{1,2}(\alpha, \beta)$ given by

$$x, y \in \mathscr{W}_n^{1,2}(\alpha, \beta) \to (x, y)_{\mathscr{W}} = (Dx, Dy)_{\mathscr{L}} + (x, y)_{\mathscr{L}}$$

and

$$x \in \mathscr{W}_n^{1,2}(\alpha, \beta) \to \|x\|_{\mathscr{W}} = (\|Dx\|_{\mathscr{L}^2}^2 + \|x\|_{\mathscr{L}^2}^2)^{1/2}\,.$$

The corresponding spaces of row vector functions will be denoted also by $\mathscr{L}_n^2(\alpha, \beta)$ and $\mathscr{W}_n^{1,2}(\alpha, \beta)$. No misunderstanding may arise.

4,2. Assumptions. Let $-\infty < a < b < +\infty$ and $r > 0$. Let $A(t)$ and $B(t)$ be $n \times n$-matrix functions essentially bounded on $[a, b]$ and $f(t) \in \mathscr{L}_n^2(a, b)$, let M and N be constant $m \times n$-matrices and $l \in R_m$. Let Λ be an arbitrary B-space, $w \in \Lambda$ and let $P : \mathscr{L}_n^2(a - r, a) \to \Lambda$ and $Q : \mathscr{W}_n^{1,2}(a, b) \to \Lambda$ be linear and bounded operators.

4,3. Problem (π). The subject of this paragraph is the following boundary value type problem (π)

Determine $x \in \mathscr{W}_n^{1,2}(a, b)$, $\xi \in \mathscr{R}_n$ and $u \in \mathscr{L}_n^2(a - r, a)$ in such a way that

$$\dot{x}(t) - A(t)\, x(t) - \begin{Bmatrix} B(t)\, u(t - r), & t < a + r \\ B(t)\, x(t - r), & t \geqq a + r \end{Bmatrix} = f(t) \quad \text{a.e. on} \quad [a, b]\,, \tag{4,1}$$

$$Pu + Qx = w\,, \tag{4,2}$$

$$M\xi + N\, x(b) = l\,, \tag{4,3}$$

$$x(a) - \xi = 0\,. \tag{4,4}$$

Let $\mathscr{W} = \mathscr{W}_n^{1,2}(a, b) \times \mathscr{R}_n \times \mathscr{L}_n^2(a - r, a)$, $\mathscr{Z} = \mathscr{L}_n^2(a, b) \times \Lambda \times \mathscr{R}_m \times \mathscr{R}_n$ and let the operators $D, A, B_1 : \mathscr{W}_n^{1,2}(a, b) \to \mathscr{L}_n^2(a, b)$ and $B_2 : \mathscr{L}_n^2(a - r, a) \to$ $\to \mathscr{L}_n^2(a, b)$ be defined analogously as in 2,3 and

$$U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix} \in \mathscr{W} \to \begin{bmatrix} Dx - Ax - B_1 x - B_2 u \\ Pu + Qx \\ M\xi + N\, x(b) \\ x(a) - \xi \end{bmatrix} \in \mathscr{Z}\,.$$

The operator U is clearly linear and bounded and the given problem (π) is equivalent to the operator equation

$$U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix} = \begin{bmatrix} f \\ w \\ l \\ 0 \end{bmatrix}.$$

4,4. Remark. The corresponding initial value problem (4,1) and (4,4) (with $u \in$ $\in \mathscr{L}_n^2(a - r, a)$ and $\xi \in \mathscr{R}_n$ fixed) was studied in [3].

4,5. Theorem. *Let* $\eta' \in \mathscr{L}_n^2(a, b)$, $\lambda \in \Lambda^*$, $\gamma' \in \mathscr{R}_m^*$ *and* $\delta' \in \mathscr{R}_n^*$. *Then* $(\eta', \lambda, \gamma', \delta') \in$ $\in \operatorname{Ker}(U^*)$ *iff there exists* $y' \in \mathscr{L}_n^2(a, b)$ *such that* $y' + (\mathrm{d}/\mathrm{d}t)(Q^*\lambda) \in \mathscr{AC}_n(a, b)$, $y(t) = \eta(t)$ *a.e. on* $[a, b]$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y' + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](t) = -y'(t)\, A(t) - \begin{Bmatrix} y'(t + r)\, B(t + r), & t < b - r \\ 0, & t > b - r \end{Bmatrix} +$$
$$+ (Q^*\lambda)(t) \quad \text{a.e. on} \quad [a, b]\,,$$

$$\left[y' + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](a) = \gamma' M\,, \quad \left[y' + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](b) = -\gamma' N\,,$$

$$y'(t + r)\, B(t + r) - (P^*\lambda)(t) = 0 \quad \text{a.e. on} \quad [a - r, a]\,,$$

while $\delta' = \gamma' M$ $(P^* : \Lambda^* \to \mathscr{L}_n^2(a - r, a)$ *and* $Q^* : \Lambda^* \to \mathscr{W}_n^{1,2}(a, b)$ *are the adjoints to* P *and* $Q)$.

Proof. Let $\eta' \in \mathscr{L}_n^2(a, b)$, $\lambda \in \Lambda^*$, $\gamma' \in \mathscr{R}_m^*$ and $\delta' \in \mathscr{R}_n^*$. Then $(\eta', \lambda, \gamma', \delta') \in$ $\in \operatorname{Ker}(U^*)$ iff for any $(x, \xi, u) \in \mathscr{W}$

$$0 = \left(\begin{bmatrix} x \\ \xi \\ u \end{bmatrix}, U^*(\eta', \lambda, \gamma', \delta')\right)_{\mathscr{W}} = \left(U \begin{bmatrix} x \\ \xi \\ u \end{bmatrix}, (\eta', \lambda, \gamma', \delta')\right)_{\mathscr{L}} =$$
$$= \int_a^b \eta'(t)\, \dot{x}(t)\, \mathrm{d}t - \int_a^b \eta'(t)\, A(t)\, x(t)\, \mathrm{d}t - \int_a^{b-r} \eta'(t + r)\, B(t + r)\, x(t)\, \mathrm{d}t -$$
$$- \int_{a-r}^a \eta'(t + r)\, B(t + r)\, u(t)\, \mathrm{d}t + \gamma'(M\xi + N\, x(b)) + \delta'(x(a) - \xi) +$$

$$+ (u, P^*\lambda)_{\mathscr{L}} + (x, Q^*\lambda)_{\mathscr{W}} =$$

$$= \int_a^b \eta'(t)\,\dot{x}(t)\,\mathrm{d}t + \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)(t)\right]\dot{x}(t)\,\mathrm{d}t - \int_a^b p'(t)\,x(t)\,\mathrm{d}t +$$

$$+ \gamma' N\, x(b) + \delta'\, x(a) - \int_{a-r}^a q'(t)\,u(t)\,\mathrm{d}t + (\gamma' M - \delta')\,\xi\,,$$

where

$$p'(t) = \eta'(t)\,A(t) + \begin{Bmatrix} \eta'(t+r)\,B(t+r), & t < b - r \\ 0, & t \geqq b - r \end{Bmatrix} - (Q^*\lambda)(t) \quad \text{on} \quad [a, b]$$

and

$$q'(t) = \eta'(t+r)\,B(t+r) - (P^*\lambda)(t) \quad \text{on} \quad [a - r, a]\,.$$

In particular, putting $\xi = 0$ and $x(t) = 0$ on $[a, b]$, we get

$$(4{,}5) \qquad \eta'(t+r)\,B(t+r) - (P^*\lambda)(t) = 0 \quad \text{a.e. on} \quad [a - r, a]\,.$$

Furthermore, putting $x(t) = 0$ on $[a, b]$ and $u(t) = 0$ on $[a - r, a]$, we get

$$(4{,}6) \qquad \gamma' M - \delta' = 0\,.$$

Let us put

$$g'(t) = \begin{Bmatrix} \int_t^b p'(s)\,\mathrm{d}s - \gamma' N - \delta', & t = a \\ \int_t^b p'(s)\,\mathrm{d}s - \gamma' N & , a < t < b \\ 0 & , t = b \end{Bmatrix} \in \mathscr{V}_n^0(a, b)\,.$$

Then, in virtue of the integration-by-parts formula,

$$0 = \int_a^b \left\{\eta'(t) + \left[\frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)(t)\right]\right\}\dot{x}(t)\,\mathrm{d}t + \int_a^b [\mathrm{d}g'(t)]\,x(t) =$$

$$= \int_a^b \left\{\eta'(t) + \left[\frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)(t)\right] - g'(t)\right\}\dot{x}(t)\,\mathrm{d}t - g'(a)\,x(a)$$

for all $x \in \mathscr{AC}_n(a, b)$. Again, we deduce that

$$(4{,}7) \quad g'(a) = \int_a^b y'(s)\,A(s)\,\mathrm{d}s + \int_{a+r}^b y'(s)\,B(s)\,\mathrm{d}s - \int_a^b (Q^*\lambda)(s)\,\mathrm{d}s - \gamma' N - \delta' = 0$$

and

$$(4{,}8) \qquad y'(t) + \left[\frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)(t)\right] = \int_t^b y'(s)\,A(s)\,\mathrm{d}s - \int_t^b (Q^*\lambda)(s)\,\mathrm{d}s - \gamma' N +$$

$$+\begin{cases}\int_{t+r}^{b} y'(s)\,B(s)\,\mathrm{d}s, & t < b - r\\ 0 & , t \geqq b - r\end{cases} \quad \text{on} \quad (a, b)$$

for some $y' \in \mathscr{L}_n^2(a, b)$, $y'(t) = \eta'(t)$ a.e. on $[a, b]$.

By (4,8), $[y' + (\mathrm{d}/\mathrm{d}t)(Q^*\lambda)](a+)$ and $[y' + (\mathrm{d}/\mathrm{d}t)(Q^*\lambda)](b-)$ exist,

$$\left[y' + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](b-) = -\gamma' N$$

and according to (4,6) and (4,7)

$$\left[y' + \frac{\mathrm{d}}{\mathrm{d}t}(Q^*\lambda)\right](a+) = \delta' = \gamma' M .$$

The theorem easily follows.

4,6. Corollary. *Let the operator Q in (4,2) be a linear and bounded mapping of $\mathscr{L}_n^2(a, b)$ into Λ. Then $(\eta', \lambda, \gamma', \delta') \in \mathrm{Ker}(U^*)$ iff there is $y' \in \mathscr{AC}_n(a, b)$ such that $y'(t) = \eta'(t)$ a.e. on $[a, b]$ and*

$$\dot{y}'(t) = -y'(t)\,A(t) - \begin{cases} y'(t + r)\,B(t + r), & t < b - r\\ 0 & , t > b - r\end{cases} + (Q^*\lambda)(t) \ \text{ a.e. on } \ [a, b],$$

$$y'(a) = \gamma' M , \quad y'(b) = -\gamma' N ,$$

$$-y'(t + r)\,B(t + r) + (P^*\lambda)(t) = 0 \quad \text{a.e. on} \quad [a - r, a]$$

($P^* : \Lambda^* \to \mathscr{L}_n^2(a - r, a)$ and $Q^* : \Lambda^* \to \mathscr{L}_n^2(a, b)$ are adjoints of P and Q).

Proof. Since for all $x \in \mathscr{L}_n^2(a, b)$ and $\lambda \in \Lambda^*$

$$\langle Qx, \lambda\rangle_\Lambda = (x, Q^*\lambda)_{\mathscr{L}} = \int_a^b (Q^*\lambda)(t)\,x(t)\,\mathrm{d}t ,$$

the term $[(\mathrm{d}/\mathrm{d}t)(Q^*\lambda)]$ does not appear in the formula (4,8).

4,7. Remark. Let $Q : \mathscr{L}_n^2(a, b) \to \Lambda$ be linear and bounded. Then Q is also bounded as an operator $\mathscr{W}_n^{1,2}(a, b) \to \Lambda$ and apparently we have two possible adjoint problems, defined in Theorem 4,5 and Corollary 4,6, respectively. We must take into account that in this case we should write $\tilde{Q}^*$ instead of Q^* in the former adjoint, where $\tilde{Q} = QE$ and $E : x \in \mathscr{W}_n^{1,2}(a, b) \to x \in \mathscr{L}_n^2(a, b)$ is a continuous imbedding of $\mathscr{W}_n^{1,2}(a, b)$ into $\mathscr{L}_n^2(a, b)$. (Given $\lambda \in \Lambda^*$ and $x \in \mathscr{W}_n^{1,2}(a, b)$),

$$\int_a^b (Q^*\lambda)(t)\,x(t)\,\mathrm{d}t = \int_a^b \left\{\left[\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{Q}^*\lambda)(t)\right]\dot{x}(t) + (\tilde{Q}^*\lambda)(t)\,x(t)\right\}\mathrm{d}t .$$

5. REMARKS ON THE CLOSEDNESS OF Im (U)

All the boundary value type problems which occur in paragraphs 2 and 3 of this paper may be formulated as operator equations of the type

$$U\xi = \eta\,,$$

where U is a linear bounded mapping of either $\mathscr{X}_c = \mathscr{AC}_n(a, b) \times \mathscr{C}_n(a - r, a)$ or $\mathscr{X}_v = \mathscr{AC}_n(a, b) \times \mathscr{BV}_n(a - r, a)$ into $\mathscr{Y} = \mathscr{L}_n(a, b) \times \Lambda \times \mathscr{R}_n$ and Λ is a B-space. The aim of this paragraph is to characterize in some special cases the range Im (U) of the operator U and, in particular, to find some conditions guaranteeing the closedness of Im (U).

Let $(\mathscr{C} + \mathscr{BV})_n\,(a - r, a)$ denote the set of all functions $w : [a - r, a] \to \mathscr{R}_n$ for which there exist functions $u \in \mathscr{C}_n(a - r, a)$ and $v \in \mathscr{BV}_n(a - r, a)$ such that $w(t) = u(t) + v(t)$ on $[a - r, a]$.

In what follows we make use of the following lemma which is a slight modification of the variation-of-constants formula due to H. T. Banks [1].

5,1. Lemma. *Let the $n \times n$-matrix function $P(t, \vartheta)$ fulfil the corresponding assumptions from* Sec. 3,4. *Given $f \in \mathscr{L}_n(a, b)$ and $u \in (\mathscr{C} + \mathscr{BV})_n\,(a - r, a)$, there is just one solution to the initial value problem* ((3,5), (3,6))

$$\dot{x}(t) = \int_{-r}^{0} [\mathrm{d}_\vartheta P(t, \vartheta)]\, x(t + \vartheta) + f(t) \quad \text{a.e. on} \quad [a, b]\,,$$
$$x(t) = u(t) \quad \text{on} \quad [a - r, a]\,.$$

There exist a linear operator $\Phi : (\mathscr{C} + \mathscr{BV})_n\,(a - r, a) \to \mathscr{AC}_n(a, b)$ and a linear bounded operator $\Psi : \mathscr{L}_n(a, b) \to \mathscr{AC}_n(a, b)$ such that this solution is given by

$$x = \Phi u + \Psi f\,. \tag{5,1}$$

The operator Φ as a mapping $\mathscr{BV}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ is completely continuous and as a mapping $\mathscr{C}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ bounded. Moreover, if $b - r \geqq a$ and if $S_b : x \in \mathscr{AC}_n(a, b) \to x/[b - r, b] \in \mathscr{C}_n(b - r, b)$, then the operator $T = S_b\Phi : \mathscr{C}_n(a - r, a) \to \mathscr{C}_n(b - r. b)$ is completely continuous.

(The compactness of $\Phi : \mathscr{BV}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ was shown in [13] and the proof of the compactness of T can be find in [7], Remark 8,9.)

5,2. Remark. It follows from the special form of the operator Φ (cf. [1]) that for any $u \in (\mathscr{C} + \mathscr{BV})_n\,(a - r, a)$

$$\Phi u = \Phi^0\, u(a) + \Phi^1 u\,, \tag{5,2}$$

where $\Phi^0 : \mathscr{R}_n \to \mathscr{AC}_n(a, b)$ is linear and bounded and $\Phi^1 : (\mathscr{C} + \mathscr{BV})_n\,(a - r, a) \to \mathscr{AC}_n(a, b)$ is linear and completely continuous as an operator $\mathscr{BV}_n(a - r, a) \to$

$\to \mathcal{AC}_n(a, b)$ and bounded as an operator $\mathcal{C}_n(a - r, a) \to \mathcal{AC}_n(a, b)$. Moreover, if v is a simple jump function $v(t) = 0$ on $[a - r, a)$ and $v(a) = d$, then $\Phi^1 v = 0$.

5,3. Problem (3,5)–(3,7). Let us turn back to the problem (3,5)–(3,7) whose adjoint was derived in Sec. 3,4. Let Λ be an arbitrary B-space and let the operators $M : \mathcal{C}_n(a - r, a) \to \Lambda$ and $N : \mathcal{AC}_n(a, b) \to \Lambda$ and the $n \times n$-matrix function $P(t, \vartheta)$ fulfil the assumptions of Sec. 3,4. Let $f \in \mathcal{L}_n(a, b)$ and $l \in \Lambda$. Let us put $\mathcal{X}_c = \mathcal{AC}_n(a, b) \times \mathcal{C}_n(a - r, a)$, $\mathcal{Y} = \mathcal{L}_n(a, b) \times \Lambda \times \mathcal{R}_n$,

$$P_1 : x \in \mathcal{AC}_n(a, b) \to \int_{\max(-r, a-t)}^{0} [\mathrm{d}_\vartheta P(t, \vartheta)]\, x(t + \vartheta) \in \mathcal{L}_n(a, b)\,,$$

$$P_2 : u \in C_n(a - r, a) \to \int_{-r}^{\max(-r, a-t)} [\mathrm{d}_\vartheta P(t, \vartheta)]\, u(t + \vartheta), \in \mathcal{L}_n(a, b)\,,$$

and

(5,3)
$$U : \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}_c \to \begin{bmatrix} Dx - P_1 x - P_2 u \\ Mu + Nx \\ u(a) - x(a) \end{bmatrix} \in \mathcal{Y}$$

(where again $D : x \in \mathcal{AC}_n(a, b) \to \dot{x} \in \mathcal{L}_n(a, b)$). The system (3,5)–(3,7) is equivalent to the operator equation

(5,4)
$$U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ 0 \end{bmatrix}.$$

5,3,1. Theorem. *Let* Im $(M + N\Phi)$ *be closed in* Λ, *then the operator* U *defined by* (5,3) *has closed range* Im (U) *in* $\mathcal{Y}$.

Proof. Let $(f, l, d) \in \mathcal{Y}$. According to the variation-of-constants formula (5,1) a couple $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{X}_c$ is a solution to the equation

$$U \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} f \\ l \\ d \end{bmatrix}$$

iff

$$x = \Phi\tilde{u} + \Psi f = \Phi^0(u(a) + d) + \Phi^1 u + \Psi f = \Phi^0 d + \Phi u + \Psi f\,,$$

where $\tilde{u} = u + u_d$, $u_d(t) = 0$ on $[a - r, a)$, $u_d(a) = d (\Phi^1 u_d = 0$, cf. Remark 5,2) and $u \in \mathcal{C}_n(a - r, a)$ is a solution to the operator equation

$$[M + N\Phi]\, u = -N\Psi f + l - N\Phi^0 d\,.$$

Let us denote

$$S : \begin{bmatrix} f \\ l \\ d \end{bmatrix} \in \mathscr{Y} \to -N\Psi f + l - N\Phi^0 d \in \Lambda .$$

Then $S(\operatorname{Im}(U)) = \operatorname{Im}(M + N\Phi)$ and since the operator S is linear and bounded, our assertion readily follows.

5,3,2. Corollary. *If* $\Lambda = \mathscr{R}_m$, *then* $\operatorname{Im}(U)$ *is closed in* $\mathscr{Y}$.

(In this case $\operatorname{Im}(M + N\Phi)$ is a k-dimencional $(0 \leqq k \leqq m)$ linear subspace of $\mathscr{R}_m$.)

5,3,3. Corollary. *Let* $0 \leqq r \leqq b - a$, $S_b : x \in \mathscr{AC}_n(a, b) \to x \mid [b - r, b] \in \mathscr{C}_n(b - r, b)$ *and let* $\tilde{N} : \mathscr{C}_n(b - r, b) \to \Lambda$ *be linear and bounded. Let the operator* U *be given by* (5,3), *where* N *is replaced by* $N_b = \tilde{N}S_b$. *Then, if the operator* M *posseses a bounded inverse* $M^{-1} : \Lambda \to \mathscr{C}_n(a - r, a)$, *the range* $\operatorname{Im}(U)$ *of* U *is closed in* $\mathscr{Y}$.

Proof. By Theorem 5,3,1 $\operatorname{Im}(U)$ is closed in $\mathscr{Y}$ if the range of the operator

$$M + \tilde{N}S_b\Phi = M + \tilde{N}T : \mathscr{C}_n(a - r, a) \to \Lambda$$

is closed. Since by Lemma 5,1 the operator $T = S_b\Phi : \mathscr{C}_n(a - r, a) \to \mathscr{C}_n(b - r, b)$ is completely continuous, the existence of a bounded M^{-1} implies the closedness of $\operatorname{Im}(M + \tilde{N}T)$ and hence also of $\operatorname{Im}(U)$.

5,3,4. Remark. Our restriction to two-point boundary value type problems in Corollary 5,3,3 does not mean an essential loss of generality (cf. [8]).

5,3,5. Corollary. *The T-periodic problem* (3,5), (3,6), (3,20) (cf. Sec. 3,9) *has a solution iff*

$$\int_0^T y'(s) f(s) \, ds = 0$$

for all T-periodic solutions $y'(t)$ (i.e., $y'(t) = y'(T + t)$ *on* $[-r, 0)$) *of the equation*

$$y'(t) + \int_t^b y'(s) P(s, t - s) \, ds = \text{const.} \quad \text{on} \quad [-r, T - r] .$$

(Proof follows from Corollaries 3,10 and 5,3,3.)

5,3,6. Remark. Let Λ_1 be a B-space and let the operators $M_1 : \mathscr{C}_n(a - r, a) \to \Lambda_1$ and $N_1 : \mathscr{AC}_n(a, b) \to \Lambda_1$ be linear and bounded. If $\Lambda = \mathscr{C}_n(a - r, a) \times \Lambda_1$ and

$$M : u \in \mathscr{C}_n(a - r, a) \to \begin{bmatrix} u \\ M_1 u \end{bmatrix} \in \Lambda , \quad N : x \in \mathscr{AC}_n(a, b) \to \begin{bmatrix} 0 \\ N_1 x \end{bmatrix} \in \Lambda ,$$

then the operator U given by (5,3) has closed range $\text{Im}(U)$ in $\mathscr{Y} = \mathscr{L}_n(a, b) \times \times \mathscr{C}_n(a - r, a) \times \Lambda_1 \times \mathscr{R}_n$. (Indeed, according to Lemma 5,1 an element (f, h, l, d) of $\mathscr{Y}$ belongs to $\text{Im}(U)$ iff

$$F(f, h, l, d) = N_1 \Psi f + (M_1 + N_1 \Phi) h - l + N_1 \Phi^0 d = 0 .$$

It is easy to see that the operator $F : \mathscr{Y} \to \Lambda_1$ is linear and bounded. Consequently, the set $\text{Im}(U) = \text{Ker}(F)$ is closed in $\mathscr{Y}$.)

5,3,7. Remark. All the assertions of this section will remain true if we replace the initial space $\mathscr{C}_n(a - r, a)$ by $\mathscr{BV}_n(a - r, a)$. Moreover, Corollary 5,3,3 could be now formulated directly for a general linear bounded operator $N : \mathscr{AC}_n(a, b) \to \Lambda$. ($N$ need not be of the two-point character $N = \tilde{N} S_b$.) This is possible in virtue of the compactness of the operator $\Phi : \mathscr{BV}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ in the variation-of-constants formula (5,1) (cf. Lemma 5,1).

5,4. Problem (2,1)–(2,3). The subject of this section is the general problem of finding $x \in \mathscr{AC}_n(a, b)$ and $u \in \mathscr{BV}_n(a - r, a)$ which satisfy (2,1)–(2,3). Let Assumptions 2,1 be fulfilled. We make use of the notation introduced in Sec. 2,3. (Only $\mathscr{C}_n(a - r, a)$ should be replaced everywhere by $\mathscr{BV}_n(a - r, a)$.)

5,4,1. Lemma. *Let* $-\infty < c < d < +\infty$ *and let* $K(t, s)$ *be an* $n \times n$*-matrix function defined and Borel measurable in* (t, s) *on* $[a, b] \times [c, d]$ *and such that* $\text{var}_c^d K(t, \cdot) < \infty$ *for any* $t \in [a, b]$, *while*

$$\int_a^b (\text{var}_c^d K(t, \cdot) + \|K(t, d)\|) \, dt < \infty .$$

Then the operator

$$K : u \in \mathscr{BV}_n(c, d) \to \int_c^d [d_s K(t, s)] \, u(s) \in \mathscr{L}_n(a, b)$$

is completely continuous.

Proof. The operator K is surely linear and bounded.

Let $\{u^j\}_{j=1}^{\infty} \subset \mathscr{BV}_n(c, d)$ and $\|u^j\|_{\mathscr{BV}} < 1$ $(j = 1, 2, \ldots)$. Then by Helly's Choice Theorem there exists a subsequence $\{u^{j_l}\} \subset \{u^j\}$ and $u^0 \in \mathscr{BV}_n(c, d)$ such that

$$\lim_{l \to \infty} u^{j_l}(s) = u^0(s) \quad \text{for all} \quad s \in [c, d] .$$

Let us put for $s \in [c, d]$ and $l = 1, 2, \ldots$

$$v^l(s) = \|u^{j_l}(s) - u^0(s)\|$$

and for $t, s \in [a, b] \times [c, d]$

$$k(t, s) = \text{var}_c^s K(t, \cdot) .$$

Then $\|v^l(s)\| \leqq \|u^0\|_{\mathscr{BV}} + 1$ on $[c, d]$ for any $l = 1, 2, \ldots$, $\operatorname{var}_c^d k(t, \cdot) = \operatorname{var}_c^d K(t, \cdot)$ for any $t \in [a, b]$ and by the unsymmetric Fubini theorem

$$\int_a^b \left\| \int_c^d [\mathrm{d}_s K(t, s)] \, (u^{j_l}(s) - u^0(s)) \right\| \mathrm{d}t \leqq \int_a^b \left(\int_c^d [\mathrm{d}_s k(t, s)] \, v^l(s) \right) \mathrm{d}t =$$
$$= \int_c^d \left[\mathrm{d}_s \int_a^b k(t, s) \, \mathrm{d}t \right] v^l(s) \, .$$

Given a subdivision $\{c = s_0 < s_1 \ldots < s_m = d\}$ of $[c, d]$,

$$\sum_{i=1}^m \left\| \int_a^b (k(t, s_i) - k(t, s_{i-1}) \, \mathrm{d}t \right\| \leqq \int_a^b \left(\sum_{i=1}^m \|k(t, s_i) - k(t, s_{i-1})\| \right) \mathrm{d}t \leqq$$
$$\leqq \int_a^b (\operatorname{var}_c^d k(t, \cdot)) \, \mathrm{d}t < \infty \, .$$

Thus

$$\operatorname{var}_c^d \left(\int_a^b k(t, \cdot) \, \mathrm{d}t \right) < \infty$$

and according to the dominated convergence theorem for Perron-Stieltjes integrals

$$\lim_{l \to \infty} \int_c^d \left[\mathrm{d}_s \int_a^b k(t, s) \, \mathrm{d}t \right] v^l(s) = 0$$

or

$$\lim_{l \to \infty} \|K u^{j_l} - K u^0\|_{\mathscr{L}} = \lim_{l \to \infty} \int_a^b \left\| \int_c^d [\mathrm{d}_s K(t, s)] \, (u^{j_l}(s) - u^0(s)) \right\| \mathrm{d}t = 0$$

which completes the proof.

5,4,2. Remark. The operator

$$u \in \mathscr{C}_n(c, d) \to \int_c^d [\mathrm{d}_s K(t, s)] \, u(s) \in \mathscr{L}_n(a, b)$$

(with $K(t, s)$ fulfilling the assumptions of Lemma 5,4,1) need not be generally completely continuous.

5,4,3. Theorem. *If the operator $M : \mathscr{BV}_n(a - r, a) \to \Lambda$ has a bounded inverse M^{-1}, then the operator U given by (2,4) (with $\mathscr{C}_n(a - r, a)$ replaced by $\mathscr{BV}_n(a - r, a)$) has closed range in $\mathscr{Y}$.*

Proof. By Lemma 5,1 applied to initial value problems of the type

$$\dot{x}(t) = A(t) \, x(t) + B(t) \, x(t - r) + g(t) \quad \text{a.e. on} \quad [a, b] \, ,$$
$$x(t) = u(t) \quad \text{on} \quad [a - r, a] \, ,$$

the triple $(f, l, d) \in \mathscr{Y}$ belongs to Im (U) iff there is a solution $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathscr{X}_v = \mathscr{AC}_n(a, b) \times \times \mathscr{BV}_n(a - r, a)$ to the system of operator equations

$$(5,5) \qquad x - \Psi G_1 x - \Phi u - \Psi G_2 u = \Psi f + \Phi^0 d ,$$
$$Mu + Nx = l ,$$

where the operator $\Phi : \mathscr{BV}_n(a - r, a) \to \mathscr{AC}_n(a, b)$ is linear and completely continuous and the operators $\Phi^0 : \mathscr{R}_n \to \mathscr{AC}_n(a, b)$ and $\Psi : \mathscr{L}_n(a, b) \to \mathscr{AC}_n(a, b)$ are linear and bounded. Since there exists a bounded inverse M^{-1} of M, the latter equation in (5,5) yields $u = M^{-1}l - M^{-1}Nx$, while the former becomes

$$x - \{\Phi M^{-1}N + \Psi G_1 + \Psi G_2 M^{-1}N\} u = \Psi f + (\Phi + \Psi G_2) M^{-1} l + \Phi^0 d .$$

Let us put $K = \Phi M^{-1}N + \Psi G_1 + \Psi G_2 M^{-1}N$, $S(f, l, d) = \Psi f + (\Phi + \Psi G_2) . . M^{-1}l + \Phi^0 d$ and let I denote the identity operator on $\mathscr{AC}_n(a, b)$. Then $S(\text{Im}(U)) = = \text{Im}(I - K)$ and since $S : \mathscr{Y} \to \mathscr{AC}_n(a, b)$ is linear and bounded, Im (U) is closed if Im $(I - K)$ is closed. The operators G_1, G_2 are completely continuous by Lemma 5,4,1 and since the operators M^{-1}, N and Ψ are bounded the operator K is also completely continuous and Im $(I - K)$ is closed.

5,4,4. Remark. As an easy consequence of Theorem 5,4,3 we obtain that in the case of the T-periodic problem (i.e. $a = 0$, $b = T$, $r < T$, $\Lambda = \mathscr{AC}_n(-r, 0)$, $M = I$, $N : x \in \mathscr{AC}_n(0, T) \to x_T(s) = x(T + s) \in \mathscr{AC}_n(-r, 0)$ and $l = 0$) the range Im (U) of U is closed in $\mathscr{Y}$.

5,5. Boundary value problems for ordinary integrodifferential equations. If $r = 0$ and $\Lambda = \mathscr{R}_m$, then the given problem (2,1)–(2,3) reduces to the boundary value problem for an ordinary integrodifferential equation of the form

$$(5,6) \qquad \dot{x}(t) = A(t)\, x(t) + \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) + f(t) \quad \text{a.e. on} \quad [a, b] ,$$

$$(5,7) \qquad Nx = l ,$$

where the $n \times n$-matrix function $A(t)$ is $\mathscr{L}$-integrable on $[a, b]$, $\text{var}_a^b\, G(t, \cdot) < \infty$ for any $t \in [a, b]$,

$$\int_a^b (\text{var}_a^b\, G(t, \cdot) + \|G(t, b)\|)\, \mathrm{d}t < \infty ,$$

$f \in \mathscr{L}_n(a, b)$, $l \in \mathscr{R}_m$ and the operator $N : \mathscr{AC}_n(a, b) \to \mathscr{R}_m$ is linear and bounded. (The initial space reduces to $\mathscr{R}_n$.)

Let us reformulate the problem (5,6), (5,7) as the operator equation

$$Ux = \begin{pmatrix} f \\ l \end{pmatrix},$$

where

$$(5,8) \qquad U : x \in \mathscr{AC}_n(a, b) \to \begin{pmatrix} Dx - Ax - Gx \\ Nx \end{pmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_m$$

and the symbols D, A, G have the obvious menaning.

5,5,1. Theorem. *The operator U defined by (5,8) has closed range in $\mathscr{L}_n(a, b) \times \times \mathscr{R}_m$.*

Proof. There exist linear and bounded operators $\Phi^0 : \mathscr{R}_n \to \mathscr{AC}_n(a, b)$ and $\Psi : : \mathscr{L}_n(a, b) \to \mathscr{AC}_n(a, b)$ such that an n-vector function $x(t)$ is a solution to the given problem iff

$$x = \Phi^0 c + \Psi h + \Psi f,$$

where the couple $(h, c) \in \mathscr{L}_n(a, b) \times \mathscr{R}_n$ ($h = Gx$) is a solution to the system

$$(5,9) \qquad \begin{aligned} h - (G\Phi^0)\, c - (G\Psi)\, h &= (G\Psi) f, \\ (N\Phi^0)\, c + (N\Psi)\, h &= l - (N\Psi) f. \end{aligned}$$

$(N\Phi^0)$ is a constant $m \times n$-matrix. Let e.g. $m < n$. Putting

$$Q = I_n - \begin{bmatrix} N\Phi^0 \\ 0_{n-m,n} \end{bmatrix}, \quad \tilde{l} = \begin{bmatrix} l \\ 0_{n-m,1} \end{bmatrix} \in \mathscr{R}_n,$$

$$R : h \in \mathscr{L}_n(a, b) \to \begin{bmatrix} (N\Psi)\, h \\ 0_{n-m,1} \end{bmatrix} \in \mathscr{R}_n$$

($0_{p,q}$ denotes the zero $p \times q$-matrix and I_n is the identity $n \times n$-matrix),

$$K : \begin{pmatrix} h \\ c \end{pmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_n \to \begin{bmatrix} (G\Phi^0)\, c + (G\Psi)\, h \\ Qc - Rh \end{bmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_n$$

and

$$S : \begin{pmatrix} f \\ l \end{pmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_m \to \begin{bmatrix} (G\Psi) f \\ \tilde{l} - Rf \end{bmatrix} \in \mathscr{L}_n(a, b) \times \mathscr{R}_n,$$

the system (5,9) becomes

$$(I - K) \begin{pmatrix} h \\ c \end{pmatrix} = S \begin{pmatrix} f \\ l \end{pmatrix}$$

and $S(\text{Im}\,(U)) = \text{Im}\,(I - K)$. Since by Lemma 5,4,1 the operator G is completely continuous, it is easy to verify that the operator K is completely continuous. It means that $\text{Im}\,(I - K)$ is closed and taking into account that the operator S is linear and bounded we complete the proof. The case $m > n$ can be treated analogously.

Let $N(t)$ be an $m \times n$-matrix function of bounded variation on $[a, b]$ and let the operator N be given by

$$(5,10) \qquad N : x \in \mathscr{AC}_n(a, b) \to \int_a^b [\mathrm{d}N(s)]\, x(s) \in \mathscr{R}_m .$$

Without any loss of generality we may assume that for any $t \in [a, b]$ the functions $G(t, \cdot)$ and N are right-continuous on (a, b). Let us put for $t \in [a, b]$

$$C(t) = G(t, a+) - G(t, a), \quad D(t) = G(t, b) - G(t, b-),$$

$$G_0(t, s) = \begin{cases} G(t, a+) & \text{for } s = a, \\ G(t, s) & \text{for } a < s < b, \\ G(t, b-) & \text{for } s = b, \end{cases} \qquad L(s) = \begin{cases} N(a+) & \text{for } s = a, \\ N(s) & \text{for } a < s < b, \\ N(b-) & \text{for } s = b, \end{cases}$$

$$M = N(a+) - N(a), \quad N = N(b) - N(b-).$$

Then similarly as in Sec. 3,3 we obtain that the adjoint problem to (5,6), (5,7) is equivalent to the problem of finding $y \in \mathscr{BV}_n(a, b)$, right-continuous on $[a, b)$ and left-continuous at b and $\lambda \in \mathscr{R}_n$ such that

$$(5,11) \quad y'(t) = y'(b) + \int_t^b y'(s)\, A(s)\, \mathrm{d}s - \int_a^b y'(s)\, (G_0(s, t) - G_0(s, b))\, \mathrm{d}s + \\ + \lambda'(L(t) - L(b)) \quad \text{on} \quad [a, b],$$

$$(5,12) \quad y'(a) = \lambda' M - \int_a^b y'(s)\, C(s)\, \mathrm{d}s, \quad y'(b) = -\lambda' N + \int_a^b y'(s)\, D(s)\, \mathrm{d}s.$$

The following theorem is then a direct corollary of Theorem 5,5,1.

5,5,2. Theorem. *The problem* (5,6), (5,7) *possesses a solution iff*

$$\int_a^b y'(s) f(s)\, \mathrm{d}s + \lambda' l = 0$$

for any solution $(y'(t), \lambda')$ *of the adjoint problem* (5,11), (5,12).

Theorem 5,5,2 generalizes Theorem 3,1 from [12].

Acknowledgement. The author wishes to express his appreciation to Professor A. Halanay for stimulating discussions which aided the clarity of exposition of Sec. 2,4 and have given rise to § 4 of this paper.

References

[1] *Banks H. T.:* Representations for Solutions fo Linear Functional Differential Equations, Journ. Diff. Eq. *5* (1969), 399—409.

[2] *Cameron R. H.* and *Martin W. T.*, An unsymmetric Fubini theorem, Bull. A.M.S. *47* (1941), 121—125.

[3] *Delfour M. C.* and *Mitter S. K.*, Hereditary Differential Systems with Constant Delays. II — A class of affine systems and the adjoint problem, to appear in Journ. Diff. Eq.

[4] *Dunford N.* and *Schwartz T.*, Linear operators, part I, Interscience Publishers, New York—London, 1958.

[5] *Halanay A.*, Periodic solutions of linear systems with time lag (in Russian), Rev. Math. pures appl. (Acad. R.P.R.) *6* (1961), 141—158.

[6] *Halanay A.*, On a Boundary Value Problem for Linear Systems with Time Lag, Journ. Diff. Eq. *2* (1966), 47—56.

[7] *Hale J. K.*, Functional Differential Equations, Applied Mathematical Sciences, vol. 3, Springer-Verlag, New York, 1971.

[8] *Henry D.*, The Adjoint of a Linear Functional Differential Equation and Boundary Value Problems, Journ. Diff. Eq. *9* (1971), 55—66.

[9] *Lifšic E. A.*, Fredholm alternative for the problem of periodic solutions to differential equations with retarded argument (in Russian), Probl. mat. analiza složnych sistem *1* (1967), 53—57.

[10] *Schwabik Št.*, On an integral operator in the space of functions with bounded variation, Čas. pěst. mat. *97* (1972), 297—330.

[11] *Schwabik Št.*, Remark on linear equations in Banach space, Čas. pěst. mat., *99* (1974), 115—122.

[12] *Tvrdý M.* and *Vejvoda O.*, General boundary value problem for an integro-differential system and its adjoint, Čas. pěst. mat. *97* (1972), 399—419 and *98* (1973), 26—42.

[13] *Tvrdý M.*, Note on functional-differential equations with initial functions of bounded variation, Czech. Math. J. *25* (*100*) (1975), 67—70.

[14] *Wexler D.*, On boundary value problems for an ordinary linear differential system, Ann. Mat. pura appl. *80* (1968), 123—134.

Remark 2,12 was added in the proofs. Its assertion was proved in [15] (Theorem 4,4). The results of sec. 5,5 were shown in another way also in [16].

[15] *Tvrdý M.*, Linear functional-differential operators: normal solvability and adjoints, to appear in the Proceedings of the Colloquium on Differential Equations (Keszthely, Hungary — September 1974),

[16] *Maksimov V. P.*, General boundary value problem for linear functional-differential equation is noetherian (in Russian), Diff. urav. *10* (1974), 2288—2291.

Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).