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Gary Haggard; Peter McWha

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DECOMPOSITION OF COMPLETE GRAPHS INTO TREES

GARY HAGGARD, Orono and PETER MCWHA, Waterloo

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I. INTRODUCTION

In 1963 at the graph theory conference in Smolenice, Czechoslovakia, GERHARD RINGEL conjectured that:

For any tree T with n edges the complete graph on $2n + 1$ vertices can be decomposed into $2n + 1$ subgraphs $T_0, T_1, \ldots, T_{2n}$ such that $T_i \cong T$ for $i = 0, 1, \ldots \ldots, 2n$.

In [4] ROSA modified the conjecture by constraining how the trees T_i for $i = 1, 2, \ldots, 2n$ were determined by the tree T_0. Rosa proved that the modified conjecture was equivalent to finding a certain type of integer valued function defined on the vertices of a tree. Further information about the progress on this problem is found in [1] and [2]. The authors of this paper give a sufficient condition for a solution to exist in terms of the adjacency matrix of the tree. Although this sufficient condition seems „easy" to apply to a given tree, there is not as yet an algorithm which tells one how to handle an arbitrary tree.

II. DEFINITIONS

The authors refer the reader to [3] for the standard definitions used in this paper.

The adjacency matrix $A = (a_{ij})$ of a graph with n vertices $\{v_1, \ldots, v_n\}$ is an $n \times n$ matrix in which $a_{ij} = 1$ if v_i is joined to v_j by an edge and $a_{ij} = 0$ otherwise. For a bipartite graph G it is possible to find an adjacency matrix of the form

$$\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}$$

where B^t is the transpose of B. Since a tree is a bipartite graph, every tree has an adjacency matrix of this form.

For an $m \times n$ matrix A the (i, j)-th entry is said to be on *diagonal* $(j - i)$. The diagonals of an $m \times n$ matrix can be represented by the numbers

$$1 - m,\ 1 - (m - 1),\ \ldots,\ 0,\ \ldots,\ n - 2,\ n - 1\,.$$

This terminology is clarified by the following example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

$$\text{Diagonal}\,(-2) = \{a_{31}\},\ \text{diagonal}\,(-1) = \{a_{21}, a_{32}\},\ \text{diagonal}\,(0) = \{a_{11}, a_{22}, a_{33}\},$$
$$\text{diagonal}\,(1) = \{a_{12}, a_{23}\},\ \text{diagonal}\,(2) = \{a_{13}\}\,.$$

Finally, a binary matrix D is *embedded* in a binary matrix F if under suitable permutations of the rows and columns of F we have

$$F = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}.$$

Let T be a tree with $n + 1$ vertices and let

$$f : V(T) \to \{0, 1, 2, \ldots, 2n\}$$

be a function. The function f is a *valuation* if f is injective. If $(v, w) \in E(T)$, then the *length* of the edge (v, w) relative to the valuation f is defined to be

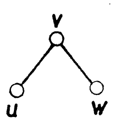
$$L_f(v, w) = \min\{|f(v) - f(w)|,\ 2n + 1 - |f(v) - f(w)|\}\,.$$

If the image of L_f is $\{1, 2, \ldots, n\}$, then by [4] it is possible to decompose K_{2n+1} into $2n + 1$ copies of T. The decomposition is effected in the following way:

For $i = 0, 1, \ldots, 2n$ label the vertex v of T_i with $f(w) + i$ where w is the vertex of T which corresponds to v and addition is done modulo $2n + 1$.

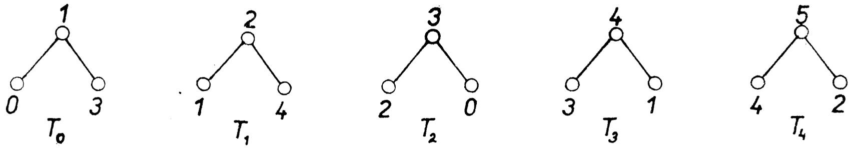
A decomposition formed in this way is called a *cyclic decomposition* in [4] and the tree T_0 is called a *starter* for the cyclic decomposition. To make this construction clearer we give an example.

Example. K_5 can be decomposed into 5 copies of P_2.



T

$$\begin{aligned} f : \{u, v, w\} &\to \{0, 1, 2, 3, 4\} \\ u &\to 0 \\ v &\to 1 \\ w &\to 3 \\ L_f : E(T) &\to \{1, 2\} \\ (u, v) &\to 1 \\ (v, w) &\to 2 \end{aligned}$$



III. THEOREM

The result of this paper is a sufficient condition for the existence of a cyclic decomposition of K_{2n+1} by a tree T with n edges. It is hoped that a procedure – other than an exhaustive search – may be found which would tell how to proceed from a given tree to the matrix we describe which determines a cyclic decomposition of the appropriate K_{2n+1}.

Theorem 1. *Let T be a tree with $n + 1$ vertices and adjacency matrix*

$$\begin{pmatrix} 0 & C \\ C^t & 0 \end{pmatrix}.$$

If C can be embedded in an $n \times n$ binary matrix E such that for $i = 0, 1, \ldots, n - 1$ the ith-diagonal of E contains exactly one non-zero entry, then there exists a cylic decomposition of K_{2n+1} by T.

Proof. Let C be embedded in an $n \times n$ binary matrix E such that the hypothesis is satisfied. Because of the particular form of C each vertex of T occurs exactly once either as a label for a row of E or as a label for a column of E, but not both. Some rows and columns of E may be unlabelled. Because of these remarks the following labelling of the rows and columns of E gives rise to a new labelling of T. Label row i of E with $i - 1$ and label column i of E with $n + i$ for $1 \leqq i \leqq n$. Relabel T as follows:

(i) If row $i - 1$ of E represents the adjacencies of v, then relabel v as $i - 1$.

(ii) If column $n + i$ of E represents the adjacencies of v, then relabel v as $n + i$. Denote by T_0 this relabelled version of T. The identity function on $V(T_0)$ is a valuation. Let L be the corresponding length function on $E(T_0)$. Because of the form of C no edge of T_0 will have both ends labelled with numbers less than n or both ends labelled with numbers greater than n. Now suppose $(l - 1, n + j) \in E(T_0)$. We note that this edge is represented by diagonal $(j - l)$ of E and hence,

$$(1) \qquad n + 1 \leqq n + j - l + 1 \leqq 2n .$$

Therefore for any edge $(l - 1, n + j)$ in T_0 we have

$$(2) \qquad L(l - 1, n + j) = 2n + 1 - (n + j - l + 1) .$$

Because of (1) and (2) we have

$$1 \leqq L(l-1,\ n+j) \leqq n \tag{3}$$

for any $(l-1,\ n+j) \in E(T_0)$. To apply the result of [4] and complete the proof we must show that L is injective. Therefore suppose $(l-1,\ n+j)$ and $(k-1,\ n+m)$ are two different edges of T_0 and

$$L(l-1,\ n+j) = L(k-1,\ n+m)\,.$$

This implies that

$$2n+1-(n+j-l+1) = 2n+1-(n+m-k+1)$$

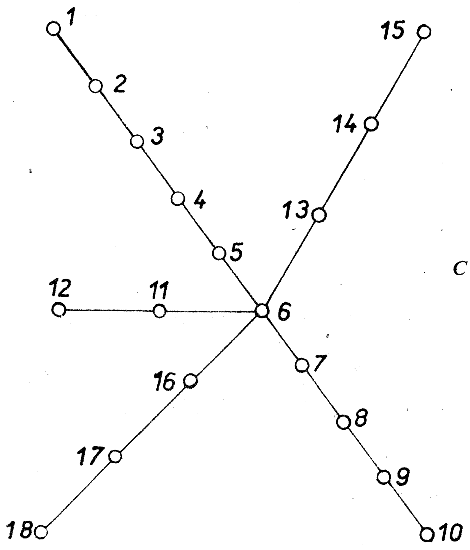
$$j-l = m-k\,. \tag{4}$$

But $0 \leqq j-l,\ m-k \leqq n-1$ and these two numbers identify the diagonals of E where we find the non-zero entries which correspond to the two given distinct edges of T_0. Therefore (4) is impossible because by hypothesis E has exactly one non-zero entry on diagonal i for each $i = 0, 1, 2, \ldots, n-1$.

Therefore T_0 is a starter for a cyclic decomposition of K_{2n+1} by T.

IV. EXAMPLE

We would now like to give an example that shows how Theorem 1 works. Figure 1 contains a tree with 17 edges and the part of its bipartite adjacency matrix represented



$$C = \begin{array}{c|cccccccccc}
 & 18 & 16 & 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 \\
\hline
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
6 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
14 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Figure 1

by C. Figure 2 shows C embedded in a binary matrix E. The second set of labels correspond to the labelling described in the proof of the theorem. Finally, the tree T_0 pictured is a starter for a cyclic decomposition of K_{35}. The labels on the edges of the tree correspond to the lengths of the edges.

			18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
							18			15	16	13	11		9	7	5	3	1
	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
	1	4		0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
	2	6			0	0	0	0	0	0	1	1	1	0	0	1	1	0	0
	3	8				0	0	0	0	0	0	0	0	0	1	1	0	0	0
	4	17					1	0	0	0	1	0	0	0	0	0	0	0	0
	5	12						0	0	0	0	0	1	0	0	0	0	0	0
$E =$	6	14							0	1	0	1	0	0	0	0	0	0	0
	7									0	0	0	0	0	0	0	0	0	0
	8										0	0	0	0	0	0	0	0	0
	9											0	0	0	0	0	0	0	0
	10	10											0	0	1	0	0	0	0
	11													0	0	0	0	0	0
	12														0	0	0	0	0
	13															0	0	0	0
	14																0	0	0
	15																	0	0
	16																		0

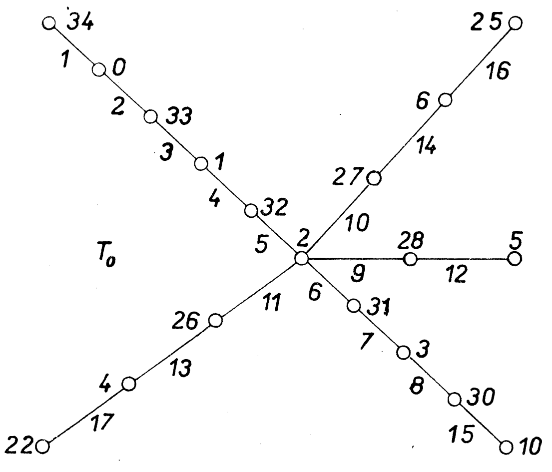


Figure 2

With a little practice the ability to embed C in E is improved. It now remains to describe a procedure that tells one how to find E without facing the task of examining all possible ways of embedding C in E.

ADDENDUM

In a preliminary version of this paper the authors suggested that Theorem 1 might remain true if the requirement that the diagonals of E with non-zero entries comprise a complete set of residues modulo n. The authors and D. A. SHEPHARD (Monthly Research Problems, 1969–1973, American Math. Monthly *80* (1973), 1120–1128) have found examples to preclude this generalization.

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Authors' adresses: G. HAGGARD, University of Maine at Orono, U.S.A.; PETER MCWHA, University of Waterloo, Canada.