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ON THE SILOV BOUNDARY INDUCED BY CERTAIN SEMIGROUP ALGEBRAS

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1. A locally compact commutative semigroup S is a locally compact Hausdorf space together with a continuous associative binary operation. We will assume that on such a semigroup S there is a non negative regular Borel measure, m, satisfying the condition that for each Borel set E in S with $m(E) = 0$, $m(x^{-1}E) = 0$, where $x^{-1}E = [y : xy \in E]$. The semigroup algebra of the locally compact commutative semigroup S is taken as those finite regular Borel measures (linear functionals) in $M(S)$ $(=C_0(S)^*)$ which are absolutely continuous with respect to the measure m, and is denoted by $L^1(S, m)$. Addition and scalar multiplication are defined pointwise and multiplication (convolution) is given by $\mu^*\nu(f) = \iint f(xy)\, \mu(dx)\, \nu(dy)$, where $f \in C_0(S)$ and $\mu, \nu \in M(S)$.

A multiplicative function τ on S is a complex valued function on S satisfying $\tau(xy) = \tau(x)\, \tau(y)$ for all x and in S and with $\tau \not\equiv 0$. A semicharacter on S is a bounded m-measurable multiplicative function on S. We denote by S^* the set of semicharacters on S. If S possessesan identity element, then S^* is a semigroup under multiplication $\tau\, \theta(x) = \tau(x)\, \theta(x)$. If S is a locally compact abelian group and m is Haar measure on S then the semigroup algebra is the known L^1 algebra and S^* is the dual group of continuous characters on S.

Some of the results of this paper are dependent on the work in [2], [4] and [6]. Furthermore, the work of COMFORT [3] on discrete semigroups is a base upon which this paper is built. It was shown in [5] that the set Δ of non trivial multiplicative linear functionals on $L^1(S, m)$ is in one to one correspondence with the elements of S^* (identified modulo equal almost everywhere). In particular, if $\tau \in S^*$, the linear functional $h_\tau(\mu) = \int \tau \, d\mu$ is a multiplicative linear functional and $\tau(x) = h_\tau(\mu * \bar{x})/h_\tau(\mu)$ $(h_\tau(\mu) \neq 0$ and $\mu * \bar{x} \in L^1(S, m))$ is such that the mapping of $\tau \to h$ is the desired correspondence.

The topology on S^* will be taken to be the Gelfand topology that S^* inherits via the above correspondence with the maximal ideal space of the Banach algebra

$L^1(S, m)$. Thus a net $\{\tau_\alpha\}$ in S^* converges to τ in S^* if and only if for each μ in $L^1(S, m)$, $\hat{\mu}(\tau_\alpha) \to \hat{\mu}(\tau)$ $(h_{\tau_\alpha}(\mu) \to h_\tau(\mu)$ or $\int \tau_\alpha \, d\mu \to \int \tau \, d\mu)$.

A fundamental neighborhood of τ in S^* is a set

$$U = \left\{\theta \in S^* : \left|\int (\theta - \tau) \, d\mu_j\right| < \varepsilon, \quad \{\mu_j\}_1^n \subset L^1(S, m)\right\}$$

where $\varepsilon > 0$. In this topology, S^* is locally compact and is compact if and only if 0 is not in the closure of the multiplicative linear functionals in the ω^* topology on $L^1(S, m)^*$.

It will be assumed from here on that S has an identity element, so that S^* is a semigroup. It follows readily from the above remarks on the topology of S^* that multiplication is continuous in S^*.

The algebra of Gelfand transforms of $L^1(S, m)$ is a separating sublagebra of the space of continuous functions vanishing at infinity on the locally compact Hausdorff space S^*, hence the Silov boundary ∂ induced by S^* exists and satisfies

(i) ∂ is a closed subset of S^*

(ii) if $\mu \in L^1(S, m)$ then $|\hat{\mu}|$ assumes its maximum on ∂

(iii) no proper closed subset of ∂ satisfies (ii).

In this paper, the boundary will be determined for a class of semigroups. In particular, for compact linearly-quasi ordered semigroups [4] we will show that

$$\partial = [\tau \in S^* : |\tau| \text{ is idempotent}] .$$

2. Let S be a compact commutative topological semigroup with identity element 1 and let m be a non negative regular Borel measure on S such that for each set E of m-measure zero,

$$x^{-1}E = [y : xy \in E]$$

is also of m-measure 0. Let K be the minimal ideal of S. Since K is a compact abelian topological group, for each $\tau \in \hat{K}$, τ a continuous character on K, the mapping $\tau'(x) = \tau(xe)$ (e the identity of K) is a continuous semicharacter on S and thus Borel measurable. We identify $\hat{K}$ then as a subset of S^*. It is clear that

$$\hat{K} = [\tau \in S^* : |\tau| \equiv 1] .$$

Lemma 2.1. *Let S be as above and let ∂ denote the Silov boundary of the maximal ideal space of $L^1(S, m)$, i.e. S^*, then $\hat{K} \subset \partial$.*

Proof. Let $\chi \in \hat{K}$ and let U be a neighborhood of χ in S^*. Without loss of generality we let

$$U = [\tau : |\hat{\mu}_i(\tau) - \hat{\mu}_i(\chi)| < \varepsilon\,, \quad \mu_i \in L^1(S, m) \quad \text{and} \quad 1 \leqq i \leqq n]$$

and also assume that $\varepsilon < \frac{1}{2}$.

We need to find $\nu \in L^1(S, m)$ such that $|\hat{\nu}|$ assumes its maximum only on U. We shall find a complex valued measurable function α, with m-finite support, on S and take $d\nu = \alpha \, dm$.

For each μ_i, let A_i be the compact support of μ_i and let $B_i = [x^2 : x \in A_i]$, note that B_i is the image under multiplication of $\Delta(S \times S) \cap (A_i \times A_i)$ and is thus compact and hence a Borel set. Let $A = \bigcup_{i=1}^{n} (A_i \cup B_i)$ and define

$$\alpha(x) = \chi_A(x) \overline{\chi(x)} \text{ for all } x \text{ in } S\,.$$

Then α is Borel measurable with compact support and $\alpha \, dm \in L^1(S, m)$. Let ν be this measure. Then

$$m(A) = \int \chi_A(x) \, dm = \int \chi_A(x) \overline{\chi(x)} \chi(x) \, dm = \int \chi(x) \, d\nu = \hat{\nu}(\chi)\,.$$

If $\theta \notin U$ then there is a j, $1 < j < n$, such that $|\hat{\mu}_j(\theta) - \hat{\mu}_j(\chi)| \geqq \varepsilon$, that is

$$\int |\theta - \chi| \, d\mu_j \geqq \left| \int (\theta - \chi) \, d\mu_j \right| \geqq \varepsilon,$$

hence there is a subset B of A_j such that $m(B) > 0$ and $|\theta - \chi| > \delta > 0$ on B, i.e. $\theta(x) \neq \chi(x)$ for all x in B.

If $\theta(x) = 0$ on a set of positive measure $C \subset B$ then

$$|\hat{\nu}(\theta)| = \left| \int \theta(x) \, \alpha(x) \, dm \right| = \left| \int_{A \setminus C} \theta(x) \, \alpha(x) \, dm \right| < m(A \setminus C) < m(A) = |\hat{\nu}(\chi)|\,.$$

On the other hand, if $\theta \mid B$ is zero only on a set of measure zero, then without loss of generality $\theta(x) \neq 0$ for all x in B. If $|\hat{\nu}(\theta)| = m(A)$, then $\hat{\nu}(\theta) = m(A) \, e^{i\varphi}$, $0 < \varphi < < 2\pi$. Let $d\lambda = e^{-i\varphi} \alpha^2(x) \, \theta(x) \, dm$ and $d\gamma = e^{-i\varphi} \alpha(x) \, dm$ and let $\mu = \gamma + \lambda$. Then $|\hat{\mu}(\theta)| = |\int e^{-i\theta}(\alpha\theta + \alpha^2\theta^2) \, dm| < 2m(A)$ since $|\alpha + \alpha^2\theta^2| < 2$ and $|\hat{\mu}(\chi)| = = |\int e^{-i\varphi}(\alpha\chi + \chi^2\chi) \, dm| = |e^{-i\varphi} \, m(A) + \int e^{-i\varphi}\alpha\theta \, dm| = 2m(A)$. Hence for $\theta \notin U$, $|\hat{\mu}(\theta)| < |\hat{\mu}(\chi)|$ and $\chi \in \partial$.

Let $\Gamma = [\chi \in S^* : |\chi| = 0 \text{ or } 1]$. Note that Γ is the set of all those elements of S^* which have an inverse with respect to some idempotent in S^* and hence that Γ is a union of groups, the maximal groups containing each idempotent element in S^*. For $\chi = \chi^2$ in S^* and $H(\chi)$ the maximal group with identity χ, $H(\chi)$ is a locally compact topological group, since inversion in $H(\chi)$ is complex conjugation, $H(\chi)$ is closed in S^* and multiplication is continuous. Now Γ is closed since S^* is closed under complex conjugation and the idempotent elements of a semigroup are a closed set, thus $\{\chi_\alpha\}$ a net in Γ with $\chi_\alpha \to \chi$ implies $\bar{\chi}_\alpha \to \bar{\chi}$ and hence $\chi_\alpha \bar{\chi}_\alpha \to \chi\bar{\chi}$. Thus since $\chi_\alpha\bar{\chi}_\alpha$ is an idempotent, $\chi\bar{\chi}$ is then an idempotent and $\chi \in \Gamma$.

Lemma 2.2. *Let S, ∂ and Γ be as above. Then $\partial \subset \Gamma$.*

Proof. We will show that for $\psi \in S^*$ and $\psi \notin \Gamma$ that $\psi \notin \partial$. Let $\psi \in S^* \setminus \Gamma$. Then there is an $\varepsilon > 0$ and a Borel set B in S of finite positive measure (m) such that $2\varepsilon < |\psi(x)| < 1 - 2\varepsilon$ for all $x \in B$. Let $v = \chi_B(x)\,\mathrm{d}m$ ($\chi_B(x)$ is the characteristic function of B), then $v \in L^1(S)$ and $U = [\theta \in S^* : |\hat{v}(\theta) - \hat{v}(\bar{\psi})| < \varepsilon]$ is an open neighborhood of ψ in S^*. Note that $\theta \in U$ implies that $\varepsilon < \hat{v}(\theta) < 1 - \varepsilon$ and $|\hat{\lambda}(\bar{\psi})| = |\int \psi \, \mathrm{d}m| < (1 - 2\varepsilon)\, m(B)$. In order to show that $\psi \notin \partial$, it suffices to show that for each $\mu \in L^1(S, m)$, $|\hat{\mu}|$ attains its maximum outside of U. Let $\theta \in S^*$ such that

$$|\hat{\mu}(\theta)| \geqq |\hat{\mu}(\varphi)|$$

for all $\varphi \in S^*$. If $\theta \in S^* \setminus U$ there is nothing to prove hence we assume $\theta \in U$. If $|\hat{\mu}(\theta)| = 0$ then $|\hat{\mu}| \equiv 0$ and since the identically 1 valued semicharacter is not in U (i.e. $U \neq S^*$) again we are finished. Thus we can assume that $0 < |\hat{\mu}(\theta)|$ and θ is an element of U. We will now construct a semicharacter not in U where $|\hat{\mu}|$ also attains its maximum value.

Since $\mu \in L^1(S, m)$, μ is absolutely continuous with respect to m and hence $\mathrm{d}\mu = f(x)\,\mathrm{d}m$. Let

$$A = [x : f(x)\,\theta(x) \neq 0] .$$

We wish to consider the function of a complex variable

$$g(z) = \int \theta(x)\, e^{(1+z)\ln|\theta(x)|}\, \mathrm{d}\mu = \int f(x)\, \theta(x)\, e^{(1+z)\ln|\theta(x)|}\, \mathrm{d}m =$$

$$= \int_A f(x)\, \theta(x)\, e^{(1+z)\ln|\theta(x)|}\, \mathrm{d}m .$$

Let $X_n = [x : |\theta(x)| > 1/n]$ and consider

$$g_n(z) = \int_{X_n} \theta(x)\, e^{(1+z)\ln|\theta(x)|}\, \mathrm{d}\mu .$$

Then

$$g(z + h) - g(z) = \frac{1}{h} \int_{X_n} \theta(x)\, e^{(1+z)\ln|\theta(x)|}\, [e^{h\ln|\theta(x)|} - 1]\, \mathrm{d}\mu .$$

For $|h|$ small, Re (h) is small and the difference quotient is

$$\int_{X_n} \theta(x)\, e^{(1+z)\ln|\theta(x)|}[\ln|\theta(x)| + h[\ldots]]\, \mathrm{d}\mu$$

and thus

$$\lim_{|h|\to 0} \frac{g(z+h) - g(z)}{h} = \int_{X_n} \theta(x)\, e^{(1+z)\ln|\theta(x)|} \ln|\theta(x)|\, \mathrm{d}\mu$$

and g_n is analytic. Now

$$|g(z) - g_n(z)| = \left|\int_{S\setminus X_n} \theta(x)\, e^{(1+z)\ln|\theta(x)|}\, \mathrm{d}\mu\right| < \int_{S\setminus X_n} |\theta(x)|\, \left|e^{(1+z)\ln|\theta(x)|}\right| \mathrm{d}\mu =$$

$$= \int_{S\setminus X_n} |\theta(x)|\, \left|e^{(1+\mathrm{Re}\, z)\ln|\theta(x)|}\right| \mathrm{d}\mu\,.$$

Now for Re $z > -1$, $e^{(1+\mathrm{Re}\, z)\ln|\theta(x)|} < 1$ and

$$|g_n(z) - g(z) < \frac{1}{n} \|\mu\|$$

hence $g_n(z)$ converges uniformly to $g(z)$ for Re $z > -1$ and g is analytic in Re $z > -1$.

We define semicharacters θ_z on S for Re $z > -1$ by

$$\theta_z(x) = 0 \quad \text{if} \quad \theta(x) = 0$$

$$\theta_z(x) = \theta(x)\, e^{(1+z)\ln|\theta(x)|} \quad \text{if} \quad \theta(x) \neq 0\,.$$

For $a, b \in S$ if $\theta(a) = 0$ or $\theta(b) = 0$ then $\theta(ab) = 0$ and $\theta_z(ab) = 0 = \theta_z(a)\,\theta_z(b)$.
If $\theta(a) \neq 0$, then $\theta(ab) \neq 0$ and

$$\theta_z(ab) = \theta(ab)\, e^{(1+z)\ln|\theta(ab)|} = \theta(a)\, e^{(1+z)\ln|\theta(a)|}\, \theta(b)\, e^{(1+z)\ln|\theta(b)|}\,.$$

Further Re $(z) > -1$ implies $|\theta_z(x)| = \left|\theta(x)\, e^{(1+z)\ln|\theta(z)|}\right| < 1$ and $\theta_z \in S^*$ since θ_z is clearly measurable.

Note that $\hat{\mu}(\theta_z) = g(z)$, and that for Re $z > -1$, $\theta_z \in S^*$ and $|\hat{\mu}|$ attains its maximum at θ i.e. $|\hat{\mu}(\theta)| = |g(0)|$ is maximum value of the analytic function g in a neighborhood of 0 and hence g is constant thus $|\hat{\mu}|$ attains its maximum also at θ_z for Re $z > -1$ and we choose z real such that $\theta_z \in S^*$ and $|\theta|^{z+1} > (1 - \varepsilon)\, m(B)$, i.e. $\theta_z \notin U$ and we see that χ is not in the boundary ∂.

3. Linarly Quasi-ordered semigroups. A general discussion of linearly quasi-ordered semigroups can be found in [4] and [6]. The existence of a measure on such semigroups satisfying the conditions in section 1 is to be found in [6]. We show in this section that the boundary ∂ of the maximal ideal space S^* of $L^1(S, m)$, where S is a compact commutative linearly quasi-ordered topological semigroup with identity is Γ the set of those semicharacters (measurable) whose absolute values are idempotent elements of the semigroup S^* of all measurable semicharacters. From lemma 2.2 we know that $\partial \subset \Gamma$ and need only the reverse inequality.

Let S be a compact commutative linearly quasi-ordered topological semigroup and $\chi \in S^*$, the measurable semicharacters on S. Now $\chi^{-1}(0)$ is a prime ideal of S and is identical with $|\chi|^{-1}(0)$. Since $|\chi|$ can also be considered as a multiplicative function on $S/\mathfrak{L}$, it is readily seen that $\chi^{-1}(0)$ either is equal to Se for some idempotent element e in S or is equal to $Se \setminus H(e)$ for some idempotent element e in S. In the following φ is the natural mapping $\varphi : S \to S/\mathfrak{L}$.

Lemma 3.1. *Let S and χ be as above and let e be such that $\varphi(e)$ is the zero of a unit thread in $S/\mathfrak{L}$, then χ in Γ implies χ is in ∂.*

Proof. Let f be the idempotent element of S such that $\varphi(f)$ is the identity of the unit thread for which $\varphi(e)$ is the zero then since χ is a measurable semicharacter on S, $\chi \mid Sf \setminus Se$ is a continuous semicharacter on $Sf \setminus Se$ and hence is the restriction of a character on the group $H(f) \times R$ to a subsemigroup and as such is in the boundary of the maximal ideal space of $L^1(Sf \setminus Se)$ [1]. Since $L^1(Sf \setminus Se)$ can be considered as a subalgebra of $L^1(S)$, we see that $\chi \in \partial$.

Lemma 3.2. *Let S and χ be as above and let e be such that the connected component containing e in E, the set of all idempotent elements, is a point. If $\varphi(e)$ is not the zero of a unit thread and χ is in Γ then χ is in ∂.*

Proof. Since the component of E containing e is $\{e\}$, there exists a linearly ordered net f_α of idempotent elements $f_\alpha \downarrow e$ such that each f_α is such that $\varphi(f_\alpha)$ is the zero of a unit thread. Define

$$\chi_\alpha = \begin{cases} \chi & \text{on} \quad S \setminus Sf_\alpha, \\ 0 & \text{on} \quad Sf_\alpha, \end{cases}$$

then by lemma 3.1 $\chi_\alpha \in \partial$. Thus we need only show that $\chi_\alpha \to \chi$ to obtain $\chi \in \partial$. Now $\chi_\alpha \to \chi$ if and only if for each $v \in L^1(S)$, $\hat{v}(\chi_\alpha) \to \hat{v}(\chi)$. We need consider only those v with support contained in $S \setminus Se$, thus

$$|\hat{v}(\chi_\alpha) - \hat{v}(\chi)| = \left|\int (\chi_\alpha - \chi)\, \mathrm{d}y\right| = \left|\int_{Sf_\alpha \setminus Se} (\chi_\alpha - \chi)\, \mathrm{d}v\right| < v(Sf_\alpha \setminus Se)\,.$$

Since v is a regular Borel measure and $f_\alpha \to e$, $v(Sf_\alpha \setminus Se) \to 0$ and $\chi \in \partial$.

Lemma 3.3. *Let S and χ be as above and let e be such that $\varphi(e)$ belongs to a nontrivial idempotent interval in $S/\mathfrak{L}$, then χ in Γ implies $\chi \in \partial$.*

Proof. Since $\varphi(e)$ is in an idempotent interval χ is equivalent to a semicharacter on S defined uniquely by extension of a character on $H(e)$ to S. For any idempotent f with $\varphi(f)$ in the same interval and $\varphi(f) > \varphi(e)$, there is an involution on $L^1(Sf \setminus Se)$ so that the algebra is self-adjoint and hence the boundary of the maximal ideal space of this algebra is the whole maximal ideal space. It then follows that χ is in ∂.

We thus have the following

Theorem 3.4. *Let S be a compact commutative linearly quasiordered topological semigroup such that $S/\mathfrak{L}$ contains no nil thread. The natural measure m on S is such that $L^1(S, m)$ is a Banach algebra and the Silov boundary of the maximal ideal space corresponds in a one to one fashion with those m-measurable semicharacters on S whose absolute values are idempotent semicharacters.*

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