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ON COMPACT N-SEMIGROUPS

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1. INTRODUCTION

A topological semigroup is a non-empty Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition $(x, y) \to xy$. When there is no possible ambiguity we shall simply refer to S as a topological semigroup. If S contains a zero, that is, an element 0 such that $x0 = 0x = 0$ for all $x \in S$, S is said to be a topological semigroup with zero. In this paper, we consider only topological semigroups with zero and hence we shall use the term"semigroup" to mean topological semigroup with zero.

If S is a semigroup, an element b of S is called nilpotent if $b^n \to 0$, that is, if for every neighbourhood U of 0 there exists an integer n_0 such that $b^n \in U$ for all $n \geqq n_0$. The set of all nilpotent elements of S shall be denoted by N. If N is an open subset of S, then S is called an N-semigroup. In addition, if S is a compact space, then S will be called a compact N-semigroup.

In [2] we studied some properties of compact commutative N-semigroups with zero and local zeros. The following definition was introduced there. If $a \in S$, the set of all right topological zero divisors of a is the set $\text{Tod}_r\, a = \{x \in S \mid ax \in N\}$. The set $\text{Tod}_l\, a$ of all left topological zero divisors of a is similarly defined. If S is commutative we shall denote them both by $\text{Tod}\, a$. We observe that $\text{Tod}\, a$ is always non-empty since $0 \in \text{Tod}\, a$. In this paper we shall study the properties of N in terms of $\text{Tod}\, e$ where e is a non-zero idempotent of S. We shall prove that in fact N is the intersection of all such $\text{Tod}\, e$. We shall also show that if e is a non-zero primitive idempotent of a compact N-semigroup S, then $\text{Tod}\, e$ is an open prime ideal of S. Finally, we show that in a compact N-semigroup, under some conditions, a nil ideal is nilpotent, thus transporting the well known Hopkins-Levitzki theorem from ring theory to compact N-semigroups, with the chain conditions being replaced by compactness.

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2. PRELIMINARIES

We shall use the following notation. Let A be any subset of a semigroup S, and let $a \in S$. Then

$\bar{A}$ = topological closure of A in S

A' = complement of A in S

$|A|$ = cardinal number of the set A

$J(A)$ = $A \cup AS \cup SA \cup SAS$, that is, the smallest ideal of S containing A

$J_0(A)$ = the union of all ideals contained in A, that is, the largest ideal contained in A if $J_0(A) \neq \emptyset$

$R(A)$ = $\{x \in S \mid x^n \in A$ for some integer $n \geqq 1\}$

$\Gamma(a)$ = $\overline{\{a^n\}_{n=1}^{\infty}}$

$K(a)$ = $\bigcap_{n=1}^{\infty} \overline{\{a^i \mid i \geqq n\}}$, that is, the set of cluster points of the sequence $\{a^n\}_{n=1}^{\infty}$.

It is well-known that if $\Gamma(a)$ is compact, it contains a unique idempotent. Moreover, $K(a)$ is a group and $K(a) = e\,\Gamma(a) = \Gamma(a)\,e$ where $e \in \Gamma(a)$ is the unique idempotent (see [6], pages 22–25).

We recall some definitions and results that we shall need.

Lemma 2.1 (Numakura [4]). *The set E of idempotents of S is a closed subspace of S which is partially ordered under the relation $e \leqq f$ if $ef = fe = e$, and this partial order is closed, that is, it has a closed graph. If $ef = fe$ for all $e, f \in E$, then E is a semigroup and ef is the greatest lower bound of $\{e, f\}$ relative to $\leqq$.*

Definition 2.2. An idempotent e is called *primitive* if $f^2 = f \in eSe$ implies that $f = 0$ or $f = e$. It is obvious that the non-zero primitive idempotents are the atoms of the partially ordered set $(E^*, \leqq)$, where $E^* = E - \{0\}$.

Definition 2.3. Two non-zero idempotents e and f of S are said *to be orthogonal* if $ef = fe = 0$. We shall denote this by $e \perp f$.

Definition 2.4. An ideal P of S is said *to be prime* if $AB \subset P$ implies that $A \subset P$ or $B \subset P$ where A and B are ideals of S. An ideal Q of S is said to be completely prime if $ab \in Q$ implies that $a \in Q$ or $b \in Q$, where a and b are elements of S.

Remark. An ideal which is completely prime is prime, but the converse need not be true. (For a counter example, see [6], page 51.) However, these concepts coincide in the case of commutative semigroups.

Theorem 2.5 (Numakura [5]). *If S is a compact semigroup with zero, then each open prime ideal $P \neq S$ has the form $J_0(S - e)$ where e is a non-zero idempotent of S. Conversely, if e is a non-zero idempotent, then $J_0(S - e)$ is an open prime ideal.*

Lemma 2.6 (Numakura [4]). *Let S be a semigroup with zero and let $a \in S$. If a^n is nilpotent for some integer $n \geqq 1$, then a itself is a nilpotent element.*

Lemma 2.7 (Hoo-Shum [2]). *If S is a compact commutative semigroup, then the set N is an ideal of S.*

3. NILPOTENT ELEMENTS AND TOPOLOGICAL ZERO DIVISORS

In this section we shall study the set N of nilpotent elements in a compact commutative semigroup S, and give a characterization of this set in terms of the sets Tod e_i where the e_i are in E^*. Some of the results are closely related to those obtained in our previous paper [2]. Throughout this section, S will denote a commutative semigroup.

Lemma 3.1. *If S is compact but not nil, then N is the intersection of all the sets* Tod e *where $e \in E$.*

Proof. Since S is a commutative semigroup, it follows from Lemma 2.7 that $N \subset \bigcap_{e \in E} \text{Tod } e$. We now show that $\bigcap_{e \in E} \text{Tod } e \subset N$. Let $x \in \bigcap_{e \in E} \text{Tod } e$. Then $ex \in N$ for all $e \in E$. Since S is compact, it follows that $\Gamma(x)$ is compact, and hence there exists an idempotent $e_1 \in \Gamma(x)$. Since $K(x) = e_1 \, \Gamma(x)$ is a group, it follows that $e_1 x \in K(x)$ has an inverse $y \in K(x)$. Hence applying Lemma 2.7 once more, since N is an ideal of S, we have $e_1 = (e_1 x) \, y \in NS \subset N$. This implies that $e_1 = 0$, that is, $K(x) = \{0\}$. But $K(x)$ is the set of all cluster points of the sequence $\{x^n\}_{n=1}^{\infty}$. Hence $x^n \to 0$, that is, $x \in N$. Therefore $N = \bigcap_{e \in E} \text{Tod } e$.

Theorem 3.2. *Let S be compact and let $E^{\#}$ be the set of all non-minimal idempotents of S. Then $N = \bigcap_{e \in E^{\#}} \text{Tod } e$.*

Proof. Since Tod $0 = S$, by Lemma 3.1, we immediately have $N = \bigcap_{e \in E^*} \text{Tod } e$. Now let e_1, e_2 be idempotents of S and let us suppose that $e_1 \leqq e_2$, that is, $e_1 e_2 = e_2 e_1 = e_1$. Then if $x \in \text{Tod } e_2$ we have $e_2 x \in N$. Thus $(e_1 e_2) \, x = e_1 (e_2 x) \in e_1 N \subset N$ by Lemma 2.7; that is, $e_1 x \in N$, or $x \in \text{Tod } e_1$. Thus, if $e_1 \leqq e_2$ we have $\text{Tod } e_2 \subset \text{Tod } e_1$. This proves the theorem.

Corollary 1. *If S is compact, then N is a closed ideal of S if and only if for each $e \in E^{\#}$,* Tod e *is a closed ideal of S.*

Proof. If for each $e \in E^{*}$, Tod e is a closed ideal of S, then by Theorem 3.2 it follows immediately that N is a closed ideal of S. The converse was proved by us in [2] and also by A. D. WALLACE in [11].

Remark. We point out that K. NUMAKURA said in [4] that the structure of semigroups in which N is not open was not known to him. The Corollary above suggests that it may be worthwhile for us to consider the sets Tod e when N is not open.

Corollary 2. (Another characterization of compact N-semigroups.) *A compact semigroup S is an N-semigroup if and only if S contains only a finite number of open ideals* Tod e *with* $e \in E^{*}$.

Proof. Since the intersection of finitely many open sets is open, in one direction, this results is obvious. The converse was proved by us in [2] and by A. D. Wallace in [11].

In [2] we proved that S is a compact N-semigroup if and only if $E^* = E - \{0\}$ is compact. The characterization above is an improvement of our previous result. Also, in [2] we called a semigroup an A-semigroup if Tod a are all open for every $a \in S$, and we asked (Colloquium Mathematicum problem P796): if S is an A-semigroup, is S an N-semigroup? If S is compact and E^* is finite, this Corollary gives an affirmative answer to this problem.

Corollary 3. *If* $e \in E^*$, *then* Tod $e = R(\text{Tod } e)$.

Proof. Clearly Tod $e \subset R(\text{Tod } e)$. Take $y \in R(\text{Tod } e)$. Then there is an integer $k \geqq 1$ such that $y^k \in \text{Tod } e$, and hence $ey^k \in N$. Since e is an idempotent and S is commutative, we have $(ey)^k \in N$. By Lemma 2.6, it follows that $ey \in N$, that is, $y \in$ $\in$ Tod e. Hence Tod $e = R(\text{Tod } e)$.

Remark 1. In general, N is properly contained in Tod e if e is a non-zero primitive idempotent. However, Tod e need not be the minimal non-nil ideal of S. The next example due to Š. SCHWARZ ([8], page 226) shows this.

Example 3.3. Let S be the discrete semigroup consisting of four elements $\{0, a, e, f\}$ with the following multiplication table:

$\cdot$	0	a	e	f
0	0	0	0	0
a	0	0	0	a
e	0	0	e	0
f	0	a	0	f

Clearly, e and f are non-zero primitive idempotents of S. The lattice of ideals of S is given by Figure I. Obviously, Tod f is not the minimal non-nil ideal of S.

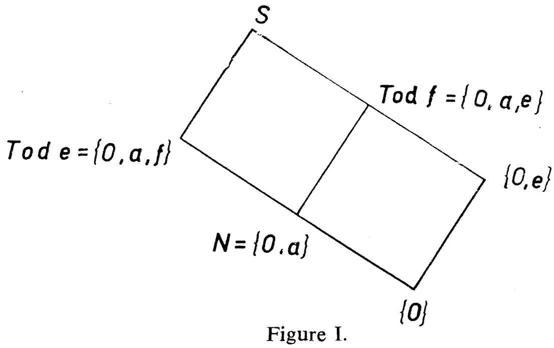

Figure I.

Remark. If S is not compact, Theorem 3.2 need not hold. This can be seen from the following example.

Example 3.4. Let S_1 be the set of all non-negative real numbers with the ordinary multiplication. Let S_2 be the set of all integers $\leqq -2$, the multiplication being the ordinary multiplication of numbers with a negative sign affixed. Define in $S_1 \cup S_2 = S$ a commutative multiplication $*$ by $x * y = 0$ if $x \in S_1$, $y \in S_2$, while the products in S_1 and S_2 are as above. Then S is a semigroup. Clearly $N = [0, 1)$ and Tod $1 = [0, 1) \cup S_2$. Thus $N \neq \bigcap_{e \in E^{\#}} \text{Tod } e$.

Proposition 3.5. *Let S be a compact N-semigroup and let e be a non-zero idempotent of S. Then*

(i) Tod *e is a nil ideal of S if there does not exist any non-zero idempotent of S which is orthogonal to e.*

(ii) *If N is itself a prime ideal of S, then $N =$* Tod *e for all non-zero idempotents e.*

(iii) *If* Tod *e is not a minimal non-nil ideal of S, then* Tod *e contains a non-zero primitive idempotent f such that $fS \not\subset N$. Conversely, if f is a non-zero primitive idempotent in* Tod *e such that $N - fS \neq \emptyset$, then* Tod *e is not a minimal non-nil ideal of S.*

Proof. The proofs of (i) and (ii) are trivial, and the proof of (iii) is similar to the arguments of Numakura in [4]. We omit the details.

4. OPEN PRIME IDEALS IN N-SEMIGROUPS

Throughout this section all semigroups under consideration are commutative compact N-semigroups. Unless otherwise specified, S will be such a semigroup.

Theorem 4.1. *If e is a non-zero primitive idempotent of S, then* Tod e *is an open prime ideal of S.*

We need the following lemma for the proof.

Lemma 4.2. *Let e be a non-zero idempotent of S. If I is an ideal of S which is not contained in* Tod e, *then there is a non-zero idempotent f such that $f \in I -$* Tod e.

Proof. Let $x \in I -$ Tod e and consider the principal ideal $J(x)$ generated by x. Clearly $\Gamma(x) \subset J(x) \subset I$. Since S is compact, $\Gamma(x)$ is a compact semigroup. Thus there is an idempotent $f \in \Gamma(x) \subset I$. Suppose, for an indirect proof, that $f \in$ Tod e. Then we have $fe \in N$, which implies that $fe = 0$. Thus, by continuity of multiplication, we have $(xe)^n \to fe = 0$. That is $xe \in N$. But this implies that $x \in$ Tod e, which is a contradiction. Hence we conclude that $f \notin$ Tod e.

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Since $e \notin$ Tod e, we have Tod $e \subset J_0(S - e)$. If Tod $e \neq J_0(S - e)$, then by Lemma 4.2, there is an idempotent $f \in J_0(S - e) -$ Tod e. Hence $ef \neq 0$. Since $(ef)\, e = ef$, we have $0 \neq ef \leqq e$. But e is a non-zero primitive idempotent of S. Hence $ef = e$. Thus $e \in J(e)\, J(f) \subset J(f) \subset J_0(S - e)$ which is a contradiction. Hence Tod $e = J_0(S - e)$. Now, applying the well-known theorem of K. Numakura (Theorem 2.5), we obtain immediately that Tod e is an open prime ideal of S.

Corollary 1. *If $E^{\#}$ consists of non-zero primitive idempotents, then N can be expressed as the itersection of a family of open prime ideals properly containing N.*

Proof. Immediate from Theorem 3.2 and Theorem 4.1.

Remark. In [5] K. Numakura proved that the set N is the intersection of all open prime ideals of S. His result is clearly strengthened here by considering the ideals Tod e in place of all open prime ideals.

Corollary 2. *Let $B_\alpha = \{x \in S \mid e_\alpha \in \Gamma(x)\}$ and let e be a non-zero primitive idempotent. Then B_α is a subsemigroup of S and* Tod e *is a union of B_α, that is,* Tod $e = \bigcup_\alpha B_\alpha$.

Proof. This follows from Schwarz's results on compact commutative semigroups [7].

Corollary 3. *Let S be a compact connected N-semigroup. If $e \in E^{\#}$ then there exists a compact group lying in the boundary of the set* Tod e.

Proof. Since S is a compact N-semigroup, Tod e is an open ideal of S. Then Tod $e \neq \overline{\text{Tod } e}$. By Lemma 4.2, we can find an idempotent $f \in \overline{\text{Tod } e} - \text{Tod } e$. Clearly f lies on the boundary Bd (Tod e) for $\text{Bd}(\text{Tod } e) = \overline{\text{Tod } e} \cap \overline{(S - \text{Tod } e)} = \overline{\text{Tod } e} \cap (S - \text{Tod } e)$. Now let $H(e)$ be the maximal group containing the idempotent e. As both Tod e and $\overline{\text{Tod } e}$ are ideals of S. We have $H(e) \subset \text{Bd}(\text{Tod } e)$, completing the proof.

Remark. We observe that the converse of Theorem 4.1 need not be true, that is, an open prime ideal of S need not correspond to an ideal Tod e for some $e \in E^{\#}$. The following example illustrates this.

Example 4.3. Let S be the teeth of the comb-space with zero adjoined, that is, $S = (\{0\} \cup \{1/n \mid n = 1, 2, \ldots\}) \times [0.1)$. The multiplication $*$ defined on S is given by

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2, \min\{y_1, y_2\}) .$$

We easily check that S is a topological semigroup with zero, and that all points lying on the lines $\{0\} \times [0,1)$ and $\{1\} \times [0,1)$ are idempotents of S. The non-zero primitive idempotent is the point $(1,0) = e$. Clearly $\text{Tod } e = J_0(S - e) = S - (\{1\} \times [0,1))$ which is an open prime ideal of S. If we consider the idempotent $e_1 = (1, \frac{1}{2})$, then for all $e \in E^*$, Tod e is not equal to $J_0(S - e_1)$.

In general, Tod e need not be a prime ideal. We have the following remark on finite semigroups.

Proposition 4.4. *Let S be a finite semigroup such that N is not equal to* Tod e *for all $e \in E^*$, then all* Tod e *must be prime ideals of S if $|S| \leqq 4$.*

Proof. If we want to construct a non-prime ideal Tod e in S, according to Theorem 4.1, we must require that e_1 to be non-primitive, that is, there is some non-zero idempotent g in S such that $g < e_1$. Moreover, we also observe that for any non-nil ideal Tod e, there exists always an idempotent $f \in \text{Tod } e$ such that $f \perp e$. Combining these two facts, one can easily derive that in order to construct a non-prime ideal Tod e in S, we must require that S contains at least one non-primitive idempotent and at least three other non-zero idempotents, or require that S contains at least one non-primitive idempotent, two non-zero primitive idempotents plus at least one other element. Thus, a non-prime ideal Tod e cannot exist unless $|S| \geqq 5$. We omit the details.

The following example shows how a non-prime ideal Tod e can be constructed in a semigroup S.

Example 4.5. Consider the semigroup with the following multiplication table.

$\cdot$	0	e	f	g	a	c
0	0	0	0	0	0	0
e	0	e	e	0	0	0
f	0	e	f	g	0	0
g	0	0	g	g	0	0
a	0	0	0	0	0	a
c	0	0	0	0	a	c

Then $N = \{0, a\}$, $\text{Tod} f = \{0, a, c\}$. Clearly $\text{Tod} f$ is not prime since $e \notin \text{Tod} f$ and $g \notin \text{Tod} f$, but $eg = 0 \in \text{Tod} f$.

Moreover $\text{Tod}\, e = \{0, g, a, c\}$, $\text{Tod}\, g = \{0, e, a, c\}$, $\text{Tod}\, c = \{0, e, f, g, a\}$. Thus $N = \text{Tod} f \cap \text{Tod}\, g \cap \text{Tod}\, e \cap \text{Tod}\, c$.

We would like to thank Dr. P. N. STEWART here for his comments which lead to the following:

Theorem 4.6. *Let $E_\sharp$ be the set of non-zero primitive idempotent s of S. Then* $N = \bigcap_{e \in E_\sharp} \text{Tod}\, e$, *where each* Tod e *is a minimal open prime ideal containing N. Conversely if P is a minimal open prime ideal containing N, then* P = Tod e *for some* $e \in E_\sharp$.

Proof. We first prove that if P is a minimal open prime ideal containing N, then $P = \text{Tod}\, e$ for some non-zero primitive idempotent e. Let P be an ideal with this property, then by Theorem 2.5 we can write $P = J_0(S - e)$ for some non-zero idempotent e. If e is not a non-zero primitive idempotent, then there exists a non-zero idempotent $e_1 < e$ such that $J_0(S - e_1) \subsetneqq J_0(S - e)$. (See [6], page 119). But then $J_0(S - e_1)$ is an open prime ideal of S, which contradicts to the minimality of P. Hence e is a non-zero primitive idempotent. Also $\text{Tod}\, e \subset J_0(S - e) = P$ and $\text{Tod}\, e$ is an open prime ideal. Thus $\text{Tod}\, e = J_0(S - e) = P$. Now $N = \bigcap$ all open prime ideals $= \bigcap$ all minimal open prime ideals $= \bigcap_{e \in E_\sharp} \text{Tod}\, e$. Our proof is completed.

Remark. If N itself is non-prime, then the set of all minimal open prime ideals of S properly containing N can be identified by the set of all non-zero primitive idempotents of S.

We now give a new version of the theorem of FAUCETT, KOCH and NUMAKURA [1].

Theorem 4.7. *Let e be a non-zero primitive idempotent of S. If the intersection of maximal ideals of S is nil, then the following conditions are equivalent.*

(1) $S - \operatorname{Tod} e$ *is a disjoint union of groups.*

(2) *For each element of* $S - \operatorname{Tod} e$ *there exists a unit element.*

(3) $a \in S - \operatorname{Tod} e$ *implies that* $a^2 \in S - \operatorname{Tod} e$.

(4) $S - \operatorname{Tod} e$ *contains an idempotent and the product of any two idempotents of* $S - \operatorname{Tod} e$ *lies in* $S - \operatorname{Tod} e$.

Proof. The proof uses a result of Schwarz [9]. It is proved there that a prime ideal of S is a maximal ideal if and only if it contains the intersection of all maximal ideal of S. Now let M be the intersection of all maximal ideals of S. By our hypothesis, M is nil. Hence $M \subset N$. Since $N \subset \operatorname{Tod} e$ we have $M \subset \operatorname{Tod} e$. By Theorem 4.1, $\operatorname{Tod} e$ is an open prime ideal; in fact, it is completely prime since S is commutative. Then, by Schwarz's result, $\operatorname{Tod} e$ is a maximal ideal of S. Hence by the theorem of Faucett, Koch and Numakura [1], the theorem follows.

Remark. If e is a non-zero idempotent of S, then (3) is always true by Corollary 3 of Theorem 3.2.

5. NIL IMPLIES NILPOTENT

The well-known theorem of Hopkins-Levitzki in ring theory states that if a ring R satisfies the descending chain condition (ascending chain condition) on its one-sided ideals, then any nil ideal of R is a nilpotent ideal of R. We show here that under some conditions, this theorem in ring theory can be transferred to compact N-semigroups without assuming the d.c.c. or a.c.c. on its ideals. In this section, the commutativity of S is not assumed.

Remark. In a compact N-semigroup, a nil ideal need not be nilpotent as the following example shows.

Example 5.1. Let S be the unit interval with the usual multiplication. Then $I = [0, 1)$ is a nil ideal (nil in the topological sense). However, I is not nilpotent since $I^n = I$ for all integers $n \geqq 1$.

Theorem 5.2. *Let S be a compact N-semigroup. If a non-nilpotent ideal I of S contains at least one closed non-nilpotent left (right) ideal of S, then I is non-nil.* (This is the Hopkins-Levitzki theorem on compact semigroups.)

The proof requires the use of the following result

Lemma 5.3. *Let S be a compact space and let $F = \{B_\lambda \mid \lambda \in \Lambda\}$ be a family of closed subspaces of S indexed by Λ. If A is an open subspace of S such that $\bigcap_{\lambda \in \Lambda} B_\lambda \subset A$, then there is a finite number of B_λ whose intersection is also contained in A.*

Proof. Since $\bigcap_{\lambda \in \Lambda} B_\lambda \subset A$ we have $A' \subset \bigcup_{\lambda \in \Lambda} B'_\lambda$. Each B'_λ is an open subspace of S and A' is compact in S. Thus $\{B'_\lambda\}_{\lambda \in \Lambda}$ is an open covering of A'. By the compactness of A', there is a finite subcovering of A', say $\{B'_\lambda\}_{\lambda=1}^{m}$. Hence $A' \subset \bigcup_{\lambda=1}^{m} B'_\lambda$ and hence $\bigcap_{\lambda=1}^{m} B_\lambda \subset A$.

Proof of Theorem 5.2. Let I be a non-nilpotent ideal of S. Let T be the collection of all closed non-nilpotent left ideals of S contained in I. Now T is partially ordered by inclusion and is non-empty by our hypothesis on I. Suppose $\{T_\alpha\}_\alpha$ is a linearly ordered subcollection of T. Then $\bigcap_\alpha T_\alpha$ is non-empty since S is compact. Hence $\bigcap_\alpha T_\alpha$ is a closed non-empty ideal of S. We claim that $\bigcap_\alpha T_\alpha$ is a non-nilpotent ideal. For if not, then $\bigcap_\alpha T_\alpha$ is nilpotent and hence is nil. Hence $\bigcap_\alpha T_\alpha \subset N$ where N is the set of all nilpotent elements of S. Since T_α is closed for all α and N is open, by Lemma 5.3, we can find finitely many T_α whose intersection is contained in N. Since $\{T_\alpha\}_\alpha$ is an inclusion tower, we have $T_\alpha \subset N$ for some α. But since T_α is a closed left ideal of S, by a result of K. Numakura ([5], page 675), T_α is nilpotent. This contradiction establishes our claim. Thus $\{T_\alpha\}$ has a lower bound, and Zorn's lemma assures us of the existence of a minimal closed non-nilpotent left ideal, say L_1 in I. We have $L_1^2 \subset L_1$, but since L_1 is non-nilpotent, we must have $L_1^2 = L_1$ by the minimality of L_1. Let $\mathscr{M}$ be the family of all left ideals J in S such that $L_1 J \neq 0$ and $J \subset L_1$. Then $\mathscr{M}$ is non-empty since $L_1 \in \mathscr{M}$. Since S is compact, applying the above arguments and Zorn's lemma, we see that $\mathscr{M}$ has a minimal closed left ideal of S, say J_1 such that $L_1 J_1 \neq 0$. Let $0 \neq x \in J_1$ be such that $L_1 x \neq 0$. Then $L_1 x$ is a closed left ideal of S, and $L_1(L_1 x) = L_1^2 x = L_1 x \neq 0$ and $L_1 x \subset L_1 J \subset L_1$. Hence $L_1 x \in \mathscr{M}$. Moreover, $L_1 x = J_1$ since $L_1 x \subset J_1$ and J_1 is minimal. Now let $a \in L_1$ be such that $ax = x$. Then for any integer $n \geqq 1$ we have $a^n x = x$, which implies that $a^n \nrightarrow 0$. Since $a \in L_1 \subset I$, I is therefore non-nil. This completes our proof.

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