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DECOMPOSITIONS OF THE STATE SPACE, HOMOMORPHISMS AND PRODUCTS OF SEMIGROUP ACTS

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1. INTRODUCTION

Let S be a (topological) semigroup and X a nonvoid T_2-space. Then an *act* [cf 4, 5, 6], denoted by the pair (X, S), is a continuous function $f : X \times S \to X$ such that $f(x, s_1 s_2) = f(f(x, s_1), s_2)$ for all $x \in X$ and all $s_1, s_2 \in S$. Throughout this paper, X and S, which are often termed as the *state space* and the *input semigroup*, respectively, will refer to an act (X, S) and $f(x, s)$ will be simply denoted by xs.

For $\emptyset \neq A \subseteq X$ and $\emptyset \neq T \subseteq S$, let $AT = \{xs : x \in A \text{ and } s \in T\}$ and $AT^{(-1)} = \{x : x \in X \text{ and } xT \cap A \neq \emptyset\}$. An *orbit* (*a point-inverse set*) is a set of the form xS ($xS^{(-1)}$) for some $x \in X$. An orbit is *maximal* if it is not properly contained in an orbit. A *minimal* orbit and a *maximal* (*minimal*) *point-inverse set* are analogously defined. An act (X, S) is *compact* if both X and S are so, and is *unitary* if $x \in xS$ for each $x \in X$. An act whose orbits, or maximal orbits (point-inverse sets) form a partition of the state space will be called a *quasi-transitive*, or *disjoint* (i-disjoint) act, respectively. For all other unexplained concepts concerning acts reference is made to DAY [4].

If S is a group and $XS = X$ the orbits partition X but if S is merely a semigroup various kinds of overlapping of orbits are possible. This paper results from an attempt to study semigroup acts from the above consideration and results concerning disjoint (and i-disjoint) acts, quasi transitive acts, how a homomorphism maps a maximal (minimal) orbit (point-inverse set), or a disjoint (i-disjoint) act onto a similar object, and how a product of acts inherit similar properties from the component acts, are presented in Sections 2, 3, 4 and 5, respectively. Some of these results were reported in [10].

2. DISJOINT (I-DISJOINT) ACTS

To start with let us state the following remarks without proof.

Remark 2.1. Let (X, S) be a compact act.

(a) Every orbit is contained in a maximal orbit [cf. 10, 1] and every orbit contains a minimal orbit. If $XS = X$, then the family F of maximal orbits form a minimal cover of X (i.e., $UF = X$ and no sub-family of F has this property).

(b) If the act is also unitary, then xS is a maximal (minimal) orbit iff $xS^{(-1)}$ is a minimal (maximal) point-inverse set. Consequently, statements similar to those in (a) hold good for maximal point-inverse sets.

Though, in general, an act need not be disjoint, the following is true.

Proposition 2.2. [cf. 10, 1]. *Let (X, S) be a compact act. Then there exists a disjoint act (X^*, S) whose homomorphic image is (X, S). Further, if the set $Y = \{x : xS$ is maximal orbit of $(X, S)\}$ is closed, then X^* is compact.*

The following gives several characterizations of disjoint acts.

Proposition 2.3. *Let (X, S) be a compact unitary act. Then the following statements are equivalent.*

(1) *The maximal orbits form a decomposition of X.*
(2) *For any distinct pair $x, y \in X$, $xS \cap yS \neq \emptyset$ implies that $xS^{(-1)} \cap yS^{(-1)} \neq \emptyset$.*
(3) *For any $\emptyset \neq A \neq B \subseteq X$, $AS \cap BS \neq \emptyset$ implies that $AS^{(-1)} \cap BS^{(-1)} \neq \emptyset$.*
(4) *Each point-inverse set contains a unique minimal point-inverse set.*
(5) *Each orbit is contained in a unique maximal orbit.*
(6) *Each maximal orbit is a union of maximal point-inverse sets.*
(7) *Each maximal orbit is a union of point-inverse sets.*
(8) *There exists a (unique) equivalence relation on X with closed graph such that each equivalence class is an orbit.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5). Easy.

(5) ⇒ (6). Suppose xS is a maximal orbit and $\{x_\alpha S\}$ are all the minimal orbits contained in xS. We claim that $\bigcup x_\alpha S^{(-1)} = xS$. If $y \in x_\alpha S^{(-1)}$, then $x_\alpha S \subseteq yS \subseteq xS$. So $y \in xS$. Conversely, if $y \in xS$, then $yS \subseteq xS$ and if $y'S$ is a minimal orbit contained in $yS \subseteq xS$, then $y \in yS^{(-1)} \subseteq y'S^{(-1)} \subseteq x_\alpha S^{(-1)}$.

(6) ⇒ (7). Trivial.

(7) ⇒ (6). Let xS be a maximal orbit which is a union of point-inverse sets $\{x_\alpha S^{(-1)}\}$. Let $\{x^\alpha S^{(-1)}\}$ be all the maximal point-inverse sets in which one or more of $x_\alpha S^{(-1)}$ are contained. We claim that $xS = \bigcup x^\alpha S^{(-1)}$. Clearly, $xS \subseteq x^\alpha S^{(-1)}$ which contains some $x_\alpha S^{(-1)}$, as $x_\alpha S^{(-1)} \subseteq xS$, $xS^{(-1)} \subseteq x_\alpha S^{(-1)} \subseteq x^\alpha S^{(-1)}$. Hence,

for some $s \in S$, $xs = x^{\alpha}$. Thus $x^{\alpha} \in xS$ and for some x_{β} such that $x_{\beta}S^{(-1)} \subseteq xS$, $x^{\alpha} \in x_{\beta}S^{(-1)}$ so that $x^{\alpha}S^{(-1)} = x_{\beta}S^{(-1)}$ as $x^{\alpha}S^{(-1)}$ is maximal. Hence, $x^{\alpha}S^{(-1)} \subseteq xS$.

(6) $\Rightarrow$ (1). We first make two observations.

(a) For any two minimal orbits xS and yS, $xS \cap ys^{(-1)} \neq \emptyset$ iff $xS = yS$. For, if $xS \cap yS^{(-1)} \neq \emptyset$, then there exist $s, t \in S$ such that $xst = y$ and, hence, $xS = yS$.

(b) If a maximal orbit xS is a union of maximal point-inverse sets $\{x_{\alpha}S^{(-1)}\}$, then $\{x_{\alpha}S\}$ are indeed all the minimal orbits contained in xS. For, (a) implies that if yS is any minimal orbit contained in xS, then $ys = x_{\alpha}S^{(-1)} \neq \emptyset$ for some α, and so $yS = x_{\alpha}S$.

Now suppose x_1S and x_2S are any two maximal orbits which intersect. Then there exists a minimal orbit $yS \subseteq x_1S \cap x_2S$ and, hence, by (b), $yS^{(-1)} \subseteq x_1S \cap x_2S$. Let $zS^{(-1)}$ be a minimal point-inverse set contained in $yS^{(-1)}$. Then zS, a maximal orbit, is contained in $x_1S \cap x_2S$ as $z \in zS^{(-1)}$ and, therefore, $zS = x_1S = x_2S$.

(1) $\Rightarrow$ (8). Define $\varrho \subseteq X \times X$ as $x \, \varrho \, y$ if x and y are contained in the same maximal orbit. Then ϱ has the required properties.

(8) $\Rightarrow$ (1). If ϱ is such an equivalence relation then note that each equivalence class is a maximal orbit. For, let $[x]$ be an equivalence class containing x and suppose $[x] = xS$. If $xS \subseteq yS \subseteq [y]$, the equivalence class containing y, for some $y \in X$, then $x \in yS \subseteq [y]$ implies that $xS = [x] \subseteq [y]$. Then it follows that $[y] = [x]$ and, hence, $xS = yS$. Further, as ϱ has closed graph each equivalence class is closed and, hence, compact.

Remark 2.4. There exist analogous characterizations for i-disjoint acts.

A characterization of acts which are both disjoint and i-disjoint is the following

Proposition 2.5. *Let (X, S) be a compact unitary act. Then the following two statements are equivalent.*

(a) *(X, S) is both disjoint and i-disjoint.*

(b) *Each maximal orbit is a maximal point-inverse set and vice-versa.*

Proof. (a) $\Rightarrow$ (b). Let xS be a maximal orbit. Then, as (X, S) is i-disjoint, by virtue of Remark 2.4, if yS is the unique minimal orbit contained in xS, we claim that $xS = yS^{(-1)}$. If $z \in xS$, then $zS \subseteq xS$ and zS contains a unique minimal orbit which must be yS and, hence, $z \in yS^{(-1)}$. Conversely, if $z \in yS^{(-1)}$, then $yS \subseteq zS$, and, hence, as (X, S) is disjoint, by virtue of Proposition 2.3 (5), the unique maximal orbit in which zS is contained in must be xS. Therefore, $z \in xS$.

To prove that each maximal point-inverse set is a maximal orbit we can apply similar arguments.

(b) $\Rightarrow$ (a). Suppose two maximal orbits x_1S and x_2S intersect and suppose $y_1S^{(-1)}$ and $y_2S^{(-1)}$ are two maximal point-inverse sets which equal x_1S and x_2S, respectively.

There exists a minimal orbit $zS \subseteq x_1S \cap x_2S$ so that $zS \subseteq y_1S^{(-1)} \cap y_2S^{(-1)}$ which implies that both y_1 and y_2 are in zS and, therefore, equivalently, $y_1S = y_2S = zS$ as zS is minimal. Therefore, $x_1S = x_2S$ and, hence, (X, S) is disjoint.

Similarly, it can be shown that (X, S) is i-disjoint.

3. QUASI-TRANSITIVE ACTS

In this section acts for which any two distinct orbits are disjoint are studied. A semigroup S acts on X *point-transitively* if $xS = X$ for some $x \in X$, *quasi-transitively* if $XS = X$ and for any $x, y \in X$, $y \in xS$ implies that $x \in yS$ and *transitively* if $xS = X$ for all $x \in X$.

First, note that an act (X, S) is quasi-transitive iff it is unitary and each orbit is minimal as well as maximal, and is transitive iff it is point-transitive and quasi-transitive. Then some characterizations for quasi-transitive compact acts are stated below.

In what follows let K, E and R stand for the minimal ideal, the set of idempotents and any minimal right ideal of S respectively and $H(e)$ stand for the maximal subgroup of S containing $e \in E$.

Proposition 3.1. *Let (X, S) be a compact act. Then the following statements are equivalent.*

(1) *S acts quasi-transitively on X.*
(2) *The orbits form a decomposition of X (i.e., the orbits partition X and each orbit is closed).*
(3) *R acts unitarily on X.*
(4) *For each $e \in K \cap E$, $(Xe, H(e))$ is a topological transformation group and $\bigcup\{Xe : e \in R \cap E\} = X$.*
(5) *For each $x \in X$, there exists $e \in R \cap E$ such that $x = xe$.*
(6) *For each $x \in X$, there exists $e \in K \cap E$ such that $x = xe$.*
(7) *K acts unitarily on X.*

Proof.

(1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (3). For any $x \in X$, $x \in xS$ and xS is a minimal orbit. Now xR is a minimal orbit and $xR \subseteq xS$. So $x \in xR = xS$.

(3) $\Rightarrow$ (4). For any $e \in K \cap E$, $Xe\,H(e) = Xe\,eSe = Xe$. $Se = XRe = Xe$. Also note that $XH(e) = Xe$. Now $X = XR = X(\bigcup\{H(e) : e \in R \cap E\}) = \bigcup\{XH(e) : e \in R \cap E\} = \bigcup\{Xe : e \in R \cap E\}$.

(4) $\Rightarrow$ (5). Since for any $x \in X$, $x \in Xe$ for some $e \in R \cap E$, we then have $x = xe$.

(5) $\Rightarrow$ (6) $\Rightarrow$ (7). Trivial.

$(7) \Rightarrow (1)$. Since $K = \bigcup R$, for any $x \in X$, $x \in xK$ implies that $x \in xR$ for some R and xR is a minimal ideal and, hence, $xR = xS$ because $x \in xR$ implies that $xS \subseteq \subseteq xRs \subseteq xR$. Thus, each orbit xS is minimal and S acts unitarily on X. Hence, (1) follows.

It is worth noting that R (or K) acts unitarily on X iff $XR = X$ (or $XK = X$) [cf. 2].

With further restrictions on X or S or both we have a few more results regarding quasi-transitive acts. Some parts of Propositions 3.2 and 3.3 are similar to results of Stadlander [7, 12].

Proposition 3.2. *Let (X, S) be a compact act. If either, S is left simple or $S^2 = S$ and S is normal, or S acts commutatively on X, then the following statements are equivalent.*

(1) *S acts quasi-transitively on X.*

(2) *For each $e \in K \cap E$, $(xS, H(e))$ is a topological transformation group for any $x \in X$ and $XS = X$.*

(3) *For each $e \in K \cap E$, $(X, H(e))$ is a topological transformation group.*

Proof. $(1) \Rightarrow (2)$. Note that $H(e)$ is a compact topological group for each $e \in K \cap \cap E$ and, by virtue of our assumptions, for each $x \in X$, $xS\,H(e) = x\,Se\,Se = xS^2e = = xSe = XeS = xR = xS$. Also $XS = X$.

$(2) \Rightarrow (3)$. For any $e \in K \cap E$, note that, $X\,H(e) = (\bigcup\{xS : x \in X\})\,H(e) = = \bigcup\{xS : x \in X\} = X$.

$(3) \Rightarrow (1)$. Since for any $e \in K \cap E$, $H(e)$ acts on X unitarily and so far any $x \in X$, $xS\,H(e) \subseteq xS$ implies that $xS = xS\,H(e) = xS\,x\,Se = x\,e\,S = xR =$ a minimal ideal. Note that S acts unitarily on X and so (1) follows.

Proposition 3.3. *Let (X, S) be a compact act. If either, S is left-simple or S is normal, or S acts commutatively on X, then the following statements are equivalent.*

(1) *S acts quasi-transitively on X.*

(2) *$\varrho_s : X \to X$, $\varrho_s(x) = xs$, is a homeomorphism for all $s \in K$.*

(3) *For any $e \in K \cap E$, $x = xe$ for all $x \in X$.*

(4) *ϱ_s, as defined in (2), is a homeomorphism for some $s \in K$.*

(5) *For some $e \in K \cap E$, $x = xe$ for all $x \in X$.*

Proof. $(1) \Rightarrow (2)$. For each $e \in K \cap E$, $Xe = Xe\,H(e) = X\,e\,S\,e = X\,e\,S = XR = X$.

Since $(Xe, H(e))$ is a topological transformation group for each $e \in E \cap K$, it follows that ϱ_s is a homeomorphism for each $s \in H(e)$ and, hence, for each $s \in K = \bigcup\{H(e) : : e \in K \cap E\}$.

$(2) \Rightarrow (3) \Rightarrow (1)$. Let $e \in K \cap E$ and $s \in H(e)$. Then ϱ_s is a homeomorphism and hence $XH(e) = X$. Therefore, $x = xe$ for all $x \in X$ and so, by Proposition 3.4, S acts quasi-transitively on X.

$(2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$. Trivial.

In the above proposition we established equivalence of the statements (2) and (4) under the assumption of normality of S or the commutativity of the act. This is, however, not necessary as we have the following simple result.

Proposition 3.4. *Let (X, S) be a compact act such that $\varrho_s : X \to X$, $\varrho_s(x) = xs$, is a homeomorphism for some $s \in K$. Then ϱ_s is a homeomorphism for all $s \in K$.*

Proof. Note that $K = \bigcup\{H(e) : e \in K \cap E\}$ and so, if, for $s\, e\, H(e)$, ϱ_s is a homeomorphism, then $(X, H(e))$ is a topological transformation group. Then, via isomorphisms of $H(e)$ and $H(f)$, $e, f \in E \cap K$ [cf. Theorem 1.2.6,9], it follows that $(X, H(f))$ is also a topological transformation group for all $f \in K \cap E$. Hence, the assertion follows.

In Proposition 3.3 we have proved equivalence of quasitransitive acts and acts where each transition map $\varrho_s : X \to X$, $\varrho_s(x) = xs$, for $s \in K$ is a homeomorphism under certain hypotheses. The implication from the homeomorphism of ϱ_s's to quasi-transitivity of the acts does not demand all these hypotheses. However, the assumption of ϱ_s's to be homeomorphisms is sufficiently strong and has some implication towards the algebraic structure of the input semigroup as seen in the following result.

Proposition 3.5. *Let a compact semigroup S act effectively on X (i.e., for $s, t \in S$, $s \neq t$ implies that for some $x \in X$, $xs \neq xt$). Then for each $s \in S$, the transition map $\varrho_s : X \to X$, $\varrho_s(x) = xs$, is $1-1$ iff (X, S) is a topological transformation group.*

Proof. It is sufficient to verify that under the hypothesis if each ϱ_s is $1-1$, then S is a topological group and $x1 = x$ for all $x \in X$ where 1 is the identity in S.

To prove that S is a topological group, by theorem 1.1.15 [9], we need only to show that S is cancellative. Now if for $s_1, s_2, t \in S$, $s_1 t = s_2 t$, then $xs_1 t = xs_2 t$ for all $x \in X$, and, as ϱ_t is $1-1$, $xs_1 = xs_2$ for all $x \in X$. Again, S acts effectively on X, and hence, $s_1 = s_2$. Similarly, $ts_1 = ts_2$ implies that $s_1 = s_2$ since $xt = x$. Thus, S is cancellative. Further, if 1 is the identity in S, then, as ϱ_1 is $1-1$, it follows that $xs = x$ for all $x \in X$. Hence, the result follows.

Note that in the above proposition the assumption of effective action can be dropped if we demand that $(X, S/\varrho)$ should be a topological transformation group where $s\varrho t$ if $xs = xt$ for all $x \in X$.

There is an analogous result in Day [4] which states that if (X, S) is effective and compact such that $Xs = X$ for all $s \in S$, then S must be a group.

There exist somewhat similar results concerning transitive acts [10]. Further, via a result on point-transitive acts [7, 8, 12] and a result on transitive acts [5, 8] it is easy to give characterizations of decompositions of a nonvoid T_2-space X induced by a disjoint or quasi-transitive action of a semigroup on it [10]. All these are simple and, hence, omitted.

4. ON HOMOMORPHISMS OF ACTS

Throughout this section we let h to be a homomorphism from a compact act (X, S) onto a compact act (Y, S), that is, h is a map from X onto Y, which need not be continuous, such that $h(xs) = h(x)s$ for all $x \in X$ and all $s \in S$. We investigate how h maps each maximal (minimal) orbit (point inverse set) or a disjoint (i-disjoint) acts onto a maximal (minimal) orbit (point inverse set) or a disjoint (i-disjoint) act respectively. This section is mainly algebraic.

Clearly, h maps an orbit onto an orbit and every maximal orbit yS of (Y, S) is h-image of some maximal orbit xS of (X, S).

But h-image of a maximal orbit need not be a maximal orbit. However, we have:

Proposition 4.1. *h maps each maximal orbit onto a maximal orbit if, for any $x_1, x_2 \in X$, (1) $h(x_1S \cap x_2S) = h(x_1S) \cap h(x_2S)$, and (2) $x_3 \notin x_1S \cap x_2S$ implies that $h(x_3) \notin h(x_1S \cap x_2S)$.*

Proof. Easy.

Proposition 4.2. *h maps each maximal orbit onto a maximal orbit if for any $x_1, x_2 \in X$, $C = h(x_1)S \cap h(x_2)S \neq \emptyset$ implies that if $C \subsetneqq h(x_1)S$, then $C \subsetneqq h(x_3)S$ for some $x_3 \in X$ such that $x_2S \subseteq x_3S$.*

Proof. Easy.

Corollary 4.3. *If h is $1-1$, then h maps maximal orbit onto a maximal orbit.*

Regarding disjoint acts, we have the following two results.

Proposition 4.4. *h maps a disjoint act (X, S) onto a disjoint act (Y, S) if, for any $y \in Y$, $h^{-1}(y) = xA$ for some $x \in X$ and $\emptyset \neq A \subseteq S$.*

Proof. Let, if possible, two maximal orbits y_1S and y_2S of (Y, S) intersect. Then suppose x_1S and x_2S are two maximal orbits of (X, S) such that $h(x_i)S = y_iS$, $i = 1, 2$. Now, for $y \in y_1S \cap y_2S \neq \emptyset$, $h^{-1}(y) \cap x_iS \neq \emptyset$, $i = 1, 2$. Now note that as (X, S) is disjoint, $h^{-1}(y) = xA$ for some $x \in X$ and $\emptyset \neq A \subseteq S$ iff $h^{-1}(y)$ is contained in a unique maximal orbit; and, furthermore, $h^{-1}(y) \subseteq x_1S \cap x_2S$ which implies that $x_1S = x_2S$. Hence, $y_1S = y_2S$.

Proposition 4.5. *The following two statements are equivalent.*

(a) (Y, S) *is disjoint and* h *maps each maximal orbit onto a maximal orbit.*

(b) *For any two maximal orbits* $x_i S$, $i = 1, 2$, *of* (X, S) $\bigcap h(x_i) S \neq \emptyset$ *implies* $h(x_1 S) = h(x_2 S)$.

Proof. Easy.

Concerning minimal orbits, note that h maps each minimal orbit onto a minimal orbit, and each minimal orbit of (Y, S) is h-image of some minimal orbit of (X, S). Therefore, a homomorphic image of a quasi-transitive (transitive) act is quasi-transitive (transitive).

We next consider maximal point-inverse (mpi) sets and homomorphisms.

Proposition 4.6. *Every mpi set* $yS^{(-1)}$ *of* (Y, S) *is* h*-image of a union of mpi sets* $\{x_\alpha S^{(-1)}\}$ *of* (X, S) *such that* $h(x_\alpha) S = yS$.

Proof. Notice that $yS^{(-1)}$ is an mpi set iff yS is a minimal orbit and there exists a minimal orbit in (X, S) whose h-image is yS. So, suppose $\{x_\alpha S\}$ are all the minimal orbits of (X, S) such that $h(x_\alpha) S = yS$. We claim that $yS^{(-1)} = \bigcup h(x_\alpha S^{(-1)})$. Note that $h(xS^{(-1)}) \subseteq h(x) S^{(-1)}$ for any $x \in X$ and $h(x_\alpha S) = yS$ if $h(x_\alpha) S^{(-1)} = yS^{(-1)}$. Therefore, $h(x_\alpha S^{(-1)}) \subseteq yS^{(-1)}$ and, hence, $\bigcup h(x_\alpha S^{(-1)}) \subseteq yS^{(-1)}$. Conversely, let $z \in yS^{(-1)}$. Then, for some $x \in X$, $h(x) = z$ and there is $s \in S$ such that $h(x) s = y$ and $h(x) s S = yS$. There exists a minimal orbit $x'S \subseteq x s S \subseteq xS$ so that $h(x'S) \subseteq$ $\subseteq h(x) s S = yS$. Now $x' = xst$ for some $t \in S$ and so $x \in x'S^{(-1)}$. So $h(x) = z \in$ $\in h(x'S^{(-1)}) \subseteq \bigcup h(x_\alpha S^{(-1)})$.

Proposition 4.7. (Y, S) *is i-disjoint iff for any two mpi sets* $x_i S^{(-1)}$, $i = 1, 2$, *of* (X, S), $\bigcap h(x_i S^{(-1)}) \neq \emptyset$ *implies that* $h(x_1) S = h(x_2) S$.

Proof. 'Only if' part follows from Proposition 4.6.

Conversely, let for any two mpi sets $x_i S^{(-1)}$, $i = 1, 2$, of (X, S), $\bigcap h(x_i S^{(-1)}) \neq \emptyset$. Then $\bigcap h(x_i) S^{(-1)} \neq \emptyset$ as $h(x, S^{(-1)}) \subseteq h(x) S^{(-1)}$ for any $x \subseteq X$. Then, as (Y, S) is i-disjoint, it follows that $\bigcap h(x_i) S \neq \emptyset$ and, hence, $h(x_1) S = h(x_2) S$.

In general, $h(xS^{(-1)}) \subseteq h(x) S^{(-1)}$ for any $x \in X$ and $h(xS^{(-1)}) = h(x) S^{(-1)}$ iff for any $a \in h(x) S^{(-1)}$, $h^{-1}(a) \cap xS^{(-1)} \neq \emptyset$. The following gives a sufficient condition for the latter to happen in case of mpi sets.

Proposition 4.8. h *maps each mpi set of* (X, S) *onto an mpi set of* (Y, S) *if for any two mpi sets* $x_i S^{(-1)}$, $i = 1, 2$ *of* (X, S), $\bigcap h(x_i S^{(-1)}) \neq \emptyset$ *implies that* $h(x_1 S^{(-1)}) =$ $= h(x_2 S^{(-1)})$.

Proof. Let $xS^{(-1)}$ be an mpi set of (X, S). Let $h(xS^{(-1)}) \subseteq yS^{(-1)}$, an mpi set in (Y, S) such that $h(x) S = yS$. Now $yS^{(-1)} = \bigcup \{h(x_\alpha S^{(-1)}) : h(x_\alpha S) = yS\}$ and, for any α, β such that $h(x_\alpha S) = h(x_\beta S) = yS$, since $\emptyset \neq yS \subseteq h(x_\alpha S^{(-1)}) \cap$

$\cap h(x_\beta S^{(-1)})$, it follows that $h(x_\alpha S^{(-1)}) = h(x_\beta S^{(-1)})$. So, $h(xS^{(-1)}) = yS^{(-1)} = h(x)\, S^{(-1)}$.

Proposition 4.9. *Let (X, S) be disjoint. Then (Y, S) is i-disjoint if for any two mpi sets $x_i S^{(-1)}$ of (X, S) that intersect $h(x_1 S^{(-1)}) = h(x_2 S^{(-1)})$. If h maps each mpi set onto an mpi set then this condition is also necessary.*

Proof. It is sufficient to show that any mpi set $yS^{(-1)}$ of (Y, S) is a union of orbits. By Proposition 4.6, $yS^{(-1)} = \bigcup h(x_\alpha S^{(-1)})$ where $x_\alpha S$ are all the minimal orbits of (X, S) such that $h(x_\alpha)\, S = yS$. Since (X, S) is disjoint, each maximal orbit xS is a union of mpi sets corresponding to the minimal orbits contained in xS, and, by the condition of the Proposition, if $xS = \bigcup x_\beta S^{(-1)}$, then $x \in \bigcap x_\beta S^{(-1)}$ implies that $h(xS) = h(x_\beta S^{(-1)})$ for each β. So, if $yS^{(-1)} = \bigcup h(x_\alpha S^{(-1)})$, from the disjointness of (X, S) and the condition of the Proposition, it follows that there exist maximal orbits $\{x^\alpha S\}$ such that $\bigcup x^\alpha S = \bigcup x_\alpha S^{(-1)}$ and $\bigcup h(x^\alpha S) = \bigcup h(x_\alpha S^{(-1)}) = yS^{(-1)}$ which is a union of orbits.

To prove the other way, suppose (Y, S) is i-disjoint and h maps each mpi set onto an mpi set. Each mpi set of (Y, S) is a union of maximal orbits. Suppose two mpi sets $x_i S^{(-1)}$, $i = 1, 2$, of (X, S) intersect. As (X, S) is disjoint, $\bigcup x_i S^{(-1)} \subseteq xS$, a maximal orbit. So, $\bigcup h(x_i S^{(-1)}) \subseteq h(x)\, S \subseteq yS$, a maximal orbit. As (Y, S) is i-disjoint yS is contained in some mpi set $y' S^{(-1)}$ and since $h(x_i S^{(-1)}) = h(x_i)\, S^{(-1)}$, an mpi set for $i = 1, 2$, it follows that $\bigcup h(x_i)\, S^{(-1)} \subseteq y' S^{(-1)}$ and, hence, $h(x_i)\, S^{(-1)} = y' S^{(-1)}$. Thus, $h(x_1 S^{(-1)}) = h(x_2 S^{(-1)})$.

5. PRODUCTS OF ACTS

Let (X_i, S) and (X, S) be two families of acts. The product acts $(\Pi X_i, \Pi S_i)$ and $(\Pi X_i, S)$ are defined in a natural way using coordinatewise operations. In this section we make note of how does a product of acts inherit a given property P from the component acts where P may be disjointness (i-disjointness), transitiveness (quasi-transitiveness) of acts, etc. We first note the following

Proposition 5.1. *Let a compact semigroup S act quasi-transitively on X. Then the equivalence relation R on X defined by xRy if $xS = yS$ is open and has a closed graph and, consequently, the quotient space X/R is Hausdorff.*

Proof. Let $A = \bigcup_{x \in A} xS \subseteq X$. Then note that $\bar{A} = \bigcup_{x \in \bar{A}} xS$ where $\bar{A}$ is the closure of A. For, $A \subseteq \bigcup_{x \in \bar{A}} xS$ since the action is unitary. Further, if $y \in \bar{A}$, then $ys \in \bar{A}$ for all $s \in S$ since there exists in A a net $y_\alpha \to y$ implies that, by the continuity of act, for any $s \in S$ in A the net $y_\alpha s \to ys$.

Then, by Proposition 6 in [p. 54, 3], R is open. That R has a closed graph needs a standard net argument.

Therefore, by Proposition 8 in [p. 79, 3], X/R is Hausdorff.

Remark 5.2. *Let $\{S_i\}$ be a family of compact semigroups. Then ΠK_i is the minimal ideal of ΠS_i iff K_i is the minimal ideal of S_i for each i.*

Remark 5.3. *$(\Pi X_i, \Pi S_i)$ is unitary iff (X_i, S_i) is so for each i.*

Proposition 5.4. *Let $\{S_i\}$ be a family of compact semigroups. Then $(\Pi X_i, \Pi S_i)$ is quasi-transitive (transitive) iff (X_i, S_i) is so for each i. Further, in that case, $(\Pi X_i/R, \Pi S_i)$ is isomorphic to $(\Pi(X_i/R_i), \Pi S_i)$ where R and R_i are the equivalences of ΠX_i and X_i induced by the quasi-transitive actions of ΠS_i and S_i, respectively.*

Proof. The first part for quasi-transitive case follows from Proposition 3.1 and Remarks 5.2 and 5.3. The second part follows from Proposition 5.1 and corollary to Proposition 8 in [p. 55, 3].

Proposition 5.5. *Let $\{(X_i, S_i)\}$ be a family of compact acts.*

(1) *For each $(x_i) \in \Pi X_i$, $(x_i)\, \Pi S_i$ is a maximal (minimal) orbit iff each $x_i S_i$ is so.*
(2) *$(\Pi X_i, \Pi S_i)$ is disjoint (i-disjoint) iff each (X_i, S_i) is so.*

Proof. Easy.

While $(\Pi X_i, \Pi S_i)$ inherits most of the properties mentioned in the beginning of this section it is not so for $(\Pi X_i, S)$. In fact, without much restriction on S nothing can be said. In view of Proposition 3.2, we can only state the following

Proposition 5.6. *Suppose a compact semigroup S acts on X_i, $i \in J$. If either (1) S is left-simple or (2) $S^2 = S$ and S is normal or S acts commutatively on X_i, then S acts quasi-transitively on ΠX_i iff S acts quasi-transitively on each X_i.*

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