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NONLINEAR EQUATIONS WITH NONINVERTIBLE LINEAR PART

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1. INTRODUCTION

Many equations of analysis have the abstract formulation as the equation of the following form

$$\lambda\, T(x) - S(x) = f\,, \tag{1}$$

where T and S are (not necessarily linear) operators acting from a Banach space X into a Banach space Z and $\lambda \neq 0$ is a real parameter. The operator S is supposed to be completely continuous. If T is a homeomorphism from X onto Z (i.e., "T behaves as the identity operator") then the equation (1) is equivalent to the equation

$$x = T^{-1}\left(\frac{1}{\lambda}\left[S(x) + f\right]\right)$$

and it is possible to use the Leray-Schauder degree theory for completely continuous mappings. The so-called Fredholm alternative (see e.g. [4, 5, 8]) deals with this case of nonlinear equations, Borsuk-Ulam theorem about the Leray-Schauder degree for odd and completely continuous operators is used and it is proved that under some additional assumptions (the operators T and S are asymptotically homogeneous with the asymptotes T_0 and S_0 – for details see e.g. [8, Chapter II]) the equation (1) is solvable for any right hand side $f \in Z$ provided the asymptotical equation

$$\lambda\, T_0(x) - S_0(x) = 0 \tag{2}$$

has only trivial solution.

The situation in the case that the asymptotical equation (2) has nontrivial solution is very complicated. Nevertheless, it is very important to study this case (to obtain a complete analogue of Fredholm alternative for linear operators in the nonlinear setting).

In this paper we give some sufficient conditions for $f \in Z$ to be an element of the range of the operator $T - S$ (Theorems 2.3.10, 2.4.1, 2.5.5), where T is a linear operator from X onto Z which is not necessarily continuous and invertible and S is a completely continuous mapping. Such sufficient conditions imply also the surjectivity of the operator $T - S$ (see Theorem 2.4.1 and Corollary 2.5.6) and yield also necessary and sufficient conditions on $f \in Z$ for the equation $T(x) - S(x) = f$ to be solvable in the case that the range of the operator $T - S$ is not the whole Z (see Theorem 2.5.8).

Such a result for boundary value problems for partial differential equations whose "linear part" has a simple spectrum was given for the first time by E. M. LANDESMAN and A. C. LAZER [11]. This result was generalized by S. A. WILLIAMS [22] to the case of multiple spectrum of the linear part. The abstract setting of the method from the papers [11, 22] is given by J. NEČAS in [15], where the range of the operator $T - S$ acting in a Hilbert space is described. A generalization of the results of [15] is given in the paper by S. FUČÍK, M. KUČERA and J. NEČAS [7], where also the surjectivity of $T - S$, S being the so-called sublinear operator (see e.g. Section 2.4), is studied. The surjectivity of $T - S$, where S is the so-called asymptotically linear operator, is investigated in the paper by S. FUČÍK [6]. All above mentioned papers deal with the operators defined on the whole Hilbert space. In this paper we prove analogous results for operators T and S densely defined in a Banach space X with values in a Banach space Z which provides better possibilities for applications. Moreover, in this paper the so-called "superlinear case" is investigated which is not included in previous papers.

Proofs of the abstract theorems are given in terms of the theory of bifurcations. A similar method was used to obtain the so-called Cesari-Lazer alternative lemma e.g. in the papers of L. CESARI [2], S. BANCROFT, J. K. HALE and D. SWEET [1], J. LOCKER [12] and also W. S. HALL [10]. Also the papers of M. SOVA [20, 21], where applications to hyperbolic partial differential equations are given, are based on the same principle. An analogue of the Cesari-Lazer alternative lemma in this paper is Theorem 2.2.3.

Another version of the "bifurcation" lemma (the so-called equivalence theorem) together with the Leray-Schauder degree are used in the paper of J. MAWHIN [13] to establish a coincidence degree theory (the coincidence degree obtained is set-valued) for couples of mappings (T, S) between locally convex topological vector spaces where the (not necessarily continuous) Fredholm linear mapping T and the (not necessarily linear) mapping S satisfy some auxiliary conditions. Note that we use from this paper the notion of a suitable mapping (see Definition 2.2.1) which is useful for the formulation of the auxiliary equation (see Theorem 2.2.3). Let us remark for the completeness that the degree theory for mappings $T - S$ is also established in the paper of L. NIRENBERG [17], from which the result of E. A. Landesman and A. C. Lazer [11] follows.

The abstract results from Part 2 may be applied to the existence of solutions for various types of boundary value problems for ordinary and partial differential equations as well as to the existence theorems for nonlinear integral equations. Part 3 is not intended to be the detailed study of applications but serves as an example of the applicability of the main theorems. Therefore the examples of sufficient and necessary-sufficient conditions for solvability of periodic and boundary value problems for ordinary differential equations are given under more restrictive assumptions than necessary and the boundary value problems for partial differential equations are not solved at all in this paper. This will be perhaps (together with the connection to previous results) the main subject of a next paper.

2. ABSTRACT RESULTS

2.1. Terminology and notation. Let X and Z be two vector spaces. If T is a mapping defined on the set $\mathrm{Dom}\,[T] \subset X$ with the values in the space Z (we write $T : \mathrm{Dom}\,[T] \subset X \to Z$), denote by $\mathrm{Im}\,[T] \subset Z$ the set of all values of the mapping T, i.e.

$$\mathrm{Im}\,[T] = T(\mathrm{Dom}\,[T]) .$$

Let

$$L : \mathrm{Dom}\,[L] \subset X \to Z$$

be a linear mapping. In this case we shall suppose that $\mathrm{Dom}\,[L]$ is a vector subspace of the vector space X and denote by $\mathrm{Ker}\,[L]$ the null space of the operator L, i.e.,

$$\mathrm{Ker}\,[L] = \{x \in \mathrm{Dom}\,[L] : L(x) = O_Z\}$$

(O_X and O_Z denote the zero elements of X and Z, respectively).

Let Y_1 be a vector subspace of the vector space Y. Then there exists at least one algebraic projection T from the space Y onto Y_1, i.e., $\mathrm{Dom}\,[T] = Y$, $\mathrm{Im}\,[T] = Y_1$ and it is $T^2 = T \circ T = T$.

If a subset Y_1 of the vector space Y contains only the zero point, it will be also considered a subspace of Y. It is clear that there exists only one algebraic projection onto such subspace, namely, identically zero mapping.

Let Y be a vector space and let T be an algebraic projection from Y onto $\mathrm{Im}\,[T] \subset Y$. Then

$$T^c = I_Y - T$$

(I_Y denotes the identity operator in the space Y) is also an algebraic projection (clearly $T^{cc} = T$) and

$$Y = \mathrm{Im}\,[T] \oplus \mathrm{Im}\,[T^c] ,$$

i.e., the space Y is an algebraic direct sum of the subspaces $\mathrm{Im}\,[T]$ and $\mathrm{Im}\,[T^c]$. In this case we shall consider (using a suitable identification) the space Y to be a product of the subspaces $\mathrm{Im}\,[T]$ and $\mathrm{Im}\,[T^c]$, i.e.,

$$Y = \mathrm{Im}\,[T] \times \mathrm{Im}\,[T^c] .$$

2.2. Auxiliary equation. Let

$$L : \mathrm{Dom}\,[L] \subset X \to Z$$

be a linear mapping and let

$$N_0 : X \to Z$$

be an operator such that Dom $[N_0] = X$. For $h \in Z$ set

(1) $$N_h : x \in X = \mathrm{Dom}\,[N_h] \mapsto N_0(x) - h \in Z\,.$$

We shall investigate the conditions under which there exists $x \in \mathrm{Dom}\,[L]$ satisfying the equation

(2) $$L(x) = N_h(x)\,.$$

Let P and Q be fixed algebraic projections in the spaces X and Z, respectively, such that

(3) $$\mathrm{Im}\,[P] = \mathrm{Ker}\,[L]\,,$$

(4) $$\mathrm{Im}\,[Q^c] = \mathrm{Im}\,[L]\,.$$

The restriction $\tilde{L}$ of the operator L to $P^c(\mathrm{Dom}\,[L])$ is one-to-one and hence $\tilde{L}$ is an algebraic isomorphism between $P^c(\mathrm{Dom}\,[L])$ and $\mathrm{Im}\,[L] = \mathrm{Im}\,[Q^c]$. Denote its inverse by K (the so-called *right inverse* of the operator L). We have

$$\mathrm{Dom}\,[K] = \mathrm{Im}\,[L]\,, \quad \mathrm{Im}\,[K] = P^c(\mathrm{Dom}\,[L])$$

and

(5) $$LK(z) = \tilde{L}K(z) = z$$

for every $z \in \mathrm{Im}\,[L]$.

(It is easily checked that K is the unique linear mapping satisfying both (5) and $PK = 0$.)

It is clear that the relation

(6) $$KLP^c(x) = P^c(x)$$

holds for every $x \in \mathrm{Dom}\,[L]$.

2.2.1. Definition (see J. Mawhin [13]). Let U and V be vector spaces. A mapping $T : U \to V$ is said to be *suitable* if $T_{-1}(O_V) = \{O_U\}$.

A necessary and sufficient condition for the existence of a suitable mapping between U and V is that the relations $V = \{O_V\}$, $U \neq \{O_U\}$ do not hold simultaneously. Indeed, if $V \neq \{O_V\}$, a suitable mapping is given by

$$T : u \mapsto \begin{cases} v & \text{if} \quad u \neq O_U\,, \\ O_V & \text{if} \quad u = O_U \end{cases}$$

with v a fixed non zero element of V.

Hence we have the following

2.2.2. Lemma. *A necessary and sufficient condition for the existence of a suitable mapping*

$$\psi : \mathrm{Im}\,[Q] \to \mathrm{Ker}\,[L]$$

is that the conditions

$$(7) \qquad \mathrm{Im}[Q] \neq \{O_Z\}$$

and

$$(8) \qquad \mathrm{Ker}\,[L] = \{O_X\}$$

do not hold simultaneously.

2.2.3. Theorem. *Let the conditions* (7) *and* (8) *not hold simultaneously and let*

$$\psi : \mathrm{Im}\,[Q] \to \mathrm{Ker}\,[L]$$

be a fixed suitable mapping.

Define the family $\{V_\varepsilon\}_{\varepsilon>0}$ *of mappings*

$$V_\varepsilon : \mathrm{Ker}\,[L] \times \mathrm{Im}\,[P^c] \to \mathrm{Ker}\,[L] \times \mathrm{Im}\,[P^c]$$

by

$$(9) \qquad V_\varepsilon : [u, v] \mapsto [u - \varepsilon\psi Q N_h(u + KQ^c N_h(u + v)),\ KQ^c N_h(u + v)]\,.$$

Let the operator $V_{\varepsilon 0}$ *for some* $\varepsilon_0 > 0$ *have a fixed point* $[u_0, v_0]$, *i.e.,*

$$V_{\varepsilon 0}(u_0, v_0) = [u_0, v_0]\,.$$

Then $[u_0, v_0] \in \mathrm{Ker}\,[L] \times P^c(\mathrm{Dom}\,[L])$ *and* $x_0 = u_0 + v_0$ *is a solution of the equation* (2).

Proof. Since

$$\psi Q\, N_h(u_0 + KQ^c\, N_h(u_0 + v_0)) = O_X\,,$$

we have

$$Q\, N_h(u_0 + KQ^c\, N_h(u_0 + v_0)) = O_Z$$

and thus

$$(10) \qquad N_h(u_0 + KQ^c\, N_h(u_0 + v_0)) \in \mathrm{Im}\,[L]\,.$$

Moreover, it is

$$KQ^c\, N_h(u_0 + v_0) = v_0$$

which together with (10) implies

$$N_h(u_0 + v_0) \in \mathrm{Im}\,[L]$$

and

$$Q\, N_h(u_0 + v_0) = O_Z\,.$$

So

$$L(u_0 + v_0) - N_h(u_0 + v_0) = L(v_0) - N_h(u_0 + v_0) =$$
$$= LKQ^c N_h(u_0 + v_0) - N_h(u_0 + v_0) = Q^c N_h(u_0 + v_0) - N_h(u_0 + v_0) =$$
$$= Q^c N_h(u_0 + v_0) - Q N_h(u_0 + v_0) - Q^c N_h(u_0 + v_0) = -Q N_h(u_0 + v_0) = O_Z$$

2.3. Main existence theorem. To obtain a fixed point of the operator V_{ε_0} (for some $\varepsilon_0 > 0$) we shall apply the Schauder fixed point theorem for completely continuous operators.

2.3.1. Definition. Let Y be a Banach space. The mapping

$$T : \mathrm{Dom}\,[T] \subset Y \to Y$$

is said to be *completely continuous*, if it is continuous and if for each bounded set $M \subset \mathrm{Dom}\,[T]$ the set $T(M)$ is a relatively compact subset of the space Y.

2.3.2. Schauder fixed point theorem (see e.g. [19]). *Let $\mathscr{K}$ be a nonempty convex bounded and closed subset of a Banach space Y. Let*

$$T : \mathscr{K} \subset Y \to Y$$

be a completely continuous operator such that

$$T(\mathscr{K}) \subset \mathscr{K} .$$

Then there exists $x_0 \in \mathscr{K}$ such that

$$T(x_0) = x_0 .$$

First we shall give sufficient conditions under which there exists $\varepsilon_0 > 0$ and a nonempty closed convex and bounded subset $\mathscr{K}$ of the space $\mathrm{Ker}\,[L] \times \mathrm{Im}\,[P^c]$ such that $V_{\varepsilon_0}(\mathscr{K}) \subset \mathscr{K}$. For the purpose we shall suppose that X and Z are Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively.

(11) Let $\mathrm{Dom}\,[L]$ be a dense subset of X and let

$$L : \mathrm{Dom}\,[L] \subset X \to Z$$

be a *closed operator* ,

i.e., $x_n \in \mathrm{Dom}\,[L]$, $x_n \to x$ in the space X and $L(x_n) \to z$ in the space Z imply $x \in \mathrm{Dom}\,[L]$ and $L(x) = z$.

(12) There exist continuous algebraic projections P and Q with the properties (3), (4).

2.3.3. Remark. On each subspace of X and Z we introduce the norm as a restriction of the norm $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. Thus we obtain a normed linear space.

A necessary condition for a continuous algebraic projection Q with the property (4) to exist is that Im $[L]$ should be closed subspace of Z. Thus, after introducing the norm $\|\cdot\|_Z$ on Im $[L]$, Im $[L]$ is a Banach space. Analogously Im $[P^c]$ is a Banach space with the norm $\|\cdot\|_X$.

2.3.4. Remark. By the previous remark and using Closed Graph Theorem we obtain immediately that the right inverse K of the operator L (for definition see the relation (5)) is a continuous linear mapping from the space Im $[L]$ onto $P^c(\text{Dom}\,[L]) \subset \text{Im}\,[P^c]$. Denote by $\|K\|$ the norm of this mapping.

Analogously as in Remark 2.3.3 we obtain from the assumption (12) that the subspace Ker $[L]$ is closed (this follows also from the assumption (11)). Moreover, we shall suppose

(13) the dimension of Ker $[L]$ is finite.

2.3.5. Remark. From the assumption (13) it follows that it is possible to introduce the inner product $(\cdot, \cdot)$ on the space Ker $[L]$ such that

$$u \in \text{Ker}\,[L] \mapsto (u, u)^{1/2} = |||u|||$$

is a norm equivalent with $\|\cdot\|_X$ on Ker $[L]$. Denote by c^* the norm of the identity mapping from the space (Ker $[L]$, $|||\cdot|||$) into (Ker $[L]$, $\|\cdot\|_X$). Thus it is

$$\|u\|_X \leqq c^*|||u|||$$

for each $u \in \text{Ker}\,[L]$.

We shall suppose that the mapping

$$N_0 : X \to Z$$

satisfies the following condition:

(14) there exist $\alpha_0 \geqq 0$, $\beta \geqq 0$, $\delta \geqq 0$ such that

$$\|N_0(x)\|_Z \leqq \alpha_0 + \beta\|x\|_X^\delta$$

for each $x \in X$.

It is clear that the operator N_h defined by the relation (1) satisfies for each $x \in X$ the inequality

$$\|N_h(x)\|_Z \leqq \alpha_h + \beta\|x\|_X^\delta\,,$$

where

$$\alpha_h = \alpha_0 + \|h\|_Z\,.$$

For easier formulation of the next result we introduce the assumption

(15) the conditions (7) and (8) do not hold simultaneously.

Thus there exists a suitable mapping

$$\psi : \text{Im}\,[Q] \to \text{Ker}\,[L]\,.$$

Moreover, suppose

(16) there exist $\vartheta_1 \geqq 0,\ \vartheta_2 \geqq 0,\ \theta \geqq 0$ such that

$$\|\psi(z)\|_X \leqq \vartheta_1 + \vartheta_2 \|z\|_Z^\theta\,, \quad z \in \operatorname{Im}[Q]\,.$$

2.3.6. Lemma. *Let* $h \in Z,\ k \in (0, 1),\ \varrho_0 > 0,\ a \geqq 0,\ b \geqq 0,\ \eta \geqq 0$. *Suppose* (11)–(16). *Let the following implications hold*:

(17)
$$\varrho \in \langle 0, \varrho_0 \rangle\,, \quad u \in \operatorname{Ker}[L]\,, \quad \|u\|_X \leqq \varrho\,, \quad v \in \operatorname{Im}[P^c]\,,$$
$$\|v\|_X \leqq a + b\varrho^\eta \Rightarrow \|KQ^c N_h(u + v)\|_X \leqq a + b\varrho^\eta\,;$$

(18)
$$\mathfrak{N} = \{(\psi Q N_h(tw + t^\eta v), w) : w \in \operatorname{Ker}[L],$$
$$v \in \operatorname{Im}[P^c],\ |||w||| = 1,\ \|v\|_X \leqq ak^{-\eta}\varrho_0^{-\eta} + bk^{-\eta},$$
$$t \in \langle k\varrho_0/c^*, \varrho_0/c^* \rangle\} \Rightarrow \inf \mathfrak{N} = \gamma > 0\,.$$

Set

$$\mathscr{K} = \{[u, v] \in \operatorname{Ker}[L] \times \operatorname{Im}[P^c] : |||u||| \leqq \varrho_0/c^*,\ \|v\|_X \leqq a + b\varrho_0^\eta\}\,.$$

Then $\mathscr{K}$ *is nonempty, bounded, convex and closed. Moreover, there exists* $\varepsilon_0 > 0$ *such that*

(19)
$$V_{\varepsilon_0}(\mathscr{K}) \subset \mathscr{K}\,.$$

Proof. The properties of the set $\mathscr{K}$ are evident. To prove (19) it is sufficient to show

$$\text{A:} \quad u \in \operatorname{Ker}[L]\,, \quad \|u\|_X \leqq \varrho_0\,, \quad v \in \operatorname{Im}[P^c]\,,$$
$$\|v\|_X \leqq a + b\varrho_0^\eta \Rightarrow \|KQ^c N_h(u + v)\|_X \leqq a + b\varrho_0^\eta\,.$$
$$\text{B:} \quad u \in \operatorname{Ker}[L]\,, \quad \|u\|_X \leqq \varrho_0\,, \quad v \in \operatorname{Im}[P^c]\,,$$
$$\|v\|_X \leqq a + b\varrho_0^\eta \Rightarrow |||u - \varepsilon_0 \psi Q N_h(u + KQ^c N_h(u + v))||| \leqq \varrho_0/c^*\,.$$

The implication A is included in the assumption (17). We prove the validity of the implication B for some $\varepsilon_0 > 0$. Let $\varepsilon > 0$. Clearly

$$|||u - \varepsilon\psi Q N_h(u + KQ^c N_h(u + v))|||^2 =$$
$$= |||u|||^2 - 2\varepsilon(\psi Q N_h(u + KQ^c N_h(u + v)), u) +$$
$$+ \varepsilon^2 |||\psi Q N_h(u + KQ^c N_h(u + v))|||^2$$

and

$$(\psi Q N_h(u + KQ^c N_h(u + v)), u) =$$
$$= \varrho\left(\psi Q N_h\left(\varrho\frac{u}{\varrho} + \varrho^\eta \frac{KQ^c N_h(u + v)}{\varrho^\eta}\right), \frac{u}{\varrho}\right) \geqq \gamma\varrho$$

for an arbitrary $u \in \operatorname{Ker}[L]$, $|||u||| = \varrho$, $v \in \operatorname{Im}[P^c]$ $\|v\|_X \leqq a + b\varrho_0^\eta$, $\varrho \in \langle k\varrho_0/c^*, \varrho_0/c^* \rangle$. Thus

$$|||u - \varepsilon\psi Q N_h(u + KQ^c N_h(u + v))|||^2 \leqq$$

$$\leqq \varrho^2 - 2\varepsilon\gamma\varrho + \varepsilon^2[c^*(\vartheta_1 + \vartheta_2\|Q\|^\theta \|N_h(u + KQ^c N_h(u + v))\|_Z^\theta]^2 \leqq$$
$$\leqq \varrho^2 - 2\varepsilon\gamma\varrho + \varepsilon^2[c^*\{\vartheta_1 + \vartheta_2\|Q\|^\theta (\alpha_h + \beta\|u + KQ^c N_h(u + v)\|_X^\delta)^\theta\}]^2 \leqq$$
$$\leqq (\varrho_0/c^*)^2 - 2\varepsilon\gamma k\varrho_0/c^* + \varepsilon^2[c^*\{\vartheta_1 + \vartheta_2\|Q\|^\theta .$$
$$. (\alpha_h + \beta(\varrho_0 + a + b\varrho_0^\eta)^\delta)^\theta\}]^2 .$$

This implies that for every $\varepsilon > 0$ satisfying

$$\varepsilon \leqq \varepsilon_1 = \frac{2\gamma k\varrho_0}{c^*[c^*\{\vartheta_1 + \vartheta_2\|Q\|^\theta(\alpha_h + \beta(\varrho_0 + a + b\varrho_0^\eta)^\delta)^\theta\}]^2}$$

and for each $\varrho \in \langle k\varrho_0/c^*, \varrho_0/c^*\rangle$ and $[u, v] \in \text{Ker}[L] \times \text{Im}[P^c]$ satisfying $|||u||| = \varrho$, $\|v\|_X \leqq a + b\varrho_0^\eta$ it is

(20) $$|||u - \varepsilon\psi Q N_h(u + KQ^c N_h(u + v))||| \leqq \varrho_0/c^* .$$

For $[u, v] \in \text{Ker}[L] \times \text{Im}[P^c]$, $|||u||| \leqq k\varrho_0/c^*$, $\|v\|_X \leqq a + b\varrho_0^\eta$ we have

(21) $$|||u - \varepsilon\psi Q N_h(u + KQ^c N_h(u + v))||| \leqq$$
$$\leqq k\varrho_0/c^* + \varepsilon c^*(\vartheta_1 + \vartheta_2\|Q\|^\theta(\alpha_h + \beta(\varrho_0 + a + b\varrho_0^\eta)^\delta)^\theta) \leqq \varrho_0/c^*$$

provided

$$0 < \varepsilon \leqq \varepsilon_2 = \frac{(1 - k)\,\varrho_0}{[c^*]^2\,(\vartheta_1 + \vartheta_2\|Q\|^\theta\,(\alpha_h + \beta(\varrho_0 + a + b\varrho_0^\eta)^\delta)^\theta)} .$$

The inequalities (20) and (21) prove the validity of the implication B for $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$.

Denote $\nu = \|K\|\,\|Q^c\|$ and

$$c(\delta) = \begin{cases} 1\,, & \delta \leqq 1 \\ 2^{\delta - 1}\,, & \delta > 1\,. \end{cases}$$

If $v \in \text{Im}[P^c]$, $\|v\|_X \leqq a + b\varrho^\eta$ and $u \in \text{Ker}[L]$, $\|u\|_X \leqq \varrho$ then

$$\|KQ^c N_h(u + v)\|_X \leqq \nu\|N_h(u + v)\|_Z \leqq$$
$$\leqq \nu\alpha_h + \nu\beta\, c(\delta)\, \|u\|_X^\delta + \nu\beta\, c(\delta)\, \|v\|_X^\delta \leqq$$
$$\leqq \nu\alpha_h + \nu\beta\, c(\delta)\, \varrho^\delta + \nu\beta\, c(\delta)\, a^\delta + \nu\beta\, c^2(\delta)\, b^\delta\varrho^{\delta\eta} .$$

Put

$$r(\varrho) = \nu\alpha_h + \nu\beta\, c(\delta)\varrho^\delta + \nu\beta\, c^2(\delta)\, b^\delta\varrho^{\delta\eta} - b\varrho^\eta\,, \quad z(a) = a - \nu\beta\, c^2(\delta)\, a^\delta .$$

One can easily see that for the implication (17) to be valid it is sufficient to prove the existence of $\varrho_0 > 0$ and $a_0 > 0$ such that $r(\varrho) \leqq z(a_0)$ for each $\varrho \in \langle 0, \varrho_0\rangle$.

Studying the dependence of the functions r and z on the parameters $\delta, \eta, a, b, \beta$ we obtain the proof of the following

2.3.7. Lemma. *Let* $h \in Z$, $b \geqq 0$, $\eta \geqq 0$, $\delta \geqq 0$, $a \geqq 0$, $\emptyset \neq \mathfrak{M}_a \subset (0, \infty)$.

Then for each $\varrho \in \mathfrak{M}_a$, $u \in \operatorname{Ker}[L]$, $\|u\|_X \leqq \varrho$, $v \in \operatorname{Im}[P^c]$, $\|v\|_X \leqq a + b\varrho^\eta$ *it is*

$$\|KQ^c N_h(u + v)\|_X \leqq a + b\varrho^\eta$$

provided one from the following conditions is fulfilled:

	δ	η	b, h	a	β	$\mathfrak{M}_a$
I	$\delta = 0$	$\eta \geqq 0$	$b \geqq 0$	$a \geqq a_1$	$\beta = 0$	$(0, \infty)$
II	$0 < \delta < 1$	$\delta < \eta$	$b > 0$	$a > a_2$	$\beta > 0$	$(0, \infty)$
III	$0 < \delta < 1$	$\delta = \eta$	$b > \nu\beta$	$a > a_2$	$\beta > 0$	$(0, \infty)$
IV	$0 < \delta < 1$	$\delta = \eta$	$0 < b \leqq \nu\beta$	$a > a_3$	$\beta > 0$	$(0, \varrho_{1,a}\rangle$
V	$0 < \delta < 1$	$0 < \delta < \eta$	$b > 0$	$a > a_3$	$\beta > 0$	$(0, \varrho_{1,a}\rangle$
VI	$0 < \delta < 1$	$\eta = 0$	$b = 0$	$a > a_3$	$\beta > 0$	$(0, \varrho_{2,a}\rangle$
VII	$\delta = 1$	$\eta > 1$	$b > 0$	$a > a_4$	$0 < \beta < \nu^{-1}$	$(0, \infty)$
VIII	$\delta = 1$	$\eta = 1$	$0 < b \leqq$ $\leqq \nu\beta(1 - \nu\beta)^{-1}$	$a > a_4$	$0 < \beta < \nu^{-1}$	$(0, \infty)$
IX	$\delta = 1$	$0 < \eta < 1$	$b > 0$	$a > a_5$	$0 < \beta < \nu^{-1}$	$(0, \varrho_{3,a}\rangle$
X	$\delta = 1$	$0 < \eta < 1$	$b > 0$	$a > a_5$	$0 < \beta < \nu^{-1}$	$(0, \varrho_{1,a}\rangle$
XI	$\delta = 1$	$\eta = 0$	$b = 0$	$a > a_5$	$0 < \beta < \nu^{-1}$	$(0, \varrho_{4,a}\rangle$
XII	$\delta > 1$	$0 < \eta$	$b > 0$, $\|h\|_Z < h_0$	$a = a_6$	$\beta > 0$	$(0, \varrho_{1,a}\rangle$
XIII	$\delta > 1$	$\eta = 0$	$b = 0$, $\|h\|_Z < h_0$	$a = a_6$	$\beta > 0$	$(0, \varrho_{1,a}\rangle$

where

$$\begin{aligned}
a_1 &= \nu\alpha_h\,,\\
a_2 &= \max\{r(\varrho) : \varrho \in \langle 0, \infty)\}\,,\\
a_3 &= \sup\{a > 0 : z(a) < \nu\alpha_h\}\,,\\
a_4 &= a_2(1 - \nu\beta)^{-1}\,,\\
a_5 &= a_1(1 - \nu\beta)^{-1}\,,\\
a_6 &= (\nu\beta\delta c^2(\delta))^{1/(1-\delta)}\,,\\
\varrho_{1,a} &= \inf\{\varrho > 0 : r(\varrho) > z(a)\}\,,\\
\varrho_{2,a} &= [(z(a) - \nu\alpha_h)/\nu\beta]^{1/\delta}\,,\\
\varrho_{3,a} &= (a(1 - \nu\beta) - \nu\alpha_h)/(\nu\beta + \nu\beta b - b)\,,\\
\varrho_{4,a} &= (a(1 - \nu\beta) - \nu\alpha_h)/\nu\beta\,,\\
h_0 &= \nu^{-1}(\nu\beta c^2(\delta))^{1/(1-\delta)}\,.\,[\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}] - \alpha_0\,.
\end{aligned}$$

Lemma 2.3.7 yields a good criterion for the validity of the implication (17) from Lemma 2.3.6. Sufficient conditions for the validity of the implication (18) will be given in next Sections. Now, application of the Schauder fixed point theorem requires the knowledge of conditions under which the mappings $\{V_\varepsilon\}_{\varepsilon>0}$ are completely continuous.

On the vector space $\operatorname{Ker}[L] \times \operatorname{Im}[P^c]$, we introduce the norm of the product of Banach spaces, thus obtaining a Banach space.

We shall suppose:

(22) there exists a continuous suitable mapping

$$\psi : \operatorname{Im}[Q] \to \operatorname{Ker}[L]$$

satisfying the assumption (16).

2.3.8. Lemma. *Suppose* (12)–(14), (22) *and*

(23) *the mapping* $KQ^c N_h : X \to \operatorname{Im}[P^c]$ *is completely continuous*;

(24) *the mapping* $N_0 : X \to Z$ *is continuous*.

Then for every $\varepsilon > 0$ *the mapping*

$$V_\varepsilon : \operatorname{Ker}[L] \times \operatorname{Im}[P^c] \to \operatorname{Ker}[L] \times \operatorname{Im}[P^c]$$

defined by the relation (9) *is completely continuous.*

Proof. Clearly V_ε is continuous. Let

$$M \subset \text{Ker}[L] \times \text{Im}[P^c]$$

be a bounded set. Then the set $\overline{PV_\varepsilon(M)}$ is bounded and closed and since we suppose (13), it is a compact subset of Ker $[L]$. The set $\overline{P^c K Q^c N_h(M)}$ is compact by the assumption (23). Since the product of compact sets is a compact set we obtain our assertion.

2.3.9. Remark. Let us note that if the following assumptions are introduced

(25) $\quad N_0$ is a completely continuous mapping ;

(26) $\quad K\colon \text{Im}[L] \to \text{Im}[P^c]$ is completely continuous ;

it is easy to check that either assumptions (11), (12), (25) or assumptions (12), (14), (26) imply the assumption (23).

Now we are ready to state the basic result of this paper, the proof of which follows from 2.2.3, 2.3.2, 2.3.6, 2.3.7 and 2.3.8.

2.3.10. Theorem. *Let X, Z be Banach spaces, let*

$$L\colon \text{Dom}[L] \subset X \to Z$$

be a closed linear operator with the domain Dom $[L]$ *dense in the space X and closed image* Im $[L]$ *in the space Z and with a finite dimensional null space* Ker $[L]$. *Suppose that there exist continuous algebraic projections P and Q with the properties*

$$\text{Im}[P] = \text{Ker}[L], \quad \text{Im}[Q^c] = \text{Im}[L].$$

Let Ker $[L] = \{O_X\}$ *and* Im $[Q] \neq \{O_Z\}$ *not hold simultaneously, let*

$$\psi : \text{Im}[Q] \to \text{Ker}[L]$$

be a continuous suitable mapping and suppose that there exist $\vartheta_1 \geqq 0$, $\vartheta_2 \geqq 0$, $\theta \geqq 0$ such that

$$\|\psi(z)\|_X \leqq \vartheta_1 + \vartheta_2 \|z\|_Z^\theta$$

for each $z \in \text{Im}[Q]$.

Let $N_0 : X \to Z$, Dom $[N_0] = X$ *be a continuous mapping and suppose that there exist $\alpha_0 \geqq 0$, $\beta \geqq 0$, $\delta \geqq 0$ such that*

$$\|N_0(x)\|_Z \leqq \alpha_0 + \beta \|x\|_X^\delta$$

for each $x \in X$.

Let

$$K : \text{Im}[L] \to P^c(\text{Dom}[L])$$

be the right inverse of the operator L and suppose that the mapping

$$KQ^c N_0 : X \to \mathrm{Im}\,[P^c]$$

is completely continuous.

Let the constants a, b, β, δ, η and $h \in Z$ *satisfy one from the conditions* I – XIII *from Lemma* 2.3.7 *and let* $k \in (0, 1)$ *and* $\varrho_0 \in \mathfrak{M}_a$.

If

$$\inf \mathfrak{N} = \gamma > 0\,,$$

where $\mathfrak{N}$ *is the set introduced in Lemma* 2.3.6, *then the equation*

$$L(x) = N_0(x) - h$$

has a solution $x \in \mathrm{Dom}\,[L]$.

2.4. The case of $\mathrm{Ker}\,[L] = \{O_X\}$. In this Section we shall suppose

$$\mathrm{Ker}\,[L] = \{O_X\} \quad \text{and} \quad \mathrm{Im}\,[L] = Z\,.$$

Thus we can assume P and Q to be the null mappings and obtain continuous algebraic projections with the properties (3), (4). Moreover, there exists only one suitable mapping

$$\psi : \mathrm{Im}\,[Q] \to \mathrm{Ker}\,[L]\,,$$

namely, the identically zero mapping which clearly satisfies the assumption (22). The mapping V_ε defined by the relation (9) has in this case the form

$$V_\varepsilon : X \to \mathrm{Dom}\,[L]\,,$$

$$V_\varepsilon : v \mapsto L^{-1}\,N_h(v)\,.$$

Since the set $\mathfrak{N}$ defined in Lemma 2.3.6 is empty, it is

$$\inf \mathfrak{N} = \infty$$

and the assumption (18) is automatically fulfilled. The above reasoning and Theorem 2.3.10 yield

2.4.1. Theorem. *Let X and Z be two Banach spaces, let*

$$L : \mathrm{Dom}\,[L] \subset X \to Z$$

be densely defined closed linear operator such that

$$\mathrm{Ker}\,[L] = \{O_X\} \quad \textit{and} \quad \mathrm{Im}\,[L] = Z\,.$$

Let

$$N_0 : X \to Z$$

be a continuous mapping, Dom $[N_0] = X$, *and suppose that there exist* $\alpha_0 \geqq 0$, $\beta \geqq 0$, $\delta \geqq 0$ *such that*

$$\|N_0(x)\|_Z \leqq \alpha_0 + \beta\|x\|_X^\delta$$

or each $x \in X$.

Let the mapping

$$L^{-1}N_0 : X \to X$$

be completely continuous and let $h \in Z$.

Then there exists at least one $x_0 \in$ Dom $[L]$ *satisfying*

$$L(x_0) = N_0(x_0) - h$$

provided one from the following conditions is fulfilled:

A. $\delta \in \langle 0, 1)$ (*the so-called sublinear case*), *h is arbitrary*;
B. $\delta = 1$ (*the so-called asymptotically linear case*), $\beta < \|L^{-1}\|^{-1}$, *h is arbitrary*;
C. $\delta > 1$ (*the so-called superlinear case*), $h \in Z$ *is such that*

$$\|h\|_Z < \|L^{-1}\|^{-1} (\|L^{-1}\| \beta c^2(\delta))^{1/(1-\delta)} [\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}] - \alpha_0 .$$

Proof.

ad A. Choose $\eta > \delta$.
ad B. Choose $\eta > 1$.
ad C. Choose $\eta = 0$.

In these cases, the condition II or VII or XIII from Lemma 2.3.7 is fulfilled and so our assertion follows from Theorem 2.3.10.

2.5. The case of Ker $[L] \neq \{O_X\}$. If L is a linear mapping between the vector spaces X and Z we denote its cokernel (i.e., the factor space $Z/\text{Im}\,[L]$) by Coker $[L]$.

2.5.1. Definition. A linear (not necessarily continuous) operator

$$L : \text{Dom}\,[L] \subset X \to Z$$

is said to be a *Fredholm mapping* if the following conditions are fulfilled:

(i) Ker $[L]$ is of finite dimension $m \geqq 0$;
(ii) Coker $[L]$ is of finite dimension $p \geqq 0$;
(iii) Im $[L]$ is a closed subspace of Z.

The *index* of a Fredholm mapping L will be defined as usual by

$$\text{Ind}\,[L] = m - p .$$

It follows from Definition 2.5.1 that there exist continuous algebraic projections P and Q with the properties (3), (4). Let us fix one couple of such projections.

Up to this time, we have known nothing about the existence of a continuous suitable mapping

$$\psi : \text{Im}\,[Q] \to \text{Ker}\,[L]\,.$$

We shall prove

2.5.2. Lemma (see J. Mawhin [13]). *Under the assumptions*

$$L : \text{Dom}\,[L] \subset X \to Z \text{ is a Fredholm mapping}\,; \tag{27}$$

$$m > 0\,; \tag{28}$$

there exists a continuous suitable mapping

$$\psi : \text{Im}\,[Q] \to \text{Ker}\,[L]$$

satisfying the condition (22).

It can be chosen one-to-one if and only if Ind $[L] \geqq 0$ *and this condition is also necessary and sufficient for the existence of a linear suitable mapping* ψ.

Proof. If Ind $[L] < 0$, i.e., $p > m$, it follows from Brouwer's invariance of domain theorem (see e.g. [19]) that there exists no continuous one-to-one mapping from Im $[Q]$ into Ker $[L]$. However, due to the fact that $m > 0$ in this case, the mapping defined by

$$\psi : \xi = \sum_{i=1}^{p} a_i \xi_i \mapsto \sum_{i=1}^{m-1} a_i x_i + \left(\sum_{i=m}^{p} a_i^2 \right) x_m\,, \tag{29}$$

with $[\xi_1, \ldots, \xi_p]$ and $[x_1, \ldots, x_m]$ being any bases in Im $[Q]$ and Ker $[L]$ respectively (note that the spaces Im $[Q]$ and Coker $[L]$ are isomorphic), is clearly continuous and suitable and satisfies the condition (22).

On the other hand, if Ind $[L] \geqq 0$, we can take for ψ the linear one-to-one mapping

$$\psi : \sum_{i=1}^{p} a_i \xi_i \mapsto \sum_{i=1}^{p} a_i x_i\,. \tag{30}$$

For the last assertion of our lemma observe that a linear mapping is suitable if and only if it is one-to-one.

2.5.3. Remark. In the sequel we shall consider suitable mappings defined by the relations (29) (if Ind $[L] < 0$) and (30) (if Ind $[L] \geqq 0$).

The inner product in the condition (18) in Lemma 2.3.6 has the form

$$(\psi Q N_h(tw + t^n v), w) = \sum_{i=1}^{p} [Q N_h(tw + t^n v)]_i\, w_i \tag{31}$$

provided Ind $[L] \geqq 0$ and

$$(32) \qquad (\psi QN_h(tw + t^\eta v), w) = \sum_{i=1}^{m-1} [QN_h(tw + t^\eta v)]_i \, w_i + \\ + \left(\sum_{i=m}^{p} [QN_h(tw + t^\eta v)]_i^2 \right) w_m$$

provided Ind $[L] < 0$, where

$$QN_h(tw + t^\eta v) = \sum_{i=1}^{p} [QN_h(tw + t^\eta v)]_i \, \xi_i$$

and

$$w = \sum_{i=1}^{m} w_i x_i \, .$$

Linearity of the suitable mapping ψ defined by the relation (30) implies in the case Ind $[L] \geqq 0$ that the necessary condition for the validity of the implication (18) is Ind $[L] = 0$, i.e. $m = p$.

2.5.4. Definition. Let

$$L : \text{Dom}\,[L] \subset X \to Z$$

be a Fredholm linear mapping with Ind $[L] = 0$ and Ker $[L] \neq \{O_X\}$. Let

$$N_0 : X \to Z$$

be a nonlinear operator and let

$$\Phi : S = \{w \in \text{Ker}\,[L] : w = \sum_{i=1}^{m} w_i x_i, \ \sum_{i=1}^{m} w_i^2 = 1\} \to R_1$$

be a lower semicontinuous function. Let $\eta \geqq 0$ and $r > 0$.

The function Φ is said to be a weak (η, r)-*subasymptote of the operator* N_0 *with respect to* Ker $[L]$ if there exists $t_0 > 0$ such that

$$\sum_{i=1}^{m} [QN_0(tw + t^\eta v)]_i \, w_i \geqq \Phi\left(\sum_{i=1}^{m} w_i x_i\right)$$

provided $t \geqq t_0$, $v \in X$, $\|v\|_X \leqq r$ and $w \in S$.

2.5.5. Theorem. *Let* X, Z *be Banach spaces, let*

$$L : \text{Dom}\,[L] \subset X \to Z$$

be a densely defined closed linear operator with Ind $[L] = 0$ *and* Ker $[L] \neq \{O_X\}$.

Let

$$N_0 : X \to Z \, ,$$

$\text{Dom}[N_0] = X$ *be a continuous mapping and suppose that there exist* $\alpha_0 \geqq 0$, $\beta \geqq 0$, $\delta \geqq 0$ *such that*

$$\|N_0(x)\|_Z \leqq \alpha_0 + \beta\|x\|_X^\delta$$

for each $x \in X$.

Let

$$K : \text{Im}[L] \to P^c(\text{Dom}[L]) \subset \text{Im}[P^c]$$

be the right inverse of the operator L and suppose that the mapping

$$KQ^c N_0 : X \to X$$

is completely continuous.

Let $\eta \geqq 0$ *and* $r > 0$. *Suppose that* Φ *is a weak* (η, r)*-subasymptote of the operator* N_0 *with respect to* $\text{Ker}[L]$. *Let* $h \in Z$.

Then the equation

$$L(x) = N_0(x) - h$$

is solvable in $\text{Dom}[L]$ *if*

$$\sum_{i=1}^{m} [Qh]_i w_i < \Phi(\sum_{i=1}^{m} w_i x_i)$$

for each $\sum_{i=1}^{m} w_i x_i \in S$, *where*

$$Qh = \sum_{i=1}^{m} [Qh]_i \xi_i ,$$

provided one from the following conditions is fulfilled:

A (*sublinear case*): $\delta = 0$, $\eta \geqq 0$, $\beta = 0$, $r > v(\alpha_0 + \|h\|_Z)$;
B (*sublinear case*): $0 < \delta < 1$, $\delta < \eta$, $\beta > 0$, $r > 0$;
C (*sublinear case*): $0 < \delta < 1$, $\delta = \eta$, $\beta > 0$, $r > 0$;
D (*asymptotically linear case*): $\delta = 1$, $\eta > 1$, $r > 0$;
E (*asymptotically linear case*): $\delta = 1$, $\eta = 1$, $0 < \beta < v^{-1}$, $r > 0$.

Proof. Since the function

$$w \mapsto \Phi(w) - \sum_{i=1}^{m} [Qh]_i w_i$$

is lower semicontinuous and the set S is compact, the assumptions of our theorem imply the existence of $\gamma > 0$ such that

$$\Phi(w) - \sum_{i=1}^{m} [Qh]_i w_i \geqq \gamma$$

on S. Choose $a > 0$, $0 \leqq b < r$, $k \in (0, 1)$ and $\varrho_1 > 0$ such that

$$ak^{-\eta}\varrho_1^{-\eta} + bk^{-\eta} < r .$$

Then the conditions from Lemma 2.3.7 are fulfilled, namely:

$$\text{A} \Rightarrow \text{I}, \quad \text{B} \Rightarrow \text{II}, \quad \text{C} \Rightarrow \text{III}, \quad \text{D} \Rightarrow \text{VII}, \quad \text{E} \Rightarrow \text{VIII}.$$

By Definition 2.5.4 there exists $t_0 = k\varrho_0/c^*$ such that $\varrho_0 \geqq \varrho_1$ and

$$(\psi Q N_h(tw + t^\eta v), w) \geqq \Phi(w) - \sum_{i=1}^{m} [Qh]_i w_i \geqq \gamma > 0$$

for each $w \in \text{Ker}[L]$, $|||w||| = 1$, $v \in \text{Im}[P^c]$, $\|v\|_X \leqq r$ and $t \geqq t_0$. Thus we obtain our assertion from Theorem 2.3.10.

2.5.6. Corollary. *Let all assumptions of Theorem 2.5.5 be fulfilled and, moreover, let every positive constant be a weak (η, r)-subasymptote of the operator N_0 with respect to* Ker $[L]$ (*we suppose only the conditions* B, C, D *and* E). *Then the equation*

$$L(x) = N_0(x) - h$$

is solvable for each $h \in Z$.

Theorem 2.5.5 gives us sufficient conditions for the solvability of the equation considered. The following definition of "maximal subasymptote" will be used in the formulation of necessary and sufficient conditions.

2.5.7. Definition. Let

$$L : \text{Dom}[L] \subset X \to Z$$

be a Fredholm linear mapping with Ind $[L] = 0$ and Ker $[L] \neq \{O_X\}$. Let $N_0 : X \to \to Z$, Dom $[N_0] = X$ be a nonlinear operator. Suppose that

$$\Phi : S \to R_1$$

is a lower semicontinuous function.

The function Φ is said to be *a weak asymptote of the operator N_0 with respect to* Ker $[L]$ if for each $\varepsilon > 0$ and $r > 0$ there exists $t_\varepsilon > 0$ such that for $t \geqq t_\varepsilon$, $v \in X$, $\|v\|_X \leqq r$ and $w \in S$ it is

$$\left|\sum_{i=1}^{m} [QN_0(tw + v)]_i w_i - \Phi(w)\right| < \varepsilon.$$

It is easy to see that if Φ is a weak asymptote of the operator N_0 with respect to Ker $[L]$ then for any $d > 0$ the function

$$w \mapsto \Phi(w) - d$$

is a weak $(0, r)$-subasymptote of the operator N_0 with respect to Ker $[L]$.

2.5.8. Theorem. *Let the condition* A *from Theorem* 2.5.5 *be fulfilled. Moreover, suppose that* Φ *is a weak asymptote of the operator* N_0 *with respect to* Ker $[L]$. *Let*

$$\sum_{i=1}^{m} [Q\, N_0(x)]_i\, w_i < \Phi(w)$$

for each $x \in X$ *and* $w = \sum_{i=1}^{m} w_i x_i \in S$, *where*

$$Q\, N_0(x) = \sum_{i=1}^{m} [Q\, N_0(x)]_i\, \xi_i\,.$$

Suppose $h \in Z$.

Then the equation

$$L(x) = N_0(x) - h$$

is solvable in Dom $[L]$ *if and only if*

$$\sum_{i=1}^{m} [Qh]_i\, w_i < \Phi(w), \left(\sum_{i=1}^{m} [Qh]_i\, \xi_i = Qh\right)$$

for each $w = \sum_{i=1}^{m} w_i x_i \in S$.

Proof. Sufficiency follows from Theorem 2.5.5 and the remark before. If the equation considered is solvable in Dom $[L]$ then for any solution $x_0 \in$ Dom $[L]$ it is

$$N_0(x_0) - h \in \text{Im}\,[L]$$

and thus

$$Q\, N_0(x_0) = Qh\,,$$

i.e.,

$$\Phi(w) > \sum_{i=1}^{m} [Q\, N_0(x_0)]_i\, w_i = \sum_{i=1}^{m} [Qh]_i\, w_i\,.$$

2.5.9. Remark. Many equations involving a linear operator

$$L : \text{Dom}\,[L] \subset X \to Z$$

are such that there exists a vector space Y with the property (using the Riesz identification)

(33) $$X \subset Y, \quad Z^* \subset Y$$

(the asterisk denotes the adjoint space) and

(34) $$\text{Ker}\,[L] = \text{Ker}\,[L^*]$$

(L^* : Dom $[L^*] \subset Z^* \to X^*$ is the adjoint operator).

Since we suppose that L is a closed densely defined Fredholm linear operator (and according to the assumptions (33), (34) the operator L has index zero) it is

$$\text{(35)} \qquad \operatorname{Im}[L] = \{z \in Z : \langle w^*, z\rangle_Z = 0\,, \quad w^* \in \operatorname{Ker}[L^*]\}\,,$$

(see e.g. K. YOSIDA [23, § 7.5]), where $\langle w^*, z\rangle_Z$ denotes the value of the continuous linear functional $w^* \in Z^*$ at the point $z \in Z$. Note that $\xi_1, \ldots, \xi_m$ is a basis of $\operatorname{Im}[Q]$. According to the relation (35) and with respect to the assumptions (33) and (34) we can suppose that the basis $x_1, \ldots, x_m$ of $\operatorname{Ker}[L]$ has the property

$$\langle x_i, \xi_j\rangle_Z = \delta_{ij} \quad (i, j = 1, \ldots, m)$$

(δ_{ij} denotes the Kronecker symbol, $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$). Then for $z \in \operatorname{Im}[Q]$ we have

$$z = \sum_{i=1}^{m} \langle x_i, z\rangle_Z\, \xi_i$$

and thus the relation (31) may be written ($h = 0$) in the form

$$\text{(36)} \qquad \sum_{i=1}^{m} [Q\, N_0(tw + t^\eta v)]_i\, w_i = \sum_{i=1}^{m} \langle x_i, Q\, N_0(tw + t^\eta v)\rangle_Z\, w_i =$$
$$= \langle \sum_{i=1}^{m} w_i x_i,\ Q\, N_0(tw + t^\eta v)\rangle_Z = \langle Q^*(\sum_{i=1}^{m} w_i x_i),\ N_0(tw + t^\eta v)\rangle_Z =$$
$$= \langle Q^*(w), N_0(tw + t^\eta v)\rangle_Z\,.$$

Hence

$$\text{(37)} \qquad \sum_{i=1}^{m} [Q\, N_0(tw + t^\eta v)]_i\, w_i = \langle w, N_0(tw + t^\eta v)\rangle_Z\,.$$

3. APPLICATIONS

3.1. Periodic solutions of nonlinear differential-difference equations. Let T (*the period*) be a positive fixed number and let $q(s, u)$ be a continuous function on R_2 such that

$$\text{(1)} \qquad q(s + T, u) = q(s, u)$$

for every $[s, u] \in R_2$. Let $\tau \in R_1$. Let us consider the differential-difference equation

$$\text{(2)} \qquad x'(s) = q(s, x(s - \tau))\,.$$

The class of equations (2) includes also ordinary differential equations (take $\tau = 0$). A T-periodic solution of (2) will be a solution $x(s)$ such that

$$\text{(3)} \qquad x(s + T) = x(s)$$

for every $s \in R_1$.

To apply the abstract results from Part 2 to the existence of a T-periodic solution of the equation (2) put

$$X = Z = C_T,$$

where C_T is the Banach space of all continuous and T-periodic functions x (i.e., satisfying (3)) with the norm

$$\|x\|_{C_T} = \sup_{s \in R_1} |x(s)| = \sup_{s \in \langle 0, T \rangle} |x(s)|.$$

Moreover, set

$$\text{Dom}[L] = \{x \in X : \text{ is from the class } C^1\}, \quad L : x \mapsto x'.$$

It is easy to see that Dom $[L]$ is a dense subspace of the space X,

$$L : \text{Dom}[L] \subset X \to Z$$

is a closed linear operator,

$$\text{Ker}[L] = \{x \in X : x \text{ is a constant function}\}$$

and

$$\text{Im}[L] = \left\{z \in Z : \int_0^T z(s)\, \mathrm{d}s = 0\right\}.$$

For $f \in C_T$ set

$$Q(f) = P(f) = \frac{1}{T}\int_0^T f(s)\, \mathrm{d}s.$$

Thus

$$\text{Im}[P] = \text{Ker}[L], \quad \text{Im}[Q^c] = \text{Im}[L]$$

and P, Q are continuous algebraic projections in the spaces X and Z, respectively.

Suppose that there exist $\alpha_0 \geqq 0$, $\beta \geqq 0$, $\delta \geqq 0$ such that

(4) $$|q(s, u)| \leqq \alpha_0 + \beta |u|^\delta$$

for each $[s, u] \in R_2$. Define

$$N_0 : x \mapsto q(s, x(s - \tau)).$$

The mapping $N_0 : \text{Dom}[N_0] = X \to Z$ satisfies

$$\|N_0(x)\|_Z \leqq \alpha_0 + \beta \|x\|_X^\delta$$

for every $x \in X$.

Thus for arbitrary $x \in X$ we have (see 2.5.3)

(5) $$(\psi Q\, N_0(x), w) = \frac{1}{T}\int_0^T q(s, x(s - \tau))\, w\, \mathrm{d}s,$$

where $w = \pm 1$.

Now Theorems 2.5.5 and 2.5.8 imply

3.1.1. Theorem. *Suppose* (1) *and* (4) *with* $\beta = \delta = 0$. *Let there exist, for each* $s \in R_1$,

(6) $$\lim_{\xi \to \infty} q(s, \xi) = a_+(s),$$

(7) $$\lim_{\xi \to -\infty} q(s, \xi) = a_-(s).$$

Then the equation (2) *has a T-periodic solution provided*

(8) $$\int_0^T a_-(s)\,\mathrm{d}s < 0 < \int_0^T a_+(s)\,\mathrm{d}s$$

(or

(8′) $$\int_0^T a_-(s)\,\mathrm{d}s > 0 > \int_0^T a_+(s)\,\mathrm{d}s).$$

If we suppose, moreover, that

(9) $$a_-(s) \leqq q(s, u) \leqq a_+(s), \quad s \in R_1, \quad u \in R_1$$

(or

(9′) $$a_-(s) \geqq q(s, u) \geqq a_+(s), \quad s \in R_1, \quad u \in R_1)$$

and the strict inequalities hold for a set of $s \in R_1$ *which has positive measure, then the condition* (8) (*or* (8′)) *is also necessary and sufficient for the existence of T-periodic solution of* (2).

Proof. Suppose (8). From remarks before Theorem 3.1.1 and from Theorem 2.5.8 it follows that it is sufficient to show:

A. The mapping

$$KQ^c N_0 : X \to \mathrm{Im}\,[P^c] \subset X$$

(K is the right inverse of the operator L) is completely continuous;

B.

$$\Phi : w \mapsto \begin{cases} \dfrac{1}{T}\displaystyle\int_0^T a_+(s)\,\mathrm{d}s, & w = 1 \\ \dfrac{1}{T}\displaystyle\int_0^T [-a_-(s)]\,\mathrm{d}s, & w = -1 \end{cases}$$

is a weak asymptote of the operator N_0 with respect to Ker $[L]$;

C.

$$[Q\,N_0(x)]_1\, w_1 < \Phi(w_1), \quad x \in X.$$

(ad A). From the Arzela-Ascoli criterion of compactness it follows that the mapping $K : \mathrm{Im}\,[L] \to X$ is completely continuous. Moreover, the mappings Q^c and N_0 are continuous and bounded. This implies the assertion A.

(ad B). From (6), (7) and (4) it follows that the functions a_+, a_- are L_1-functions. Suppose that Φ is not a weak asymptote of the operator N_0 with respect to Ker $[L]$. Thus there exist $\varepsilon > 0$, a sequence $v_n \in X$, $\|v_n\|_{C_T} \leqq r$ and a sequence $t_n \in R_1$, $\lim_{n\to\infty} t_n = \infty$, such that either

$$\left|\frac{1}{T}\int_0^T q(s, t_n + v_n(s - \tau))\,\mathrm{d}s - \Phi(1)\right| \geqq \varepsilon$$

or

$$\left|\frac{1}{T}\int_0^T [-q(s, -t_n + v_n(s - \tau))]\,\mathrm{d}s - \Phi(-1)\right| \geqq \varepsilon\,.$$

From Lebesgue's Dominated Convergence Theorem we obtain a contradiction.

The assertion C is obvious.

Note that assuming the condition (8′) we consider the equation

$$-x'(s) = -q(s, x(s - \tau))$$

and use the same argument.

3.1.2. Theorem. *Suppose* (1) *and* (4). *For each* $s \in R_1$, *let there exist*

(10) $$\lim_{\xi\to\infty} q(s, \xi) = +\infty\,, \quad q(s, \xi) \geqq 0 \quad \textit{for} \quad \xi \geqq 0$$

(*or*

(10′) $$\lim_{\xi\to\infty} q(s, \xi) = -\infty\,, \quad q(s, \xi) \leqq 0 \quad \textit{for} \quad \xi \geqq 0)\,,$$

(11) $$\lim_{\xi\to\infty} q(s, \xi) = -\infty\,, \quad q(s, \xi) \leqq 0 \quad \textit{for} \quad \xi \leqq 0$$

(*or*

(11′) $$\lim_{\xi\to\infty} q(s, \xi) = +\infty\,, \quad q(s, \xi) \geqq 0 \quad \textit{for} \quad \xi \leqq 0)\,.$$

Then the equation (2) *has a T-periodic solution provided one from the following conditions is fulfilled*:

A (*sublinear case*): $0 < \delta < 1$;

B (*asymptotically linear case*): $\delta = 1$, $0 < \beta < 1$.

Proof. By contradiction, using the Fatou lemma we can prove that every constant is a weak (η, r)-subasymptote of the operator N_0 with respect to Ker $[L]$, where $\eta \in (\delta, 1)$ in the case A, $\eta = 1$ and $r < 1$ in the case B.

The superlinear case is solved in Theorem 3.1.3 the proof of which follows immediately from the reasoning before Theorem 3.1.1, from Theorem 2.3.10 and from Remark 2.5.3.

3.1.3. Theorem. *Suppose* (1), (4) *with* $\delta > 1$ *and*

(12) $$0 \leqq \alpha_0 < T^{-1}(T\beta\, 2^{2(\delta-1)})^{1/(1-\delta)} \left[\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)} - 2^{1-\delta}\delta^{\delta/(1-\delta)}\right];$$

(13) *there exists* $\gamma > 0$ *such that*

$$q(s, u) \geqq \gamma\,, \quad q(s, -u) \leqq -\gamma$$

for each $s \in R_1$ *and* $u \in \langle \tau_1, \tau_2 \rangle$, *where*

$$\tau_1 < \{T^{-1}\beta^{-1}\, 2^{1-\delta}[(T\beta\, 2^{2(\delta-1)})^{1/(1-\delta)}\,(\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}) - T\alpha_0]\}^{1/\delta} - $$
$$- (T\beta\delta\, 2^{2(\delta-1)})^{1/(1-\delta)}$$

and

$$\tau_2 \geqq \{T^{-1}\beta^{-1}\, 2^{1-\delta}[(T\beta\, 2^{2(\delta-1)})^{1/(1-\delta)}\,(\delta^{1/(1-\delta)} - \delta^{\delta/(1-\delta)}) - T\alpha_0]\}^{1/\delta} +$$
$$+ (T\beta\delta\,.\,2^{2(\delta-1)})^{1/(1-\delta)}\,.$$

Then the equation (2) (*or the equation*

(2′) $$-x'(s) = q(s, x(s - \tau)))$$

has a T-periodic solution.

3.1.4. Remark. Analogously as in Theorem 3.1.3 we can solve the second order ordinary differential equation

$$x'' = f(s, x, x')$$

(or

$$-x'' = f(s, x, x'))$$

under periodic boundary conditions

$$x(a) = x(b)\,, \quad x'(a) = x'(b)\,,$$

where f is a continuous real valued function on $\langle a, b \rangle \times R_2$. Note that

$$X = \{x \in C^1\langle a, b \rangle : x(a) = x(b),\ x'(a) = x'(b)\}\,,$$
$$Z = C\langle a, b \rangle\,,$$
$$\text{Dom}\,[L] = \{x \in X : x \text{ is of the class } C^2\}\,,$$
$$L\ : x \mapsto x''(\text{or } L : x \mapsto -x'')\,,$$
$$N_0 : x \mapsto f(\cdot, x(\cdot), x'(\cdot))\,.$$

If the function f does not depend on the last variable then we can give immediately the analogue of Theorems 3.1.1 and 3.1.2.

3.2. Classical solutions of boundary value problems for nonlinear second order ordinary differential equations. Let us consider the second order ordinary differential equation

$$x'' = f(s, x, x') \tag{14}$$

(or

$$-x'' = f(s, x, x')), \tag{14'}$$

where $f : \langle a, b\rangle \times R_2 \to R_1$ is continuous. If ξ_a, ξ_b are fixed real numbers, a solution of the boundary value problem for the equation (14) (or (14′)) will be a mapping $x : \langle a, b\rangle \to R_1$ of class C^2 which satisfies (14) (or (14′)) and the boundary conditions

$$x(a) = \xi_a\,, \quad x(b) = \xi_b\,. \tag{15}$$

Analogously as in J. Mawhin [14], the application of Part 2 is possible if we put

$$X = C^1\langle a, b\rangle \quad \text{with the norm} \quad \|x\|_X = \max\{\sup_{s\in\langle a,b\rangle} |x(s)|, \sup_{s\in\langle a,b\rangle} |x'(s)|\}\,.$$

If $C\langle a, b\rangle$ is equipped with the norm

$$\|z\|_C = \sup_{s\in\langle a,b\rangle} |z(s)|\,,$$

then the norm in the set $Z = C\langle a, b\rangle \times R_2$ is given by

$$\|z\|_C + |y_1| + |y_2|\,.$$

$$\operatorname{Dom}[L] = \{x \in X : x \text{ is of the class } C^2\}\,,$$

$$L : x \mapsto [x'', x(a), x(b)] \text{ (or } L: x \mapsto [-x'', x(a), x(b)])\,,$$

$$N_0 : x \mapsto [f(\cdot, x(\cdot), x'(\cdot)), \xi_a, \xi_b]\,.$$

Moreover, L is closed, $\operatorname{Ker}[L] = \{O_X\}$, $\operatorname{Im}[L] = Z$, the right inverse $K : Z \to X$ of L is completely continuous (this is an easy application of Arzela-Ascoli theorem) and N_0 is continuous and maps bounded sets into bounded sets.

We shall suppose that there exist $c \geqq 0$, $\beta \geqq 0$, $\delta \geqq 0$ such that

$$|f(s, u, v)| \leqq c + \beta[\max(|u|, |v|)]^\delta \tag{16}$$

for $s \in \langle a, b\rangle$ and $[u, v] \in R_2$. Thus

$$\|N_0(x)\|_Z \leqq \alpha_0 + \beta\|x\|_X^\delta\,,$$

where

$$\alpha_0 = c + |\xi_a| + |\xi_b| .$$

Now we apply Theorem 2.4.1:

3.2.1. Theorem. *The problem* (14)–(15) (*or* (14')–(15)) *has a solution if one from the following conditions is fulfilled*:

A (*sublinear case*): $\delta \in \langle 0, 1)$;
B (*asymptotically linear case*): $\delta = 1$, $\beta < [\max((b - a)^2, 2)]^{-1}$;
C (*superlinear case*): $\delta > 1$,

$$\alpha_0 < [\max((b - a)^2, 2)]^{-1} (\max((b - a)^2, 2)\, \beta\, 2^{2(\delta - 1)})^{1/(1 - \delta)} . \\ . [\delta^{1(1 - \delta)} - \delta^{\delta/(1 - \delta)}] .$$

3.2.2. Example. As an example of the superlinear case we shall consider the boundary value problem

$$(17) \qquad \begin{aligned} -x'' &= x^3 - h , \\ x(0) &= x(1) = 0 . \end{aligned}$$

It is easy to see that $\beta = 1$, $\delta = 3$, $\alpha_0 = \|h\|_C$. Then from the part C of Theorem 3.2.2 we obtain that (17) has a solution provided

$$\|h\|_C < 12^{-1} . 6^{-1/2} .$$

The existence of a solution of (17) for "a small right hand side h" follows also from the theorem about the local diffeomorphism (see e.g. J. Dieudonné [3, Theorem 10.2.5]) since the mapping

$$x \in X \mapsto -x'' - x^3 \in C\langle 0, 1\rangle$$

has invertible Fréchet derivative at the origin of the space

$$X = \{x \in C^2\langle 0, 1\rangle : x(0) = x(1) = 0\} .$$

An analogous upper bound for "the smallness" of $\|h\|_C$ as in this example may be obtained by the method of the upper and lower solutions which is explained in the paper of K. Schmitt [18]. It seems that it remains an open problem whether the boundary value problem (17) has a solution for any right hand side $h \in C\langle 0, 1\rangle$ (see also [8, Appendix VII]).

Note that it is possible to generalize the result from this example to the problems

$$\begin{aligned} -x'' &= x^{2n+1} - h , \\ x(0) &= x(1) = 0 , \end{aligned}$$

n a positive integer, or more generally to the equations

$$x'' + x\,F(x^2, t) = h$$

studied for example by Z. NEHARI [16] and G. B. GUSTAFSON - K. SCHMITT [9].

3.2.3. Remark. Other existence theorems for boundary value problems for higher order ordinary differential equations may be obtained from the papers [6, 7, 15].

3.3. Remarks. I. The existence of a solution of boundary value problems for nonlinear (partial or ordinary) differential equations in the sublinear or asymptotically linear case under the assumption of the existence of weak asymptotes or weak sub-asymptotes is established in the papers [6, 7, 15]. The proofs in these papers are based on a Hilbert space analogue of Theorems 2.5.5 and 2.5.8 for a selfadjoint bounded linear operator L. The same idea can be also applied to nonlinear integral equations of the Hammerstein type.

II. It is possible to apply existence theorems 2.5.5 and 2.5.8 (and also Remark 2.5.9) to boundary value problems for partial differential equations of the type

$$-\Delta u + g(u) = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i},$$

where $f_i \in L_p(\Omega)$, $1 < p < 2$ ("very bad right hand sides").

III. Analogously as in Theorem 3.1.3 we can apply the superlinear case to boundary value problems for partial differential equations as well as to nonlinear integral equations provided the linear part has simple eigenvalues.

IV. If the operator N_0 has a weak asymptote with respect to Ker $[L]$ equal to zero then the solvability of the equation

$$L(x) = N_0(x) - h$$

cannot be obtained by applying Theorem 2.5.8 but we must go back to the general Theorem 2.3.10. A typical example is the problem

$$x'' + x = \frac{x}{1 + x^2} - h\,, \tag{18}$$

$$x(0) = x(\pi) = 0\,.$$

To solve (18), we put

$$X = \{x \in C^2\langle 0, \pi\rangle : x(0) = x(\pi) = 0\}\,,$$

$$Z = C\langle 0, \pi\rangle\,,$$

$$\text{Dom}\,[L] = X\,, \quad L : x \mapsto x'' + x$$

and

$$N_0 : x \mapsto \frac{x(\cdot)}{1 + x^2(\cdot)}\,.$$

Then

$$\operatorname{Ker}[L] = \{\lambda \sin\tau : \lambda \in R_1\}\,,$$

$$\operatorname{Im}[L] = \left\{z \in Z : \int_0^\pi z(\tau)\sin\tau\,d\tau = 0\right\}.$$

The mappings

$$Q : z \mapsto \frac{2\sin\tau}{\pi}\int_0^\pi z(\tau)\sin\tau\,d\tau\,,\quad z \in C\langle 0, \pi\rangle\,,$$

$$P : x \mapsto \frac{2\sin\tau}{\pi}\int_0^\pi x(\tau)\sin\tau\,d\tau\,,\quad x \in C^2\langle 0, \pi\rangle$$

are continuous algebraic projections in the spaces considered and $\operatorname{Im}[P] = \operatorname{Ker}[L]$, $\operatorname{Im}[Q^c] = \operatorname{Im}[L]$. For each $x \in X$ it is

$$\|N_0(x)\|_Z \leqq \tfrac{1}{2}$$

and it is easy to see that the mapping $N_0 : X \to Z$ is completely continuous. So the mapping $KQ^c N_0 : X \to \operatorname{Im}[P^c]$, where K is the right inverse of L, is completely continuous and $\|K\| = \frac{4}{3}$. According to Remark 2.5.3 it is

$$(\psi Q\, N_0(tw + v), w) = \frac{2}{\pi}\mu\int_0^\pi \frac{tw + v(\tau)}{1 + (tw + v(\tau))^2}\sin\tau\,d\tau\,,$$

where $\mu = \pm 1$, $w = \mu\sin\tau$.

Since the condition **I** from Lemma 2.3.7 is fulfilled with $a = 2 + 4\|h\|_Z$ we shall investigate the set $\mathfrak{N}$ introduced in Lemma 2.3.6. Let $\delta = (9a^2 + 16a + 9)^{-1}$ (i.e., $(34a^2)^{-1} \leqq \delta \leqq 9^{-1}$). For $\|v\|_X \leqq a$, $\mu = 1$ (and analogously for $\mu = -1$), $(1 + a)\,\delta^{-1} \leqq t \leqq 2(a + 1)\,\delta^{-1}$ it is

$$\frac{2}{\pi}\int_0^\pi \frac{t\sin\tau + v(\tau)}{1 + (t\sin\tau + v(\tau))^2}\sin\tau\,d\tau = \frac{2}{\pi}\left(\int_\delta^{\pi-\delta} \cdot/\cdot + \int_0^\delta \cdot/\cdot + \int_{\pi-\delta}^\pi \cdot/\cdot\right) \geqq$$

$$\geqq \frac{2\delta}{\pi}\left\{2\,\frac{2(a + 1) + a\delta}{\delta^2 + [2(a + 1) + a\delta]^2}\cos\delta - \delta\right\} \geqq$$

$$\geqq \frac{2\delta}{\pi}\left\{2\,\frac{2(a + 1) + a\delta}{\delta^2 + [2(a + 1) + a\delta]^2}(1 - \delta^2) - \delta\right\} \geqq \frac{81}{17a^2\pi}\,\frac{4a - 3}{1 + [19a + 1]^2} = \varphi(a)\,.$$

Thus (by Theorem 2.3.10) the boundary value problem (18) is solvable provided

$$\frac{2}{\pi}\left|\int_0^\pi h(\tau)\sin\tau\,d\tau\right| < \varphi(2 + 4\|h\|_Z)\,.$$

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