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ON SOME PROPERTIES OF THE CANTOR SET AND THE CONSTRUCTION OF A CLASS OF SETS WITH CANTOR SET PROPERTIES

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1. It has been shown by RANDOLPH [8] and BOSE MAJUMDER [4] that each point in $(0, 1)$ is the mid point of at least one pair of Cantor points and it has further been shown by Bose Majumder [4] that except for a set of measure zero, each point of $(0, 1)$ is the mid point of continuum number of pairs of points of the Cantor set and that no point of $(0, 1)$ is the mid point of countably infinite number of pairs of Cantor points.

Theorem 1. *Each point d in $(0 < x < 1)$ is a point of trisection on a segment of the interval $0 \leqq x \leqq 1$, the two end points of which are Cantor points.*

Proof. Let $x \in [0, 1]$ be represented in its triadic expansion:

$$x = \frac{x_1}{3} + \frac{x_2}{3^2} + \ldots + \frac{x_i}{3^i} + \ldots,$$

where $x_i = 0, 1, 2$ for all i.

We take [6]

$$f_i(x) = 2\delta(x_i, 2)$$

and

$$v_i(x) = 2\delta(x_i, 1)$$

where

$$\delta(a, b) = 1\,, \quad \text{if} \quad a = b$$
$$= 0\,, \quad \text{if} \quad a \neq b\,.$$

Hence

$$f_i(x) = v_i(x) = 0 \quad \text{if} \quad x_i = 0$$

whereas

$$f_i(x) \neq v_i(x) \qquad \text{if} \quad x_i = 2 \quad \text{or} \quad 1\,.$$

$$\left[\begin{matrix} f_i(x) = 2 \\ v_i(x) = 0 \end{matrix}\right\} \text{ when } \ x_i = 2 \ \text{ and } \ \left.\begin{matrix} f_i(x) = 0 \\ v_i(x) = 2 \end{matrix}\right\} \text{ when } \ x_i = 1 \Bigg]$$

For a given $x \in (0, 1)$ let

$$f(x) = \frac{f_1(x)}{3} + \frac{f_2(x)}{3^2} + \frac{f_3(x)}{3^3} + \ldots$$

and

$$v(x) = \frac{v_1(x)}{3} + \frac{v_2(x)}{3^2} + \frac{v_3(x)}{3^3} + \ldots .$$

It follows that

$$x = f(x) + \frac{v(x)}{2} \quad \text{where} \quad f(x) \in C\,, \quad v(x) \in C\,,$$

C being the Cantor set. Indeed

$$x_i = 0 \Rightarrow \frac{f_i(x) + \frac{1}{2}\,v_i(x)}{3^i} = \frac{0 + \frac{1}{2} \times 0}{3^i} = 0,$$

$$x_i = 1 \Rightarrow \frac{f_i(x) + \frac{1}{2}\,v_i(x)}{3^i} = \frac{0 + \frac{1}{2} \times 2}{3^i} = \frac{1}{3^i},$$

$$x_i = 2 \Rightarrow \frac{f_i(x) + \frac{1}{2}\,v_i(x)}{3^i} = \frac{2 + \frac{1}{2} \times 0}{3^i} = \frac{2}{3^i}.$$

[For instance, let

$$x = \frac{1}{3} + \frac{2}{3^2} + \frac{0}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{2}{3^6} = \cdot\,1\,2\,0\,1\,1\,2 \ (\text{scale } 3)\,.$$

Hence

$$f(x) = \frac{0}{3} + \frac{2}{3^2} + \frac{0}{3^3} + \frac{0}{3^4} + \frac{0}{3^5} + \frac{2}{3^6} = \cdot\,0\,2\,0\,0\,0\,2 \ (\text{scale } 3) \in C\,,$$

$$v(x) = \frac{2}{3} + \frac{0}{3^2} + \frac{0}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \frac{0}{3^6} = \cdot\,2\,0\,0\,2\,2\,0 \ (\text{scale } 3) \in C\,.$$

Hence

$$\frac{v(x)}{2} = \cdot\,1\,0\,0\,1\,1\,0 \ (\text{scale } 3)\,.$$

Thus

$$f(x) + \frac{v(x)}{2} = \cdot 1\,2\,0\,1\,1\,2 = x \Big] .$$

It follows that if d is any point of $(0, 1)$ then

$$(1) \qquad d = f(d) + \frac{v(d)}{2}$$

where $f(d)$ and $v(d)$ are Cantor points.

Now let d be any point in $0 < x < \frac{2}{3}$. We now choose d' such that

$$d = \tfrac{2}{3}d' \quad \text{i.e.} \quad d' = \tfrac{3}{2}d .$$

Since $0 < d < \frac{2}{3}$, we have

$$0 < \tfrac{2}{3}d' < \tfrac{2}{3} \quad \text{or} \quad 0 < d' < 1 .$$

By (1)

$$d' = f(d') + \frac{v(d')}{2} = \frac{2c_2 + c_1}{2} ,$$

where $c_2[=f(d')]$ and $c_1[=v(d')]$ are two Cantor points depending on d' and hence on d.

Therefore

$$\tfrac{3}{2}d = \frac{2c_2 + c_1}{2} \quad \text{or} \quad d = \frac{2c_2 + c_1}{3}$$

i.e. d trisects the segment $[c_1, c_2]$.

If $\frac{2}{3} \leqq d < 1$ then $1 - \frac{2}{3} \geqq 1 - d > 0$ or $0 < 1 - d \leqq \frac{1}{3}$.

Hence by previous argument

$$1 - d = \frac{2c_2' + c_1'}{3}$$

where c_1' and c_2' are Cantor points.

Thus

$$3 - 3d = 2c_2' + c_1' \quad \text{or} \quad 3d = 2(1 - c_2') + (1 - c_1') = 2c_1'' + c_1''$$

$$\therefore d = \frac{2c_2'' + c_1''}{3}$$

where c_1'' and c_2'' are Cantor points and thus d is a point of trisection of the segment $[c_1'', c_2'']$ with Cantor end points. Thus the theorem is proved.

2. A linear set S is said to have the property (S_n) if there exists an η_n such that if

$$X_1 < X_2 < \ldots < X_n, \quad X_n - X_1 < \eta_n$$

are any n real numbers, there exist n elements $Y_1, Y_2, \ldots, Y_n \in S$ congruent to $X_1, X_2, \ldots, X_n$.

E. MARCZEWSKI proposed the following problem: does there exist a perfect set S of measure zero having the property (S_3)? It may be mentioned in this connection that the Cantor middle third set C, which is perfect and of Lebesgue measure zero has the property (S_2) (STEINHAUS, [13]; RANDOLPH [8]; UTZ [14]; ŠALÁT, [10]; BOSE MAJUMDER [4]). ERDÖS and KAKUTANI [5] constructed a set S of measure zero having the property (S_n), $n > 1$. It is known that the Cantor set C does not possess the property (S_3) (Šalát [10], cross-reference, Steinhaus [12]).

In this article we have tried to investigate the reasons as to why the set C fails to possess the property (S_3) and our results are embodied in Theorem 2.

Theorem 2. *Let $X_1, X_2, X_3 (X_1 < X_2 < X_3)$ be any triad of three points on the real line such that*

$$X_2 - X_1 = d_1 = \sum_{k=1}^{\infty} \frac{2v_k^{(1)}}{3^k},$$

$$X_3 - X_1 = d_2 = \sum_{k=1}^{\infty} \frac{2v_k^{(2)}}{3^k}, \quad 0 < d_2 \leqq \tfrac{1}{3},$$

where $v_k^{(i)} = -1, 0$ or 1, $i = 1, 2$ and $k = 1, 2, 3, \ldots$.

A necessary and sufficient condition that there exists a triad of Cantor points congruent to X_1, X_2, X_3 is that

$$|v_k^{(1)} - v_k^{(2)}| \neq 2$$

for any k; and when there exists one such triad belonging to C, then there exists either a finite or continuum number of such triads (and never "a" number of such triads, "a" being the power of the rational set).

Proof. That any d $(0 \leqq d \leqq 1)$ can be expressed as

$$d = \sum_{k=1}^{\infty} \frac{2v_k}{3^k}, \quad v_k = -1, 0, 1, \quad k \geqq 1$$

has been shown by Bose Majumder [4].

Now let

$$d_1 = \sum_{k=2}^{\infty} \frac{2v_k^{(1)}}{3^k} \quad \text{and} \quad d_2 = \sum_{k=2}^{\infty} \frac{2v_k^{(2)}}{3^k}.$$

Choose

$$d_0 = \sum_{k=2}^{\infty} \frac{2v_k^{(0)}}{3^k}, \quad v_k^{(0)} = \binom{0}{1}, \quad k = 2, 3, \ldots$$

such that

$$v_k^{(i)} + v_k^{(0)} \neq 2, \quad i = 1, 2, \quad k = 1, 2, 3, \ldots.$$

That such a choice of $v_k^{(0)}$ is possible may be seen from the table.

$v_k^{(1)}$	$v_k^{(2)}$	$v_k^{(0)}$
-1	-1	1
-1	0	1
-1	1	*
0	-1	1
0	0	(0 or 1) †
0	1	0
1	-1	*
1	0	0
1	1	0

By hypothesis

$$|v_k^{(1)} - v_k^{(2)}| \neq 2$$

hence the possibilities (*) are excluded. Hence it follows that

$$d_0 \in C, \quad d_0 + d_i \in C, \quad i = 1, 2.$$

Therefore the first part of the theorem follows. The conclusion in the second part follows from (†) shown in the table, since the choices of $v_k^{(0)}$ are either 2^m, m finite or $2^a = c$.

3. If the distance set of any point set E fills an interval with origin as its left hand end point, then the set E is called an S-set. It is known that any set E with positive measure is an S-set [13].

If the distance set of any point set E fills an interval with origin as its left hand end point, the length of the interval being equal to the diameter of the set, then the set E is called an SD-set [2].

Cantor set C even though it is of measure zero is an S-set, in fact, it is an SD-set [12], [8], [14], [1], [10], [3].

The distance $\varrho(A, B)$ between two non-empty sets A and B in a metric space is defined by

$$\varrho(A, B) = \inf \{\varrho(a, b) \mid a \in A, \ b \in B\} \quad [9].$$

For a class Λ of sets we can define its diameter $\delta(\Lambda)$ as

$$\delta(\Lambda) = \sup \{\varrho(A, B) \mid A \in \Lambda, \ B \in \Lambda\} .$$

If the distance set $\{\varrho(A, B)\}$ of any class Λ of point set fills an interval with origin as its left hand end point, the length of the interval being equal to the diameter $\delta(\Lambda)$ of the class Λ, then the class Λ will be called an *SD*-class.

Now we ask: does there exist a class Λ of linear point sets, such that it is an *SD*-class? We answer this question in affirmative in the following theorem.

Theorem 3. *There exists a class Λ of sets, where Λ consists of continuum number of pairwise disjoint non-empty linear sets such that the distance set $\{\varrho(A, B)\}$ of Λ fills an interval of length $\delta(\Lambda)$ i.e. Λ is an SD-class.*

Proof. SIERPIŃSKI [11] gave the following theorem.

"*If $2^{\aleph_0} = \aleph_1$, then each linear measurable (in the Lebesgue sense) set E, neither empty nor containing all the real numbers, admits an infinity of linear distinct sets of the power of the continuum superposable by translation on E*".

Suppose we consider the Cantor middle third set C (which stands for E in Sierpiński's theorem). This linear set C satisfies all the conditions of the aforesaid theorem. Hence there exists a set K of real numbers, of the power c of the continuum, such that the class $\Gamma = \{C(a)\}$ of sets [where $C(a)$ represents for a real number $a \in K$, the translation of the set C along the straight line by length a i.e. $C(a)$ is the set of all real numbers $x + a$, $x \in C$] are pairwise disjoint.

Now consider, the class of all sets $\Lambda = \{K(x)\}$ where x is any element of the Cantor set i.e. K is translated separately by each of the points of the Cantor set to form Λ.

Obviously

$$\overline{\overline{\Lambda}} = \overline{\overline{C}} = c ;$$

thus the power of the class Λ is that of the continuum.

Now, we propose to show that the sets of Λ are pairwise disjoint. If possible, let

$$K(x) \cap K(y) \neq \emptyset ,$$

where x and y are two distinct Cantor points.

Let

$$z \in K(x) \cap K(y) ,$$

$$\therefore z \in K(x) \quad \text{and} \quad z \in K(y) \ \text{also} ,$$

$$\therefore z = \lambda + x \quad \text{and} \quad z = \eta + y ,$$

where $\lambda \in K$, $\eta \in K$ and $x, y \in C$.

$$\therefore z \in C(\lambda) \quad \text{and} \quad z \in C(\eta),$$

$$\therefore C(\lambda) \cap C(\eta) \neq \emptyset$$

which contradicts Sierpiński's theorem that the class Γ consists of pairwise disjoint sets and thus Λ consists of pairwise disjoint sets.

We shall now find the distance between two sets $K(x)$ and $K(y)$ of the class Λ.

Now

$$\varrho(K(x), K(y)) = \inf \{|r_x - r_y|, r_x \in K(x), \ r_y \in K(y)\}.$$

But $r_x = r' + x$ and $r_y = r'' + y$, where $r' \in K$, $r'' \in K$ and x and y are fixed Cantor points (as far as $K(x)$ and $K(y)$ are concerned).

$$\therefore \ |rx - r_y| = |r' + x - r'' - y| \geqq |x - y| - |r' - r''|.$$

It follows that the greatest lower bound of the set $\{|r_x - r_y|\}$ is $|x - y|$.

Therefore

$$\varrho(K(x), K(y)) = |x - y|, \quad x \in C, \quad y \in C.$$

It thus follows that the diameter $\delta(\Lambda)$ of the class Λ is 1, which is equal to the diameter of the Cantor set C.

Finally we propose to show that the distance set of the class Λ fills an interval $0 \leqq x \leqq 1$.

Let d be any real number in the interval $0 \leqq x \leqq 1$. Then we know that there exists at least one pair (x, y) of Cantor points such that $d = |x - y|$. It follows that there exist sets $K(x)$ and $K(y)$ of the class Λ such that

$$d = |x - y| = \varrho(K(x), K(y)).$$

Hence Λ is an SD-class.

Corollary. *Except for a set* $\{d\} \subset [0, 1]$ *of measure zero, for every* $d \in [0, 1]$ *there exists continuum number of pairs* $K(x), K(y)$ *of sets of the class* Λ, *such that*

$$\varrho(K(x), K(y)) = d$$

for each pair.

Also for any $d \in [0, 1]$ the cardinal number of the set $\{(K(x), K(y))\}$ such that

$$\varrho(K(x), K(y)) = d$$

is either a finite number or c but never "a" (these results follow from the corresponding results of the Cantor set as given by Bose Majumder [4]).

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