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CONSTRUCTION OF A NET WITHOUT TRANSVERSALS OVER A NON-PLANAR NEAR-FIELD

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Primary motivation of this work was a remark in T. G. OSTROM [1] (cited below in the introduction) concerning the existence of nets without transversals, which are not affine planes. It comes out that it is rather difficult to find examples of such nets because no general procedures for finding transversals or for the verification that a given net contains no transversal are known. In [1], transversal is defined only for a finite net but even if we extend the definition to infinite nets the problem remains the same. From this extended point of view I have succeeded in finding a type of examples, namely using geometric structures over non-planar near-fields. It appeared fruitful to use the projective version of definitions from [1] in connection with [3], pp. 87–93 and with the duality principle. The proof showing that the net constructed has no transversal can be applied even to some other cases (not only for nets over non-planar near-fields). In all these cases we get necessarily infinite nets.

I am indebted to V. HAVEL for calling my attention to the subject and for his constant help and encouragement during the preparation of this paper.

0. INTRODUCTION: MOTIVATION OF THE PROBLEM

In [1], pp. 2–5 the following definitions and Remark are given (we reproduce them freely):

Definition 0,1. (Affine definition of net.) *A net* is a set of objects, called points, together with certain designated subsets, called lines. The lines occur in classes, called *parallel classes*, such that:

(a) Each point bellongs to exactly one line of each parallel class.

(b) If p_1 and p_2 are lines of different parallel classes, then p_1 and p_2 have exactly one point in common.

(c) There are at least three parallel classes and at least two points on a line.

A net with a finite number of points is called *finite net*. Each finite net is characterized by a parameter n, called *the order* of the net, such that:

(a) each line contains exactly n points;

(b) each parallel class consists of exactly n lines;

(c) the total number of points is n^2.

Definition 0,2. An affine plane is a net in which each pair of points belongs to a line. (As it is well-known, a finite affine plane of order n is a finite net of order n with $n + 1$ parallel classes.)

Definition 0,3. If $\mathcal{N}$ is a net of order n and T is a set of n points such that no two points of T lie on a line of $\mathcal{N}$, then T is said to be a transversal of $\mathcal{N}$.

Remark. An affine plane has no transversal, since every pair of points belongs to a line of the plane. However, there are nets which are not planes and have no transversals.

The last remark became the motivation of this work. Later on all the above definitions will appear once more, because we shall need them in the projective version (so that also the definition of transversal must be extended to the infinite case). I shall not deal with relations between the both versions of definitions (they are very simple) because I shall work with the projective version only. Notice here that from the point of view of the incidence relation this projective version is a little more general.

1. BASIC DEFINITIONS

Definition 1,1. An ordered triplet $\mathcal{S} = (\mathcal{A}, \mathcal{B}, \mathrm{I})$, where $\mathcal{A}, \mathcal{B}, \mathrm{I}$ are non-empty sets, is called an *incidence structure* if $\mathrm{I} \subseteq \mathcal{A} \times \mathcal{B}$ and

$$P_i \,\mathrm{I}\, g_k (P_i \in \mathcal{A},\ g_k \in \mathcal{B};\ i, k = 1, 2) \quad \text{imply} \quad P_1 = P_2 \quad \text{or} \quad g_1 = g_2 .$$

Elements of $\mathcal{A}$ are called *points*, elements of $\mathcal{B}$ are called *lines*, relation I is said to be the *incidence relation* (*incidence*) of the given incidence structure, $A \,\mathrm{I}\, b$ is read "A is incident with b", negation of $A \,\mathrm{I}\, b$ will be denoted by A non I b. Two different points which are incident with the same line will be called *joinable*, otherwise *non-joinable*. Two different lines are said to be *intersecting* if there is a point incident with each of them. In the converse case the term *non-intersecting* will be used.

Definition 1,2. A line p of an incidence structure $\mathcal{S}$ is called *principal*, when for each line $g \neq p$ there exists a point A such that $A \,\mathrm{I}\, p, g$. (Such a point is just one.)

Definition 1,3. A point A of an incidence structure $\mathscr{S}$ is called *principal*, when for each point $B \neq A$ a line p exists such that $A, B \mathrm{I} p$.

Remark. For all lines h of an incidence structure $\mathscr{S} = (\mathscr{A}, \mathscr{B}, \mathrm{I})$ put $\tilde{h} := \{P \in \in \mathscr{A} \mid P \mathrm{I} h\}$. Analogously for each $H \in \mathscr{A}$, denote $\tilde{H} := \{p \in \mathscr{B} \mid p \mathrm{I} H\}$.

Definition 1,4. *A net* is an incidence structure with the following properties:

(g 1) each line is principal;

(g 2) there exists at least one line h of this incidence structure such that

a) there are at least three principal points which are incident with h;

b) each point which is incident with h is principal;

(g 3) there are at least two points, not incident with h.

Remark. This definition is more general than that of the affine net. There is no basic ("ideal") line here and its role can be taken by any line which satisfies condition (g 2). The projective version of the definition of the transversal can be modified as follows:

Definition 1,5. Let $\mathscr{G} = (\mathscr{A}, \mathscr{B}, \mathrm{I})$ be a net containing a line h satisfying condition (g 2) from Definition 1,4 (a "basic" line). A transversal T of $\mathscr{G}$ with respect to h is a subset of $\mathscr{A} \setminus \tilde{h}$ such that for each $p \in \mathscr{B}$ there is exactly one $P \in \mathscr{A}$ such that $P \mathrm{I} p$, $P \in T$.

Definition 1,6. A projective parallel structure is an incidence structure $(\mathscr{A}, \mathscr{B}, \mathrm{I})$ such that:

(p 1) each point is principal;

(p 2) there is at least one principal line;

(p 3) there is a principal line $h \in \mathscr{B}$ and points $X, Y, Z \in \mathscr{A}$ such that X, Y, Z non I h and there is no line $a \in \mathscr{B}$ such that $X, Y, Z \mathrm{I} a$;

(p 4) there are at least two points which are incident with each line of $\mathscr{B}$.

Remark. In [3], p. 89 it is proved that condition (p 3) si satisfied in each projective parallel structure.

Let us study connections between nets and projective parallel structures. For this purpose we construct the structure dual to a net. In what follows we shall use the duality principle (cf. [5], p. 3).

Definition 1,7. *A dual net* is an incidence structure such that

(g* 1) each point is principal;

(g* 2) there is at least one point H possessing the properties:

a) it is incident with at least three principal lines;

b) if it is incident with a line p then p is principal.

(g* 3) there are at least two lines such that point H is not incident with them.

Remark. The notion of the dual net can be obtained from the notion of the net by means of the duality.

Proposition 1,1. *The incidence structure from Definition* 1,7 *is a projective parallel structure.*

Proof. [3], p. 94.

Now let us introduce dual versions of projective parallel structures.

Definition 1,8. A *dual projective parallel structure* is an incidence structure $(\mathscr{A}^*, \mathscr{B}^*, \mathrm{I}^*)$ satisfying the following conditions:

(p* 1) each line is principal;

(p* 2) there is at least one principal point;

(p* 3) there is a principal point $H \in \mathscr{A}^*$ such that there exist $x, y, z \in \mathscr{B}^*$ with H non I^* x, y, z and for every $A \in \mathscr{A}^*$ at least one of the relations $A \mathrm{I}^* x$, $A \mathrm{I}^* y$, $A \mathrm{I}^* z$ is violated;

(p* 4) each point of $\mathscr{A}^*$ is incident with at least two lines of $\mathscr{B}^*$.

Remark. The verification of the duality with a projective structure is very easy. By the remark following Definition 1,6, in a dual projective parallel structure the condition (p* 3) holds for an arbitrary point.

Proposition 1,2. *Let there exist a line h of a dual projective parallel structure $\mathscr{P}^*$ such that each point of $\mathscr{P}^*$ incident with h is principal. Then $\mathscr{P}^*$ is a net.*

Proof. Let $\mathscr{P}^*$ be a dual projective parallel structure satisfying the above property. We are going to verify that $\mathscr{P}^*$ satisfies the conditions from Definition 1,4. As $\mathscr{P}^*$ is by Definition 1,8 an incidence structure we verify directly that the properties (g 1), (g 2), (g 3) of Definition 1,4 are consequences of the properties (p* 1), (p* 2), (p* 3), (p* 4) (of Definition 1,8) and the hypothesis of the proposition: (p* 1) obviously implies (g 1). For the line h which occur in (g 2) we take exactly the line h from Proposition 1,2 and we prove that the conditions a) and b) from (g 2) are satisfied. The existence of h and the property b) follow from the hypothesis of the proposition. Thus it remains to prove a). The existence of at least three points incident with h follows from (p* 3) and (p* 1). If x, y, z are lines and H is the point occuring in (p* 3), then at least two of the points $Z \mathrm{I} x, y$; $X \mathrm{I} y, z$; $Y \mathrm{I} x, z$ which exist according to (p* 1) and H satisfy the property (g 3).

Corollary. *Let $\mathcal{P}$ be a projective parallel structure such that*

(pS) *there is a point with the following property:*

if it is incident with a line p, then p is principal.

Then the structure $\mathcal{P}^*$ dual to $\mathcal{P}$ is a net, so that $\mathcal{P}$ is a dual net.

Proof follows from Proposition 1,2 by means of duality.

Definition 1,9. Let $\mathcal{G}^* = (\mathcal{A}^*, \mathcal{B}^*, \mathrm{I}^*)$ be a dual net containing a point H satisfying condition (g* 2) from Definition 1,7 (a "basic" point). *A dual transversal* T^* of $\mathcal{G}^*$ with respect to H is a subset T^* of $\mathcal{B}^* \setminus \tilde{H}$ such that to any point $P \neq H$ there exists just one line $p \in T^*$ with $P \, \mathrm{I}^* \, p$.

Remark. The notion of the dual transversal T^* of a dual net $\mathcal{G}^*$ with respect to some point H is evidently dual to the notion of the transversal T of the net $\mathcal{G} = (\mathcal{A}, \mathcal{B}, \mathrm{I})$ with respect to some line h.

2. CONSTRUCTION OF A NET OVER A NON-PLANAR NEAR-FIELD

Definition 2,1. *A right near-field* is a set N provided with two binary operations $+, .$ such that

(1) $(N, +)$ is an Abelian group with the neutral element 0;

(2) $(N \setminus \{0\}, .)$ is a group with the neutral element 1;

(3) $0 \, . \, a = a \, . \, 0 = 0 \; \forall a \in N$;

(4) $a \, . \, (b + c) = a \, . \, b + a \, . \, c \;\; \forall a, b, c \in N$.

Such a near-field will be denoted by $\mathbf{N} = (N, +, .)$. We shall say that a near-field is non-planar if there exist $a_0, b_0 \in N$; $a_0 \neq 1$ such that $x \neq a_0 x + b_0 \; \forall x \in N$.

Definition 2,2. Let $\mathbf{N} = (N, +, .)$ be a non-planar near-field. Define $\mathcal{A}_{\mathbf{N}}$ to be the set of exactly all triplets $(a, b, 1)$ or $(1, m, 0)$ or $(0, 1, 0)$ $\forall a, b, m \in N$ and $\mathcal{B}_{\mathbf{N}}$ to be the set of exactly all subsets of $\mathcal{A}_{\mathbf{N}}$ of the form $\{(x_1, x_2, x_3) \in \mathcal{A}_{\mathbf{N}} \mid c_1 x_1 + c_2 x_2 + c_3 x_3 = 0\}$ $\forall (c_1, c_2, c_3) \in N^3$; $(c_1, c_2, c_3) \neq (0, 0, 0)$.

Proposition 2,1. *If* $\mathbf{N}$ *is a non-planar near-field, then*

a) $\mathcal{P}_{\mathbf{N}} = (\mathcal{A}_{\mathbf{N}}, \mathcal{B}_{\mathbf{N}}, \in)$ *is a projective parallel structure*

and

b) *it satisfies* (pS).

Proof. a) $\mathscr{P}_{\mathbf{N}}$ is an incidence structure. Show that there is at most one $t \in \mathscr{B}_{\mathbf{N}}$ for any $P_1, P_2 \in \mathscr{A}_{\mathbf{N}}$; $P_1 \neq P_2$ with $P_1, P_2 \in t$.

1) Let $P_1 = (a, b, 1)$, $P_2 = (a', b', 1)$ and let

$$(1) \qquad \begin{aligned} c_1 a + c_2 b + c_3 &= 0 \qquad c'_1 a + c'_2 b + c'_3 = 0 \\ c_1 a' + c_2 b' + c_3 &= 0 \qquad c'_1 a' + c'_2 b' + c'_3 = 0\,; \end{aligned}$$

then

$$c_1(a - a') + c_2(b - b') = 0$$
$$c'_1(a - a') + c'_2(b - b') = 0\,.$$

It is easy to verify that we can choose arbitrarily one of c_1, c_2, c_3 (or c'_1, c'_2, c'_3). This fact will be used in the sequel. If $a = a'$, $b = b'$ there is nothing to verify. If $a = a'$, $b - b' \neq 0$, the last equations read

$$c_2(b - b') = 0 \qquad c'_2(b - b') = 0\,,$$

so that $c_2 = c'_2 = 0$, and if we choose $c_1 = c'_1$ we get $c_3 = c'_3$.

The case $a - a' \neq 0$, $b = b'$ can be considered analogously. If $a - a' \neq 0$, $b - b' \neq 0$, we get equations

$$c_1 = c_2(b - b')(a - a')^{-1}$$
$$c'_1 = c'_2(b - b')(a - a')^{-1}$$

which after substituting $c_2 = c'_2$ imply $c_1 = c'_1$, $c_3 = c'_3$.

2) Let $P_1 = (a, b, 1)$, $P_2 = (1, m, 0)$. The system analogous to (1) admits exactly one solution for $c_2 = c'_2$; then $c_1 = c'_1$, $c_3 = c'_3$. Evidently the case $P_1 = P_2$ cannot occur.

3) Let $P_1 = (a, b, 1)$, $P_2 = (0, 1, 0)$. If we choose $c_1 = c'_1$ then $c_2 = c'_2$, $c_3 = c'_3$.

4) Let $P_1 = (1, m, 0)$, $P_2 = (1, m', 0)$. Equations similar to (1) imply

$$c_2(m - m') = 0\,, \quad c'_2(m - m') = 0\,,$$

i.e. either $m = m'$ or $c_2 = c'_2 = 0$ so that $c_1 = c'_1 = 0$.

5) The last case $P_1 = (1, m, 0)$, $P_2 = (0, 1, 0)$ implies $c_2 = c'_2 = 0$. So $c_1 = c'_1 = 0$.

(p 1) We shall show that each point of the structure $\mathscr{P}_{\mathbf{N}}$ is principal, i.e. that each two different points P_1, P_2 of $\mathscr{P}_{\mathbf{N}}$ are joinable. According to a) we can find a triplet $(c_1, c_2, c_3) \neq (0, 0, 0)$ (which is uniquely determined up to the multiplication from the left).

(p 2) All other lines are intersected by the line h with the equation $x_3 = 0$, for $c_1x_1 + c_2x_2 = 0$ has the solution $x_2 = -c_2^{-1}c_1$ for $c_2 \neq 0$ and $x_1 = 1$. For $c_2 \neq 0$, $x_1 = 0$ we get $x_2 = 0$. Evidently x_2 remains arbitrary if $c_2 = 0$, $x_1 = 1$. For $c_2 = 0$, $x_1 = 0$ we get $x_2 = 1$.

(p 3) None of the points $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$ incides with the line h having the equation $x_3 = 0$. Moreover, the above points are simultaneously incident with no line.

(p 4) Now investigate the equation

$$c_1x_1 + c_2x_2 + c_3x_3 = 0\,.$$

Let $x_3 = 0$ or $x_3 = 1$. Then for any choice of x_2 we can find x_1 by easy calculation.

b) Let $(0, 1, 0)$ have the desired property from (p S). Let us study the set of just all lines having the equations $\bar{c}_1x_1 + \bar{c}_3x_3 = 0$. It is evident that

α) $\bar{c}_1 \neq 0$ implies $x_1 = kx_3$ for a suitable $k \in N$,

β) $\bar{c}_1 = 0$ implies $x_3 = 0$.

Let us verify that each line t of the set under consideration intersects each line $t' \neq t$ with the equation

$$(*) \qquad c_1x_1 + c_2x_2 + c_3x_3 = 0\,.$$

For this purpose it is sufficient to verify the case α). Substituting $x_1 = kx_3$ into (*) we get $c_1kx_3 + c_2x_2 + c_3x_3 = 0$. For $x_3 = 1$ we obtain $c_2 \neq 0$, $x_2 = c_2^{-1}(-c_1k - c_3)$.
If $c_2 = 0$ we get a line $\bar{l}$ such that $(0, 1, 0)$ is incident with it and that $\bar{l}$, t intersect in $(0, 1, 0)$. In the case $c_1 = 0$ we proceed similarly as for (p 2).

Corollary. *$\mathcal{P}_N$ from Proposition 2,1 is a dual net.*

Proof follows from Corollary of Proposition 1,4.

Lemma 2,1. *In $\mathcal{P}_N$ of Proposition 2,1 there exists a point $R \neq (0, 1, 0)$ such that each line such that R is incident with it is principal.*

Proof of the lemma. Choose $R = (1, 0, 0)$. Quite analogously as in b) in the proof of the previous proposition we verify the assertion of the lemma.

Proposition 2,2. *$\mathcal{P}_N$ from Proposition 2,1 contains no dual transversal with respect to the base point $(0, 1, 0)$.*

Proof. According to Proposition 2,1 and Corollary of Proposition 1,2 $\mathscr{P}_{\mathsf{N}}$ is a dual net. Suppose that there exists a dual transversal T^* of the dual net $\mathscr{P}_{\mathsf{N}}$ with respect to the basic point H (as in Definition 1,9), when taking $H = (0, 1, 0)$. If the point R from Lemma 2,1 is incident with each line of T^*, then there exists at most one line in T^*, and all points of $\mathscr{P}_{\mathsf{N}}$ (according to Definition 1,9) are incident with this one line, a contradiction to (g* 3). So we can assume that there exists a line $g \in T^*$ such that R is not incident with it. Then R is incident with some line $r \in T^*$, $r \neq g$, which is principal (according to Lemma 2,1). So there exists a point M which is incident with g and r, a contradiction.

Corollary. *Let a projective parallel structure satisfying* (p S) *contain besides a basic point S another point $R \neq S$ such that if R incides with l then l is principal. Then it contains no dual transversal with respect to S.*

Proof is included in the proof of the above proposition.

Before describing examples of nets over a non-planar near-field we will mention a few simple properties of nets; these properties will be used in the final paragraph of the paper.

Lemma 2,2. *The net $\mathscr{G}$ from Definition* 1,4 *contains besides h (from* (g 2)) *at least three other lines such that no point of the net is incident with all of them simultaneously.*

Proof. According to (g 2) there exist at least three principal points X, Y, Z; $X, Y, Z \mathrel{I} h$. According to (g 3) there exist at least two points A, B which are not incident with h. Then it is easy to verify that among lines XA, XB, YA, YB, ZA, ZB there are always three lines satisfying the hypothesis of Lemma 2,2.

Corollary of the lemma. *A net $\mathscr{G}$ from Definition* 1,4 *contains at least three points which are not incident with the same line and such that none of them is incident with h (with the same meaning as in* (g 2)).

Proof follows from the above lemma by virtue of (g 1).

Proposition 2,3. *Let a net $\mathscr{G}$ of Definition* 1,4 *contain besides the line h (as in* (g 2)) *one more line r such that each point incident with r is principal. Then $\mathscr{G}$ contains no transversal with respect to h.*

Proof. Suppose that the net $\mathscr{G}$ contains a transversal T and that A is a point of this transversal.

a) Let A be not incident with r. There exists a point $B \neq A$ (see Definition 1,5) such that $B \in T$ and $B \mathrel{I} r$. As B is principal there exists a line m such that $A, B \mathrel{I} m$, a contradiction.

b) Assume that T contains points which are incident with r only. According to the definition of the transversal, T contains just one point C. By the lemma there exists a line p, with C non I p, a contradiction to the definition of the transversal.

Now we describe an example of a net over a non-planar near-field.

Definition 2,3. Let $\mathbf{N} = (N, +, .)$ be a non-planar near-field. Let $\mathscr{K}_{\mathbf{N}}$ be the set of exactly all equivalence classes of triplets (x_1, x_2, x_3) $(\forall (x_1, x_2, x_3) \in N^3, (x_1, x_2, x_3) \neq (0, 0, 0)$, where (x_1, x_2, x_3) is equivalent to $(\lambda x_1, \lambda x_2, \lambda x_3)$ for each $\lambda \neq 0$, $\lambda \in N$, $\mathscr{L}_{\mathbf{N}}$ the set

$$\{\{\{(x_1, x_2, x_3) \in \mathscr{K}_{\mathbf{N}} | x_1 \alpha + x_2 \beta + x_3 = 0\} | \alpha, \beta \in N\} \cup$$

$$\cap \{\{(x_1, x_2, x_3) \in \mathscr{K}_{\mathbf{N}} | x_1 + x_2 m = 0\} | m \in N\} \cup \{(x_1, x_2, x_3) \in \mathscr{K}_{\mathbf{N}} | x_2 = 0\}\} .$$

Proposition 2,4. *Let* $\mathbf{N} = (N, +, .)$ *be a non-planar near-field. Then* $(\mathscr{K}_{\mathbf{N}}, \mathscr{L}_{\mathbf{N}}, \in)$ *is a net which has no transversal with respect to the line h with the equation* $x_2 = 0$ *and which is not a plane.*

Proof. $\mathscr{G}_{\mathbf{N}} = (\mathscr{K}_{\mathbf{N}}, \mathscr{L}_{\mathbf{N}}, \in)$ is evidently a dual of $\mathscr{P}_{\mathbf{N}}$ from Proposition 2,1. Hence according to Proposition 2,1 and Consequence of Proposition 1,2 $\mathscr{G}_{\mathbf{N}}$ is a net. The proof of non-existence of a transversal of this net with respect to h may be accomplished by means of Proposition 2,3. Let us choose $r = \{(x_1, x_2, x_3) \in \mathscr{K}_{\mathbf{N}} \mid x_1 = 0\}$ and show that every point $Z \in r$ (i.e. $Z = (0, z_2, z_3)$) is principal, i.e. that for each point Z and each $Y = (y_1, y_2, y_3) \in \mathscr{K}_{\mathbf{N}}$ there exists $p \in \mathscr{L}_{\mathbf{N}}$ such that $Z, Y \in p$.

If $z_2 \neq 0$, $y_1 \neq 0$, then p has the equation $x_1(y_1^{-1}(y_2 z_2^{-1} z_3 - y_3)) - x_2 z_2^{-1} z_3 + z_3 = 0$. If $z_2 \neq 0$, $y_1 = 0$ then $p = r$. If $z_2 = 0$, $y_3 \neq 0$, then p has the equation $x_1 - x_2 y_2^{-1} y_1 = 0$. Finally if $z_2 \neq 0$, $y_2 = 0$ then p has the equation $x_2 = 0$, so that $p = h$. We have exhausted all possibilities. We verify easily that for suitable choices of a, b the points $(1, -1, 0)$, $(a, -1, b) \in \mathscr{K}_{\mathbf{N}}$ is incident with no line simultaneously. The corresponding calculation leads to the equation of the form $x = ax + b$, which has no solution for $a = a_0$, $b = b_0$ from Definition 2,1.

Remark. Proposition 2,4 provides us with the desired example of a net without transversals which is not a plane. To obtain a direct analogy with the remark quoted from [1] it suffices to proclaim the basic line h as ideal one. In this way we get an infinite affine net without transversals which is not plane. Similarly it is possible to obtain further infinite affine nets without transversals which are distinct from planes using non-associative and non-distributive non-planar quasifields.*)

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*) Cf. *J. Bureš,* Construction of an infinite Quasifield, Geometriae Dedicata (in press).

3. REMARKS CONCERNING FINITE NETS

Lemma 3,1. *Let $\mathcal{G}$ be a net from Definition* 1,4 *of finite order n, with basic line h* (*as in* (g 2)). *If $\mathcal{G}$ contains a line $p \neq h$ such that there are exactly $n + 1$ points incident to it, then for each line of $\mathcal{G}$ there are just $n + 1$ points incident with it.*

Proof. Let us use the properties (g 1) and (g 2) from Definition 1,4, which enable us to construct a bijection between the set of all points incident with p and the set of all points incident with p' (p' is arbitrary).

Definition 3,1. Let $\mathcal{G}$ be a net from Definition 1,4 of finite order n, with basic line h (as in (g 2)). Let there exist a line $p \neq h$ of $\mathcal{G}$ with which exactly $n + 1$ points are incident and let exactly k points are incident with the line h (n, k positive integers). Then we say that $\mathcal{G}$ is of degree k.

Proposition 3,1. *Let $\mathcal{G}$ be a finite net of order n and of degree k from Definition* 3,1 *and let $\mathcal{G}$ contain besides the basic line h one more line r such that each point which is incident with r is principal. Then $\mathcal{G}$ has degree $n + 1$.*

Proof. According to Corollary of Lemma 2,2 there exists a point P of a net $\mathcal{G}$ not incident with the line r. Let us denote by R the point incident with both h, r. Denote by r' the line such that both P, R are incident with it. Let us choose a point $W \mathrm{I}\, r'$, $W \neq R$. Obviously W is principal, so that for each point $K_i \mathrm{I}\, r$ $(i = 1, 2, \ldots, n + 1)$ there exists a line w_i; $W, K_i \mathrm{I}\, w_i$ and a point $M_i \mathrm{I}\, w_i, h$. Because $\mathcal{G}$ is an incidence structure and the points $K_1, \ldots, K_{n+1}$ are pairwise distinct, the lines $w_1, \ldots, w_{n+1}$ are also pairwise distinct and thus the points $M_1, \ldots, M_{n+1}$ are pairwise distinct, too. So h contains exactly $n + 1$ points $M_1, \ldots, M_{n+1}$ and $k = n + 1$.

Remark. It is evident from the last proposition that in Proposition 2,3 one infinite net can occur.

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