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OSCILLATION OF SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS

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We consider the nonlinear delay differential equation

(1) $$y^{(n)}(t) + F(t, y[h_1(t)], \ldots, y[h_m(t)], \ldots, y^{(n-2)}[h_m(t)]) = 0$$

where

(2) $$h_i(t) \in C[R_+ \equiv [0, \infty), R]\,, \quad h_i(t) \leqq t \quad \text{for} \quad t \in R_+\,,$$
$$\lim_{t\to\infty} h_i(t) = \infty\,, \quad (i = 1, \ldots, m)\,,$$

(3) $$F(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \in C[D \equiv R_+ \times R^m \times R^{n-1}, R]\,,$$
$$y_{10}y_{i0} > 0\,, \quad i = 2, \ldots, m\,, \quad \text{implies}$$
$$y_{10}\, F(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) > 0 \quad \text{for all sufficiently large } t\,.$$

We shall assume that under the initial conditions $y^{(k)}(t) = \Phi^{(k)}(t)$, $t \leqq t_0$ $(k = 0, 1, \ldots, n-2)$, $y^{(n-1)}(t_0) = y_0^{(n-1)}$, the equation (1) has a solution which exists for all $t \geqq t_0 \in R_+$.

A solution $y(t)$ of the equation (1) is called oscillatory if the set of zeros of $y(t)$ is not bounded from the right. A solution $y(t)$ of the equation (1) is called nonoscillatory if it is eventually of constant sign. A nonoscillatory solution is said to be strongly monotone if it tends monotonically to zero as $t \to \infty$ together with its first $n-1$ derivatives. We consider only such solutions that are not trivial for all sufficiently large t.

The purpose of this paper is to give, under appropriate restriction on F, a necessary and sufficient condition for all solutions of the equation (1) to be oscillatory in the case n is even and to be either oscillatory or strongly monotone when n is odd. Our results are generalisations of those due to KUSANO and ONOSE [2, 3], ŠEVELO and VARECH [6].

Theorem 1. *Let the functions in* (1) *satisfy* (2), (3) *and, in addition, suppose that*

$$(4) \qquad |F(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2})| \leqq \sum_{i=1}^{m} \sum_{j=2}^{n} P_{ij}(t) \, |y_{i,n-j}|^{\alpha_{ij}}$$

for $\qquad (t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \in D \; ; \quad 0 \leqq \alpha_{ij} \leqq 1 \, ,$

$$P_{ij}(t) \in C[R_+, R_+] \, , \quad (i = 1, \ldots, m, \; j = 2, \ldots, n)$$

and such that

$$(5) \qquad \sum_{i=1}^{m} \int^{\infty} [h_i(t)]^{(j-1)\alpha_{ij}} \, P_{ij}(t) \, \mathrm{d}t < \infty \, , \quad j = 2, \ldots, n-1 \, .$$

Then a necessary condition for all solutions of (1) *to be oscillatory if n is even and to be either oscillatory or strongly monotone when n is odd is that*

$$(6) \qquad \sum_{i=1}^{m} \int^{\infty} [h_i(t)]^{(n-1)\alpha_{in}} \, P_{in}(t) \, \mathrm{d}t = \infty \, .$$

The following lemma [1, Lemma 1] will be needed.

Lemma 1. *Let* $a_i \geqq 0$, $b_i \geqq 0$, $r_i > 0$ *and* $r = \max_i \{r_i\}$ $(i = 1, \ldots, m)$. *If* $b_i > 1$ *for some i, then*

$$\sum_{i=1}^{m} a_i b_i^{r_i} \leqq \Big[\sum_{i=1}^{m} a_i\Big] \Big[\sum_{i=1}^{m} b_i\Big]^r \, .$$

Proof of Theorem 1. Our proof is an adaptation of the arguments developed by Hallam [1] and it is similar to that used in [3], [4].

We assume that (6) does not hold and

$$(\bar{6}) \qquad \sum_{i=1}^{m} \int^{\infty} [h_i(t)]^{(n-1)\alpha_{in}} \, P_{in}(t) \, \mathrm{d}t < \infty \, .$$

Then we demonstrate that the equation (1) has a nonoscillatory solution $y(t)$ which is asymptotic to at^{n-1} $(a \neq 0)$ as $t \to \infty$. Choose t_0 so large that $h_i(t) > 1$ for all $t \geqq t_0 > 1$, $(i = 1, \ldots, m)$ and integrate (1) k-times (k is a fixed number from $\{1, \ldots, n\}$) on $[t_0, t]$; we obtain

$$(7) \qquad y^{(n-k)}(t) = \sum_{v=0}^{k-1} \frac{y^{(n-k+v)}(t_0)}{v!} (t - t_0)^v -$$

$$- \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} F(s, y[h_1(s)], \ldots, y[h_m(s)], \ldots, y^{(n-2)}[h_m(s)]) \, \mathrm{d}s \, .$$

From this and in view of (4) we get

$$(8)\qquad |y^{(n-k)}(t)| \leqq t^{k-1}\left[A_k + \sum_{i=1}^{m}\sum_{j=2}^{n}\int_{t_0}^{t} P_{ij}(s)\,|y^{(n-j)}[h_i(s)]|^{\alpha_{ij}}\,ds\right], \quad t \geqq t_0,$$

where

$$A_k = \sum_{v=0}^{k-1}\frac{y^{(n-k+v)}(t_0)}{v!}.$$

Define the function

$$(9)\qquad F_k(t) = A_k + \sum_{i=1}^{m}\sum_{j=2}^{n}\int_{t_0}^{t} P_{ij}(s)\,|y^{(n-j)}[h_i(s)]|^{\alpha_{ij}}\,ds.$$

Then

$$(10)\qquad |y^{(n-k)}(t)| \leqq t^{k-1}F_k(t), \quad t \geqq t_0.$$

Choose $t_1 \geqq t_0$ so large that $h_i(t) \geqq t_0$ for $t \geqq t_1$, $i = 1, \ldots, m$. Then (10) and the monotone character of F_k $(F'_k > 0)$ imply

$$(11)\qquad |y^{(n-k)}[h_i(t)]| \leqq [h_i(t)]^{k-1}F_k(t), \quad t \geqq t_1, \quad i = 1, \ldots, m.$$

Putting (11) in (9) and then summing up from $k = 1$ to $k = n$, we get

$$\sum_{k=1}^{n} F_k(t) \leqq \sum_{k=1}^{n} A_k + \sum_{k=1}^{n}\int_{t_1}^{t}\sum_{i=1}^{m}\sum_{j=2}^{n} P_{ij}(s)\,[h_i(s)]^{(j-1)\alpha_{ij}}\,[F_j(s)]^{\alpha_{ij}}\,ds.$$

If we choose $y^{(n-1)}(t_0)$ such that $A_k > 1$ for some $k \in \{1, \ldots, n\}$ and use Lemma 1, then we have from the last inequality

$$(12)\qquad \sum_{k=1}^{n} F_k(t) \leqq \sum_{k=1}^{n} A_k + n\sum_{i=1}^{m}\int_{t_1}^{t}\left(\sum_{j=2}^{n} P_{ij}(s)\,[h_i(s)]^{(j-1)\alpha_{ij}}\right)\left(\sum_{j=1}^{n}[F_j(s)]^{r}\right)ds,$$

where $r = \max_{i,j}\{\alpha_{ij}\} \leqq 1$, $i \in \{1, \ldots, m\}$, $j \in \{2, \ldots, n\}$.

Then from (12), with regard to $[F_j(t)]^r \leqq F_j(t)$, $t \geqq t_0$, Gronwall's inequality and (5), ($\bar{6}$),

$$\sum_{k=1}^{n} F_k(t) \leqq \left(\sum_{k=1}^{n} A_k\right)\exp n\sum_{i=1}^{m}\sum_{j=2}^{n}\int_{t_1}^{\infty}[h_i(s)]^{(j-1)\alpha_{ij}}P_{ij}(s)\,ds \leqq K < \infty$$

follows.

The inequality (11) now becomes

$$(13)\qquad |y^{(n-k)}[h_i(t)]| \leqq K[h_i(t)]^{k-1}, \quad t \geqq t_1, \quad (i = 1, \ldots, m,\ k = 1, \ldots, n).$$

Integrating the equation (1) from t_1 to t_0, we get

$$y^{(n-1)}(t) = y^{(n-1)}(t_1) - \int_{t_1}^{t} F(s, y[h_1(s)], \ldots, y[h_m(s)], \ldots, y^{(n-2)}[h_m(s)]) \, ds$$

from which, in view of (4), (6), (13) we conclude that a finite limit $\lim y^{(n-1)}(t)$ exists as $t \to \infty$.

If we choose t_1 so large that

$$1 > K \sum_{i=1}^{m} \sum_{j=2}^{n} \int_{t_1}^{\infty} P_{ij}(s) \, [h_i(s)]^{(n-j)\alpha_{ij}} \, ds$$

and consider a solution such that $y^{(n-1)}(t_1) = 1$, then this solution has the desired asymptotic property.

We shall show that a sufficient condition for the oscillation of the equation (1) can be established by means of the differential inequalities

$$\text{(A)} \qquad y^{(n)}(t) + p(t) f(y[h(t)]) \leqq 0 \,, \quad t \geqq 0$$

$$\text{(B)} \qquad y^{(n)}(t) + p(t) f(y[h(t)]) \geqq 0 \,, \quad t \geqq 0 \,.$$

With regard to the inequalities assume that the following conditions are satisfied:

(a) $p \in C[R_+, R_+]$,

(b) $f \in C[R, R]$, $z f(z) > 0$ for $z \neq 0$, $f(z)$ is nondecreasing on R,

(c) there exists α: $0 < \alpha < 1$ such that

$$\liminf_{|z| \to \infty} \frac{|f(z)|}{|z|^{\alpha}} > 0 \,.$$

Theorem 2. *Let the inequality* (A) [(B)] *satisfy* (a)–(c) *and* (2), *and, in addition,*

$$\text{(14)} \qquad \int^{\infty} [h(t)]^{(n-1)\alpha} \, p(t) \, dt = \infty \,.$$

Then for n even the inequality (A) [(B)] *has no positive* [*negative*] *solution on* $[t_0, \infty)$, $t_0 \in R_+$, *while for n odd all positive* [*negative*] *solutions of* (A) [(B)] *are strongly monotone.*

For convenience of the reader, before proving Theorem 2, a modification of Kiguradze lemma [5] will be introduced.

Lemma 2. If $y(t), y'(t), \ldots, y^{(n-1)}(t)$ are absolutely continuous and of constant sign in the interval (t_0, ∞) and $y(t)\, y^{(n)}(t) \leq 0$, then there exists an integer k with $0 \leqq k < n$, $n + k$ being odd and such that

$$y(t)\, y^{(i)}(t) \geqq 0\,, \quad (i = 0, 1, \ldots, k)\,, \tag{15}$$

$$(-1)^{n+i-1}\, y(t)\, y^{(i)}(t) \geqq 0\,, \quad (i = k+1, \ldots, n)\,, \quad t \geqq t_0\,,$$

$$\left|y^{(k)}(t)\right| \geqq t^{n-k-1}\left|y^{(n-1)}(2^{n-k-1}t)\right|\,, \quad t \geqq t_0\,, \tag{16}$$

$$\left|y^{(k-i)}(t)\right| \geqq B_i t^{n-k+i-1}\left|y^{(n-1)}(t)\right|\,, \quad (i = 1, \ldots, k)\,, \quad t \geqq 2^{n-k}t_0\,, \tag{17}$$

where
$$B_i = \frac{2^{-(n-k+i)^3}}{(n-k)\ldots(n-k+i-1)}\,.$$

Proof of Theorem 2. Suppose $y(t) > 0$ for $t \geqq t_0 \in R_+$ [the case $y(t) < 0$ is treated similarly]. Since $\lim h(t) = \infty$ as $t \to \infty$, there exists a $t_1 \geqq t_0$ such that $y[h(t)] > 0$ for $t \geqq t_1$. In view of (a), (b) we get from (A)

$$y^{(n)}(t) \leqq -p(t)\, f(y[h(t)]) \leqq 0\,, \quad t \geqq t_1\,.$$

Therefore $y^{(n-1)}(t)$ is decreasing and the derivatives of $y(t)$ of orders up to $n-1$ are eventually of constant sign for large t, say $t \geqq t_2 \geqq t_1$. Then by Lemma 2 for $y(t)$ and its derivatives (15)–(17) hold, where $k \in \{1, 3, \ldots, n-1\}$ if n is even and $k \in \{0, 2, \ldots, n-1\}$ if n is odd.

I. Let n be either even or odd and $k \in \{1, 2, \ldots, n-1\}$. Since $y'(t) > 0$ for $t > t_2$, $\lim_{t\to\infty} y(t)$ exists either as a finite or infinite limit. In either case, in view of (b), (c) there exists $t_3 \geqq t_2$ such that

$$\frac{f(y[h(t)])}{[y[h(t)]]^\alpha} \geqq d > 0\,, \quad t \geqq t_3\,. \tag{18}$$

Therefore, using (18) we obtain from (A)

$$y^{(n)}(t) + \mathrm{d}p(t)\,[y(h(t))]^\alpha \leqq 0\,, \quad t \geqq t_3\,. \tag{19}$$

If $k \in \{1, 2, \ldots, n-1\}$, then by (17) and the monotonicity of $y^{(n-1)}(t)$ we have

$$y(t) \geqq B_k t^{n-1}\, y^{(n-1)}(t)\,, \quad t \geqq 2^{n-k}t_2 = t_3'$$

and

$$y[h(t)] \geqq B_k[h(t)]^{n-1}\, y^{(n-1)}(t)\,, \quad t \geqq t_4' \geqq t_3'\,.$$

From (19) using the last inequality we have

$$y^{(n)}(t) + dB_k^\alpha\, p(t)\,[h(t)]^{(n-1)\alpha}\,[y^{(n-1)}(t)]^\alpha \leqq 0\,, \quad t \geqq t_4\,, \tag{20}$$

where $t_4 = \max\{t_3, t_4'\}$.

Further we shall use the method by Ševelo and Varech [6, for even order linear delay equations] which is used in the proof of Theorem 2 [3], too.

Dividing (20) by $[y^{(n-1)}(t)]^{\alpha}$ and integrating from t_4 to t, we obtain

$$\int_{t_4}^{t} [h(s)]^{(n-1)\alpha} p(s)\, \mathrm{d}s < \infty \quad \text{as} \quad t \to \infty\,,$$

which contradicts (14).

II. Let n be odd and $k = 0$. If $y(t)$ does not approach zero as $t \to \infty$, then according to (16), we get

$$y(2^{1-n}t) \geqq 2^{-(n-1)^2} t^{n-1}\, y^{(n-1)}(t)\,, \quad t \geqq 2^{n-1} t_2 = t_5\,.$$

Then

$$y(t) = [y(t)/y(2^{1-n}t)]\, y(2^{1-n}t) \geqq A t^{n-1}\, y^{(n-1)}(t)\,, \quad t \geqq t_5\,,$$

where $A = \inf\limits_{t \geqq t_2} |y(t)/y(2^{1-n}t)|\, 2^{-(n-1)^2} > 0$.

Now, if we proceed in the proof exactly as in the case I, we get a contradiction with the existence of a positive solution $y(t)$ of (A), which does not approach zero as $t \to \infty$. Hence it follows that a positive solution of (A) and its first $n - 1$ derivatives must approach zero as $t \to \infty$.

The proof of Theorem 2 is complete.

Corollary. *Let the equation* (1) *satisfy* (2), (3) *and*

(21) $$F(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \geqq p_1(t) f_1(y_{10}) \quad \textit{if} \quad y_{i0} > 0\,,$$
$$(i = 1, \ldots, m) \quad \textit{and such that} \quad (t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \in D\,,$$

(22) $$F(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \leqq p_2(t) f_2(y_{10}) \quad \textit{if} \quad y_{i0} < 0\,,$$
$$(i = 1, \ldots, m) \quad \textit{and such that} \quad (t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \in D\,,$$

where

(a) $$p_i \in C[R_+, R_+]\,, \quad (i = 1, 2)\,,$$

(b) $$f_1 \in C[(0, \infty), (0, \infty)]\,, \quad f_2 \in C[(-\infty, 0), (-\infty, 0)]$$
are nondecreasing functions,

(c) *there exist* $\alpha_i : 0 < \alpha_i < 1$ $(i = 1, 2)$ *such that*

$$\liminf_{z \to \infty} \frac{f_1(z)}{z^{\alpha_1}} \neq 0\,, \quad \liminf_{z \to -\infty} \frac{f_2(z)}{|z|^{\alpha_2}} \neq 0\,.$$

In addition, suppose that

$$(23)\qquad \int^{\infty} [h_1(t)]^{(n-1)\alpha_1}\, p_1(t)\, \mathrm{d}t = \int^{\infty} [h_1(t)]^{(n-1)\alpha_2}\, p_2(t)\, \mathrm{d}t = \infty\,.$$

Then for n even all solutions of (1) are oscillatory, while for n odd all solutions of (1) are either oscillatory or strongly monotone.

Proof. Let us suppose that there exists a nonoscillatory solution $y(t)$ of (1). Let $y(t) > 0$ for $t \geqq t_0 \in R_+$ and such that $y(t)$ is not strongly monotone for n odd. The case $y(t) < 0$ is treated similarly. Since $\lim h_i(t) = \infty$ as $t \to \infty$, $(i = 1, \ldots, m)$, there exists $t_1 \geqq t_0$ such that $y[h_i(t)] > 0$, $(i = 1, \ldots, m)$ for $t \geqq t_1$. Then from the equation (1), in view of (21), (a), (b) we have

$$(24)\qquad y^{(n)}(t) + p_1(t) f_1(y[h_1(t)]) \leqq 0\,, \quad t \geqq t_1$$

and $y(t)$ satisfies (24), which by Theorem 2 yields a contradiction. The proof of Corollary is complete.

Combining Theorem 1 and Corollary we obtain the following theorem, which is an extension of Theorem 3 [3].

Theorem 3. *Let the equation (1) satisfy (2), (3) and, in addition, suppose that there exist functions p_k, f_k $(k = 1, 2)$, $P_{ij}(t)$ $(i = 1, \ldots, m, j = 2, \ldots, n)$ and positive constants $\alpha_k < 1$ $(k = 1, 2)$, $\alpha_{ij} \leqq 1$ $(i = 1, \ldots, m, j = 2, \ldots, n)$ such that*

$$\text{(i)}\qquad p_1(t) f_1(y_{10}) \leqq F(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \leqq$$
$$\leqq \sum_{i=1}^{m} \sum_{j=2}^{n} P_{ij}(t) \left|y_{i,n-j}\right|^{\alpha_{ij}}, \quad y_{i0} > 0\,, \quad (i = 1, \ldots, m)\,,$$
$$(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \in D\,,$$

$$\text{(ii)}\qquad p_2(t) f_2(y_{10}) \geqq F(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \geqq$$
$$\geqq -\sum_{i=1}^{m} \sum_{j=2}^{n} P_{ij}(t) \left|y_{i,n-j}\right|^{\alpha_{ij}}, \quad y_{i0} < 0\,, \quad (i = 1, \ldots, m)\,,$$
$$(t, y_{10}, \ldots, y_{m0}, \ldots, y_{m,n-2}) \in D\,,$$

where p_k, f_k $(k = 1, 2)$ satisfy the assumptions (a)–(c) of Corollary and P_{ij} $(i = 1, \ldots, m, j = 2, \ldots, n)$ satisfy (4) of Theorem 1.

Then a necessary and sufficient condition for all solutions of the equation (1) to be oscillatory when n is even and to be either oscillatory or strongly monotone if n is odd is that (6) and (23) are valid.

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