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NON-ANALYTIC LOCAL FUNCTIONAL CALCULUS

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1. Introduction. Consider a complex Banach space X and let $\mathscr{L}(X)$ be the algebra of all linear operators on X.

We recall that an operator $T \in \mathscr{L}(X)$ is said to have the *single valued extension property* [2] if for any open set $\omega \subset \mathbf{C}$, the only analytic X-valued solution of the equation $(\lambda - T) f(\lambda) = 0$ is the function $f = 0$. It is known that if T is *spectral* [2], or more generally *decomposable* [4], [1], then T has the single valued extension property. Whenever T has the single valued extension property it is possible to define for each $x \in X$ its *local spectrum* $\sigma_T(x)$ (see [2]). We denote by $X_T(F)$ the set of all $x \in X$ having their local spectrum contained in $F \subset \mathbf{C}$. It is known that $X_T(F)$ is a linear manifold and if T is decomposable and F is closed then $X_T(F)$ itself is closed. In this case $X_T(F)$ is a *spectral maximal space* for T; conversely, any spectral maximal space has such a form, for a certain $F \subset \mathbf{C}$ [4], [1].

In case when T is spectral, the proof of having the single valued extension property [2, Th. XV. 3.2] can be immediately adapted in order to show that T has a stronger property: Namely, the equation $(\lambda - T) f(\lambda) = 0$ has no continuous X-valued solution in any open set $\omega \subset \mathbf{C}$, except $f = 0$. One aim of our paper is to study similar phenomena for more general classes of operators, namely for *generalized scalar operators* [3], [1].

For the convenience of the reader let us recall some definitions. Denote by C^∞ the locally convex algebra of all scalar functions, infinitely differentiable in the complex plane.

A *spectral distribution* U is a continuous homomorphism of the algebra C^∞ into $\mathscr{L}(X)$, such that $U(1) = 1_X$.

An operator is said to be *generalized scalar* if there exists a spectral distribution U such that $T = U(z)$, where z stands for the function $z \to z$.

It is known that the support of any spectral distribution U is equal to the spectrum $\sigma(T)$ of T, where $T = U(z)$. The condition of continuity of a spectral distribution means the existence of a constant $M > 0$, of an integer $m \geqq 0$ (the least such m is called the *order* of the distribution) and of a compact neighbourhood Δ of $\sigma(T)$ such that

$$\|U(\varphi)\| \leqq M\|\varphi\|_{m,\Delta}, \quad \varphi \in C^{\infty},$$

where

$$\|\varphi\|_{m,\Delta} = \sum_{0 \leqq k+l \leqq m} \frac{1}{k!\, l!} \sup_{z \in \Delta} \left| \frac{\partial \varphi^{k+l}}{\partial z^k\, \partial \bar{z}^l}(z) \right|,$$

and, as usual, if $z = x + iy$, we put

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

When φ is in C^{∞} and has compact support (i.e. $\varphi \in C_0^{\infty}$), then the symbol $\|\varphi\|_m$ will mean $\|\varphi\|_{m,\Delta}$, with any $\Delta \supset \operatorname{supp} \varphi$.

In what follows we shall show first that if T is a generalized scalar operator having a spectral distribution of order m then in any open set $\omega \subset \mathbf{C}$, the equation

$$(\lambda - T) f(\lambda) = 0$$

has no m-times continuously differentiable X-valued solution, except $f = 0$. Then we prove some results concerning the algebraic character of the structure of spectral maximal spaces of a generalized scalar operator. This type of research has been initiated by P. VRBOVÁ [9] for generalized scalar operators (see also [7], [6]) and, essentially, we reprove her results. However, we get her statements in a different manner and our evaluations seem almost the best possible.

2. Generalized single valued extension property. First we need some results concerning scalar functions and their consequences for spectral distributions. Denote by $D_{r,\lambda}$ the set $\{z \in \mathbf{C};\ |z - \lambda| \leqq r\}$, for any $r > 0$ and $\lambda \in \mathbf{C}$. When $\lambda = 0$ then we put $D_{r,0} = D_r$.

2.1. Lemma. *For any $r > 0$, $r \leqq 1$, there is a function $\varphi_r \in C_0^{\infty}$ such that* $\operatorname{supp} \varphi_r \subset D_r$, $\|\varphi_r\|_m \leqq M r^{-m-2}$, *where $M > 0$ does not depend on r, and such that for any spectral distribution U the integral*

$$(2i)^{-1} \iint \psi(\lambda)\, U(\varphi_r(\lambda - z))\, \mathrm{d}\bar{\lambda} \wedge d\lambda$$

converges to $U(\psi)$ as $r \to 0$ in the norm operator topology, for any $\psi \in C^{\infty}$.

Proof. It is known [5] that there is a function $\varphi \in C_0^{\infty}$, $\varphi \geqq 0$, $\operatorname{supp} \varphi \subset D_1$ and $(2i)^{-1} \iint \varphi(w)\, \mathrm{d}\bar{w} \wedge \mathrm{d}w = 1$. Take now $\varphi_r(z) = r^{-2} \varphi(z/r)$. Then we have

$$\left| \frac{\partial^{k+s} \varphi_r}{\partial z^k\, \partial \bar{z}^s}(z) \right| \leqq \frac{M_{k,s}}{r^{2+k+s}}$$

where $M_{k,s}$ depends only on φ. Taking $0 \leqq k + s \leqq m$ and $r \leqq 1$, it is easy to get

$$\|\varphi_r\|_m \leqq Mr^{-m-2},$$

where $M > 0$ depends only on φ.

Consider now the integral

$$(2i)^{-1} \iint \psi(\lambda)\, \varphi_r(\lambda - z)\, d\bar{\lambda} \wedge d\lambda,$$

where $\psi \in C^\infty$ is arbitrary. It is known [5] that this integral converges to $\psi(z)$ when $r \to 0$ in the topology of C^∞. As U is a continuous spectral distribution, we have

$$U\left((2i)^{-1} \iint \psi(\lambda)\, \varphi_r(\lambda - z)\, d\bar{\lambda} \wedge d\lambda\right) =$$

$$= (2i)^{-1} \iint \psi(\lambda)\, U(\varphi_r(\lambda - z))\, d\bar{\lambda} \wedge d\lambda \to U(\psi).$$

2.2. Lemma. *For any $r > 0$, $r \leqq 1$ there is a function $\psi_r \in C_0^\infty$ such that* supp $\psi_r \subset$ $\subset D_{3r}$, *$\psi_r = 1$ in D_r and $\|\psi_r\|_m \leqq Mr^{-m}$, where $M > 0$ does not depend on r.*

Proof. Let φ_r be the function given by the previous lemma. Define

$$\psi_r(z) = (2i)^{-1} \iint_{D_{2r}} \varphi_r(z - w)\, d\bar{w} \wedge dw.$$

It follows then by [5, Th. 1.5.4] that $\psi_r \in C_0^\infty$, sup $\psi_r \subset D_{3r}$ and $\psi_r = 1$ in D_r. Furthermore, on account of the estimations of the derivatives of φ_r (see the proof of the previous Lemma), we obtain easily

$$\|\psi_r\|_m \leqq Mr^{-m},$$

where $M > 0$ does not depend on r.

2.3. Definition. We say that $T \in \mathscr{L}(X)$ has the *m-single valued extension property* if in any open set $\omega \subset \mathbf{C}$ the only m-times continuously differentiable X-valued solution of the equation $(\lambda - T) f(\lambda) = 0$ is $f = 0$.

2.4. Theorem. *Let T be a generalized scalar operator having a spectral distribution U of order $m \geqq 0$. Then T has the m-single valued extension property.*

Proof. Let $f : \omega \to X$ be a function m-times continuously differentiable such hat $(\lambda - T) f(\lambda) = 0$ for each $\lambda \in \omega$. Take a point $\lambda_0 \in \omega$ and suppose that $f(\lambda_0) \neq 0$.

Take also a sequence $\lambda_n \to \lambda_0$ ($\lambda_n \neq \lambda_0$) and set $r_n = 1/4|\lambda_n - \lambda_0|$. Denote by $\alpha_n(\lambda)$ the sequence $\psi_{r_n}(\lambda - \lambda_0)$, where ψ_{r_n} are given by Lemma 2.2. Since $\lambda_n \notin D_{3r_n,\lambda_0}$ and $\operatorname{supp} \alpha_n \subset D_{3r_n,\lambda_0}$, we have $U(\alpha_n) f(\lambda_n) = 0$ (see [4]). On the other hand, since α_n is equal to 1 in D_{r_n,λ_0}, we have $U(\alpha_n) f(\lambda_0) = f(\lambda_0)$ (see also [4]). Therefore we may write

$$0 = U(\alpha_n) f(\lambda_n) = U(\alpha_n)\,(f(\lambda_n) - f(\lambda_0)) + f(\lambda_0)\,.$$

By Taylor's formula we have

$$f(\lambda_n) - f(\lambda_0) = \sum_{1 \leqq k+s \leqq m} \frac{\partial^{k+s} f}{\partial \lambda^k\, \partial \bar{\lambda}^s}(\lambda_0)\,\frac{(\lambda_n - \lambda_0)^k\,(\bar{\lambda}_n - \bar{\lambda}_0)^s}{k!\, s!} + \theta_m(\lambda_n)\,,$$

where $\lim_n \|\theta_m(\lambda_n)\|\, r_n^{-m} = 0$.

Consider now the space $X_T(\{\lambda_0\})$, which is $\neq \{0\}$ from our hypothesis. Notice that $(\partial^{k+s} f/\partial \lambda^k\, \partial \bar{\lambda}^s)(\lambda_0) \in X_T(\{\lambda_0\})$, for any pair (k, s). Indeed, from $\lambda f(\lambda) - Tf(\lambda) = 0$ we get easily, for any $k \geqq 1$, $s \geqq 0$

$$k\,\frac{\partial^{k-1+s} f}{\partial \lambda^{k-1}\, \partial \bar{\lambda}^s}(\lambda) + (\lambda - T)\,\frac{\partial^{k+s} f}{\partial \lambda^k\, \partial \bar{\lambda}^s}(\lambda) = 0\,,$$

for each $\lambda \in \omega$. Since $X_T(\{\lambda_0\})$ is T-absorbing [8], we have $f(\lambda_0) \in X_T(\{\lambda_0\})$. If we assume that $(\partial^{k+s} f/\partial \lambda^k\, \partial \bar{\lambda}^s)(\lambda_0) \in X_T(\{\lambda_0\})$ for $0 \leqq k + s \leqq q$, then we may assert that $(\partial^{k+s} f/\partial \lambda^k\, \partial \bar{\lambda}^s)(\lambda_0) \in X_T(\{\lambda_0\})$ for $k + s \leqq q + 1$, on account of the above relation, according again to the fact that $X_T(\{\lambda_0\})$ is T-absorbing. Therefore $U(\alpha_n)\,((\partial^{k+s} f/\partial \lambda^k\, \partial \bar{\lambda}^s)(\lambda_0)) = (\partial^{k+s} f/\partial \lambda^k\, \partial \bar{\lambda}^s)(\lambda_0)$ and we may write

$$U(\alpha_n)\,(f(\lambda_n) - f(\lambda_0)) =$$

$$= \sum_{1 \leqq k+s \leqq m} \frac{\partial^{k+s} f}{\partial \lambda^k\, \partial \bar{\lambda}^s}(\lambda_0)\,\frac{(\lambda_n - \lambda_0)^k\,(\bar{\lambda}_n - \bar{\lambda}_0)^s}{k!\, s!} + U(\alpha_n)\,\theta_m(\lambda_n)\,.$$

On account of Lemma 2.2 we have

$$\|U(\alpha_n)\,\theta_m(\lambda_n)\| \leqq M r_n^{-m} \|\theta_m(\lambda_n)\|\,,$$

hence

$$\lim_{n \to \infty} \|U(\alpha_n)\,(f(\lambda_n) - f(\lambda_0))\| = 0\,,$$

consequently $f(\lambda_0) = 0$, which is impossible, and the proof is complete.

As a matter of fact, Theorem 2.4 may be stated for a slightly larger class of operators:

2.5. Theorem. *Let V be an operator quasi-nilpotent equivalent [1] to a generalized scalar operator T having a spectral distribution of order m. Then V has the m-single valued extension property.*

Proof. Suppose that $f(\lambda)$ is an X-valued m-times continuously differentiable function, defined in an open set $\omega \subset \mathbf{C}$, such that $(\lambda - V) f(\lambda) = 0$. Since V is quasi-nilpotent equivalent to T, it follows that V is decomposable [1]. Furthermore, for any closed set $F \subset \mathbf{C}$ we have $X_V(F) = X_T(F)$. In particular, $f(\lambda) \in X_T(\{\lambda\})$ for any $\lambda \in \omega$. Since T is generalized scalar with a spectral distribution of order m we have $(\lambda - T)^{m+1} \mid X_T(\{\lambda\}) = 0$, therefore $(\lambda - T)^{m+1} f(\lambda) = 0$ in ω. According to the fact that T has the m-single valued extension property (Theorem 2.4) we get by recurrence that $f(\lambda) = 0$ in ω.

2.6. Corollary. *Let V be a generalized spectral operator and T its scalar part* [1]. *If T has a spectral distribution of order m then V has the m-single valued extension property.*

2.7. Example. The index given by Theorem 2.4 is the best possible. Indeed, let X be the dual of the space $C^m(D_1)$ (i.e. the Banach space of all complex functions, m-times continuously differentiable in D_1) and T the adjoint of the operator Z defined by

$$Zf(z) = z f(z), \quad f \in C^m(D_1).$$

Then it is easy to see that T is a generalized scalar operator and has a spectral distribution of order m, say U, given by

$$(U(\psi)\, u)(f) = u(\psi f), \quad u \in X, \quad f \in C^m(D_1), \quad \psi \in C^\infty.$$

Let us denote for any $\lambda \in D_1$

$$\delta_\lambda(f) = f(\lambda), \quad f \in C^m(D_1),$$

i.e. δ_λ is the δ-Dirac measure concentrated in $\{\lambda\}$. Notice that $\lambda \to \delta_\lambda$ is an X-valued $(m - 1)$-continuously differentiable function in D_1. Indeed, we have

$$\frac{\partial^{k+s}\delta_\lambda}{\partial z^k\, \partial \bar{z}^s}(f) = \frac{\partial^{k+s} f}{\partial z^k\, \partial \bar{z}^s}(\lambda),$$

for any pair (k, s), $0 \leqq k + s \leqq m$, and consequently for $k + s \leqq m - 1$ we infer easily

$$\left\| \frac{\partial^{k+s}\delta_\lambda}{\partial z^k\, \partial \bar{z}^s} - \frac{\partial^{k+s}\delta_\mu}{\partial z^k\, \partial \bar{z}^s} \right\| \leqq C|\lambda - \mu|,$$

where $C > 0$ depends neither on λ nor on μ.

On the other hand, $(\lambda - T)\, \delta_\lambda = 0$ for all $\lambda \in D_1$.

2.8. Remark. If we denote by $\mathcal{O}_X$ the sheaf of germs of analytic X-valued functions defined in $\mathbf{C}$ and by m_T the action of $\lambda - T$ in $\mathcal{O}_X$, where $T \in \mathscr{L}(X)$, then it is obvious that T has the single valued extension property if and only if m_T acts injectively on $\mathcal{O}_X$.

Suppose now that T is a generalized scalar operator having a spectral distribution of order m. If $\mathscr{D}_X^m$ stands for the sheaf of germs of X-valued functions on $\mathbf{C}$, m-times continuously differentiable, Theorem 2.4 shows that m_T is injective on $\mathscr{D}_X^m$.

For any integer $k \geqq 0$, denote by $\sigma_T^{(k)}(x)$ $(x \in X)$ the complement of the open set $\varrho_T^{(k)}(x)$ with the property that for any $\lambda_0 \in \varrho_T^{(k)}(x)$ there is a neighbourhood V_0 of λ_0 and an X-valued function $f_x(\lambda)$, k-times continuously differentiable in V_0, such that $(\lambda - T) f_x(\lambda) = x (\lambda \in V_0)$. If T is a generalized scalar operator and it has a spectral distribution of order m then, on account of Theorem 2.4, it follows that for $k \geqq m$ there is only one function $x_T(\lambda)$ in $\varrho_T^{(k)}(x)$ such that $(\lambda - T)\, x_T(\lambda) = x$.

2.9. Proposition. *Let T be a generalized scalar operator having a spectral distribution of order m. Then for any $k \geqq m + 1$ and any $x \in X$ one has*

$$\sigma_T^{(k)}(x) = \sigma_T(x)\,.$$

Proof. It is obvious that $\sigma_T^{(k)}(x) \subset \sigma_T(x)$ for any $k \geqq 0$ and $x \in X$. Conversely, if $k \geqq m + 1$ and $(\lambda - T)\, x_T(\lambda) = x$ in an open set ω then $(\lambda - T)\,(\partial x_T/\partial\bar{\lambda})\,(\lambda) = 0$ and by Theorem 2.4 it follows $(\partial x_T/\partial\bar{\lambda})\,(\lambda) = 0$, hence x_T is actually analytic, thus $\varrho_T^{(k)}(x) \subset \varrho_T(x)$.

Proposition 2.9 fails to be true when $k \leqq m$. This fact will follow from the following

2.10. Example. Suppose that p is a real number such that $1 \leqq p < 2$ and let X be the Banach space of all complex Borel functions, p-integrable with respect to the planar Lebesgue measure $\mathrm{d}v(z) = (2i)^{-1}\, \mathrm{d}\bar{z} \wedge \mathrm{d}z$ in the unit disc D_1, with the usual identification of the functions equal v-almost everywhere. Consider on X the operator T defined by

$$(Tf)\,(z) = z\, f(z)\,, \quad f \in X\,.$$

The operator T is scalar in Dunford's sense, hence it has a spectral distribution of order 0.

Take now in D_1 the functions

$$f_\lambda(z) = \begin{cases} (\lambda - z)^{-1} & z \neq \lambda \\ 0 & z = \lambda\,. \end{cases}$$

It is easy to see that $\iint_{D_1} |f_\lambda(z)|^p\, \mathrm{d}v(z) < \infty$ for any $\lambda \in \mathbf{C}$, therefore f_λ are elements of X. Furthermore, the map $\lambda \to f_\lambda$ $(\lambda \in \mathbf{C})$ is continuous for the topology of X. With no loss of generality we may consider this continuity in int D_1. Take $\lambda_n, \lambda_0 \in \operatorname{int} D_1$, $\lambda_n \to \lambda_0$ and $r > 0$ arbitrary. Then for a sufficiently small r we have

$$\|f_{\lambda_n} - f_{\lambda_0}\|^p = \iint_{D_1} \frac{|\lambda_0 - \lambda_n|^p}{|\lambda_n - z|^p |\lambda_0 - z|^p} \mathrm{d}v(z) =$$

$$= \iint_{D_{r,\lambda_0}} \frac{|\lambda_0 - \lambda_n|^p}{|\lambda_n - z|^p |\lambda_0 - z|^p} \mathrm{d}v(z) + \iint_{\complement D_{r,\lambda_0}} \frac{|\lambda_0 - \lambda_n|^p}{|\lambda_n - z|^p |\lambda_0 - z|^p} \mathrm{d}v(z)\,.$$

Notice that $\lim_n |\lambda_0 - \lambda_n|^p |\lambda_n - z|^{-p} |\lambda_0 - z|^{-p} = 0\,(v - a.\mathrm{e.})$ if $z \notin D_{r,\lambda_0}$, therefore by Lebesgue theorem of dominated convergence we have

$$\lim_{n\to\infty} \iint_{\complement D_{r,\lambda_0}} \frac{|\lambda_0 - \lambda_n|^p}{|\lambda_n - z|^p |\lambda_0 - z|^p} \mathrm{d}v(z) = 0\,.$$

On the other hand

$$\left(\iint_{D_{r,\lambda_0}} \frac{|\lambda_0 - \lambda_n|^p}{|\lambda_n - z|^p |\lambda_0 - z|^p} \mathrm{d}v(z)\right)^{1/p} \leqq$$

$$\leqq \left(\iint_{D_{r,\lambda_0}} \frac{\mathrm{d}v(z)}{|\lambda_n - z|^p}\right)^{1/p} + \left(\iint_{D_{r,\lambda_0}} \frac{\mathrm{d}v(z)}{|\lambda_0 - z|^p}\right)^{1/p}.$$

When n is sufficiently large, we have

$$\iint_{D_{r,\lambda_0}} \frac{\mathrm{d}v(z)}{|\lambda_n - z|^p} = \iint_{D_{r_n,\lambda_n}} \frac{\mathrm{d}v(z)}{|\lambda_n - z|^p} + \iint_{D_{r,\lambda_0} \setminus D_{r_n,\lambda_n}} \frac{\mathrm{d}v(z)}{|\lambda_n - z|^p}\,,$$

where $r_n = r - |\lambda_n - \lambda_0|$. We have again by Lebesgue theorem

$$\lim_{n\to\infty} \iint_{D_{r,\lambda_0} \setminus D_{r_n,\lambda_n}} \frac{\mathrm{d}(z)}{|\lambda_n - z|^p} = 0\,.$$

An easy direct calculus gives

$$\iint_{D_{r,\lambda_0}} \frac{\mathrm{d}v(z)}{|\lambda_0 - z|^p} = \frac{2\pi r^{2-p}}{2-p}\,, \quad \iint_{D_{r_n,\lambda_n}} \frac{\mathrm{d}v(z)}{|\lambda_n - z|^p} = \frac{2\pi r_n^{2-p}}{2-p} \leqq \frac{2r^{2-p}}{2-p}\,.$$

Summarizing, we obtain

$$\overline{\lim_{n\to\infty}} \|f_{\lambda_n} - f_{\lambda_0}\| \leqq 2\left(\frac{2\pi r^{2-p}}{2-p}\right)^{1/p}.$$

As $r > 0$ is arbitrary, letting $r \to 0$ we obtain that $f_{\lambda_n} \to f_{\lambda_0}$ in X, therefore the map $\lambda \to f_\lambda$ is continuous.

Now, let us remark that for any $\lambda \in \mathbf{C}$ we have $(\lambda - T) f_\lambda = 1$, therefore, with the notations of Proposition 2.9, $\sigma_T^{(0)}(1) = \emptyset$, while $\sigma_T(1) = D_1$.

3. The algebraic structure of spectral maximal spaces. In this section we intend to describe, following Vrbová [9], the structure of spectral maximal spaces of a generalized scalar operator, pointing out its algebraic character.

3.1. Proposition. *Let T be a generalized scalar operator. Assume that T has the property*

$$\bigcap_{\lambda \in \mathbf{C}} (\lambda - T)^q X = \{0\},$$

for a certain natural number q. Then we have:

1) *For any closed $F \subset \mathbf{C}$*

$$X_T(F) = \bigcap_{\lambda \notin F} (\lambda - T)^q X;$$

2) *If V is another operator such that V commutes with T and $(V - T)^{k+1} = 0$ then*

$$\bigcap_{\lambda \in \mathbf{C}} (\lambda - V)^{q+k} X = \{0\}.$$

Proof. 1) It is clear that $X_T(F) \subset \bigcap_{\lambda \notin F} (\lambda - T)^q X$. Conversely, let x be in $\bigcap_{\lambda \notin F} (\lambda - T)^q X$ and take $\varphi \in C_0^\infty$, $\varphi = 1$ in a neighbourhood of F. Let U be a spectral distribution of T. Then $y = U(1 - \varphi) x \in X_T(\operatorname{supp}(1 - \varphi))$. As $x = (\lambda - T)^q x_\lambda$ for any $\lambda \notin F$, then we can define

$$y_\lambda = \begin{cases} U(1 - \varphi) x_\lambda & \lambda \notin F \\ (\lambda - T \mid X_T(\operatorname{supp}(1 - \varphi)))^{-q} y & \lambda \in F, \end{cases}$$

and we have $(\lambda - T)^q y_\lambda = y$ for any $\lambda \in \mathbf{C}$, hence $y = 0$. We get that $x = U(\varphi) x$ for any φ such that $\varphi = 1$ in a neighbourhood of F, hence $x \in X_T(F)$ [3].

2) Let us remark that

$$(\lambda - V)^{k+q} = (\lambda - T)^q \sum_{j=0}^{k} (-1)^j \binom{k+q}{j} (\lambda - T)^{k-j} (V - T)^j.$$

Suppose that $x \in \bigcap_{\lambda} (\lambda - V)^{k+q} X$, hence $x = (\lambda - V)^{k+q} y_\lambda$ $(\lambda \in \mathbf{C})$. If

$$x_\lambda = \sum_{j=0}^{k} (-1)^j \binom{k+q}{j} (\lambda - T)^{k-j} (V - T)^j y_\lambda$$

then $(\lambda - T)^q x_\lambda = x$ $(\lambda \in \mathbf{C})$, hence $x = 0$.

3.2. Theorem. *Let T be a generalized scalar operator having a spectral distribution of order m. If q is an integer such that $q \geqq m + 3$ then*

$$\bigcap_{\lambda \in \mathbf{C}} (\lambda - T)^q X = \{0\} .$$

Proof. First, let us notice that we have in fact to show that $\bigcap_{\lambda \in \sigma(T)} (\lambda - T)^q X = \{0\}$. Then we need the following

3.3. Lemma. *Suppose that there is an integer $q \geqq 1$ and an element $x \in X$ such that $(\lambda - T)^q y_\lambda = x$ for any $\lambda \in \sigma_T(x)$ $(x \neq 0)$. Denote by Y_λ the set $\{y_\lambda \in X; (\lambda - T)^q y_\lambda = x\}$. Then there is an open disc D such that $D \cap \sigma_T(x) \neq \emptyset$, a constant $C > 0$ and $x_\lambda \in Y_\lambda$ with $\|x_\lambda\| \leqq C$ for λ in a dense subset of $D \cap \sigma_T(x)$.*

Proof of the lemma. Consider the sets

$$B_n = \text{the closure of } \{\lambda \in \sigma_T(x);\ \inf_{x_\lambda \in Y_\lambda} \|x_\lambda\| \leqq n\} .$$

We have obviously $\sigma_T(x) = \bigcup_n B_n$, therefore by Baire's theorem at least one set B_n has a non-void (relative) interior. If B_{n_0} is such a set, we take $C = n_0 + 1$, an open disc D such that $\emptyset \neq D \cap B_n \subset \sigma_T(x)$ and then we may choose $x_\lambda \in Y_\lambda$ such that $\|x_\lambda\| \leqq C$, for λ running through a dense subset in $D \cap \sigma_T(x)$; the proof of the lemma is finished.

Let us return to the proof of our theorem. Assume that there is an $x \in \bigcap_{\lambda \in \sigma(T)} (\lambda - T)^q X$ such that $x \neq 0$. Let D be the disc given by Lemma 3.3; suppose that $(\lambda - T)^q x_\lambda = x$ with x_λ chosen according to this lemma. Denote by U the spectral distribution of T and take $\lambda_0 \in D \cap \sigma_T(x)$. There exists $\varphi \in C_0^\infty$ such that $\varphi = 1$ in a neighbourhood of λ_0, $\operatorname{supp} \varphi \subset D$ and $U(\varphi)\, x \neq 0$ (otherwise $\sigma_T(x) \not\ni \lambda_0$). Obviously $(\lambda - T)^q U(\varphi)\, x_\lambda = U(\varphi)\, x$ and $\|U(\varphi)\, x_\lambda\| \leqq C\|U(\varphi)\|$ for λ in a dense subset of $D \cap \sigma_T(x)$, therefore if we take instead of x the element $U(\varphi)\, x$ and instead of X and T the space $X_T(F)$ and $T \mid X_T(F)$, where $F = \sigma_T(U(\varphi)\, x)$, we may suppose $(\lambda - T)^q x_\lambda = x$, for any $\lambda \in \mathbf{C}$, and $\|x_\lambda\| \leqq C$ for λ in a set B whose closure contains $\sigma_T(x) = \sigma(T)$.

Now, consider for any $\lambda \in \mathbf{C}$ and $r > 0$, $r \leqq 1$, the function $\varphi_r(\lambda - z)$ given by Lemma 2.1. We want to evaluate the absolute value of the map

$$\lambda \to (\lambda - T)^q\, U(\varphi_r(\lambda - z))\, x_\lambda ,$$

defined for $\lambda \in \mathbf{C}$. Note that $U(\varphi_r\lambda - z))\, x_\lambda \in X_T(D_{r,\lambda})$, therefore if $D_{r,\lambda} \cap \sigma(T) = \emptyset$ then $U(\varphi_r(\lambda - z))\, x_\lambda = 0$. If $D_{r,\lambda} \cap \sigma(T) \neq \emptyset$ then we choose $\mu_\lambda \in B$ such that $D_{3r,\mu_\lambda} \supset D_{r,\lambda}$. Let us notice that

$$(\lambda - T)^q\, U(\varphi_r(\lambda - z))\, x_\lambda = (\mu_\lambda - T)^q\, U(\varphi_r(\lambda - z))\, x_{\mu_\lambda} =$$
$$= U((\mu_\lambda - z)^q\, \varphi_r(\lambda - z))\, x_{\mu_\lambda} .$$

On account of Leibniz' formula and the proof of Lemma 2.1 we obtain

$$\left|\frac{\partial^{k+h}}{\partial z^k\,\partial\bar{z}^h}(\mu_\lambda - z)^q\,\varphi_r(\lambda - z)\right| \leqq$$

$$\leqq \sum_{t,s} C_{t,s} \sup_{z\in D_{3r,\mu_\lambda}} \left|\frac{\partial^{k+h-s-t}}{\partial z^{k-s}\,\partial\bar{z}^{h-t}}(\mu_\lambda - z)^q\,\frac{\partial^{s+t}}{\partial z^s\,\partial\bar{z}^t}\varphi_r(\lambda - z)\right| \leqq$$

$$\leqq \sum_{s=0}^{k} C_s r^{q-k+s} r^{-s-h-2} \leqq M_{k,h} r^{q-k-h-2}\,,$$

where $M_{k,h}$ are constants (independent on λ and r). Consequently,

$$\|(\lambda - T)^q\,U(\varphi_r(\lambda - z))\,x_\lambda\| \leqq M r^{q-m-2}\,,$$

for $0 < r \leqq 1$, where $M > 0$ is a constant independent on λ and r. Take now a function $\chi \in C_0^\infty$ such that $\chi = 1$ in a neighbourhood of $\sigma(T)$. We have then

$$\iint \chi(\lambda)\,(\lambda - T)^q\,U(\varphi_r(\lambda - z))\,x_\lambda\,\mathrm{d}\bar{\lambda} \wedge \mathrm{d}\lambda = \iint \chi(\lambda)\,U(\varphi_r(\lambda - z))\,x\,\mathrm{d}\bar{\lambda} \wedge \mathrm{d}\lambda$$

(the left part is integrable as being equal to the right, which is obviously integrable).

On account of Lemma 2.1,

$$\lim_{r\to 0} \iint \chi(\lambda)\,U(\varphi_r(\lambda - z))\,x\,\mathrm{d}\bar{\lambda} \wedge \mathrm{d}\lambda = 2i\,U(\chi)\,x = 2ix\,.$$

On the other hand, if $q \geqq m + 3$ then

$$\lim_{r\to 0} \left\|\iint \chi(\lambda)\,(\lambda - T)^q\,U(\varphi_r(\lambda - z))\,x_\lambda\,\mathrm{d}\bar{\lambda} \wedge \mathrm{d}\lambda\right\| = 0\,,$$

hence $x = 0$, which is a contradiction. The proof is finished.

3.4. Corollary (Vrbová). *Let T be a generalized scalar operator. Then there is an integer q such that for any closed F one has*

$$X_T(F) = \bigcap_{\lambda\notin F} (\lambda - T)^q\,X\,.$$

The proof follows directly from Proposition 3.1 (1) and Theorem 3.2. Moreover, q can be any integer $\geqq m + 3$, where m is the order of a spectral distribution of T.

3.5. Remark. If T is a generalized scalar operator with real spectrum then the minimal index given by Theorem 3.3 can be improved. Indeed, in such a case the function φ_r, which appears in Lemma 2.1, may be taken on the real line and then it satisfies an estimation of the form $\|\varphi_r\|_m \leqq Mr^{-m-1}$, hence in Theorem 3.2 the minimal index is then $m + 2$.

4. The case of spectral operators. In what follows, we intend to give a variant of Theorem 3.2 with a better minimal index, valid for some scalar operators, and its immediate consequences. The proof is based on Badé's multiplicity theory [2, Ch. XVIII].

4.1. Theorem. *Let S be a scalar operator in Dunford's sense. If the Boolean algebra corresponding to its spectral measure is complete then for any integer $q \geqq 2$*

$$\bigcap_{\lambda \in \mathbf{C}} (\lambda - S)^q X = \{0\}\,.$$

Proof. Without loss of generality we may take $q = 2$. Suppose that $x \in \in \bigcap_{\lambda}(\lambda - S)^2 X$, $x \neq 0$; then we have $(\lambda - S)^2 z_\lambda = x$ $(z_\lambda \in X)$ for any $\lambda \in \mathbf{C}$. Let $\sigma \to E(\sigma)$ be the spectral measure of S (σ Borel set in $\mathbf{C}$) and $\mathfrak{M}(x)$ the cyclic subspace spanned by x i.e.

$$\mathfrak{M}(x) = \text{c.l.m.}\,\{E(\sigma)\,x;\ \sigma \text{ Borel set}\}\,.$$

Notice that $\mathfrak{M}(x)$ is invariant for the spectral measure of S, hence for the functional calculus of S with Borel functions. Let us remark that the solutions of the equation $(\lambda - S)^2 z_\lambda = x$ may be chosen in $\mathfrak{M}(x)$. To see that, let us denote $B_{n,\lambda} = \complement D_{1/n,\lambda}$ and let us define $x_\lambda = \lim_n E(B_{n,\lambda})\, z_\lambda$. We have that $E(B_{n,\lambda})\, z_\lambda = (\lambda - S)^{-2} E(B_{n,\lambda})\, x \in \in \mathfrak{M}(x)$, therefore $x_\lambda \in \mathfrak{M}(x)$. Moreover, $(\lambda - S)^2 x_\lambda = x$; indeed, if λ is an eigenvalue then $E(\{\lambda\})\, x = (\lambda - S)^2 E(\{\lambda\})\, z_\lambda = 0$, hence

$$(\lambda - S)^2 x_\lambda = (\lambda - S)^2 \lim_n E(B_{n,\lambda})\, z_\lambda = \lim_n E(B_{n,\lambda})\, x =$$
$$= E(\complement\{\lambda\})\, x = E(\complement\{\lambda\})\, x + E(\{\lambda\})\, x = x\,.$$

If λ is not an eigenvalue then $\lim_n E(B_{n,\lambda}) = 1_X$ and similarly $(\lambda - S)^2 x_\lambda = x$.

Let us recall some facts concerning the structure of $\mathfrak{M}(x)$, taken from [2, Ch. XVIII]. Let f be a scalar Borel function and let us consider the set

$$\mathscr{D}(S(f)) = \left\{y;\ \lim_n \int_{\sigma_n} f(\lambda)\, E(\mathrm{d}\lambda)\, y \text{ exists}\right\},$$

where $\sigma_n = \{\lambda;\ |f(\lambda)| \leqq n\}$. Define then the operator

$$S(f)\, y = \lim_n \int_{\sigma_n} f(\lambda)\, E(\mathrm{d}\lambda)\, y\,, \quad y \in \mathscr{D}(S(f))\,,$$

not necessarily bounded. It is known that

$$\mathfrak{M}(x) = \{S(f)\, x;\ x \in \mathscr{D}(S(f))\} .$$

Moreover, there exists a positive Borel measure $\sigma \to \mu(\sigma)$ (σ Borel set) which dominates the vector measure $\sigma \to E(\sigma)\, x$, such that if $x \in \mathscr{D}(S(f))$ then $f \in L^1(\mathrm{d}\mu)$ and the mapping

$$\mathfrak{M}(x) \ni S(f)\, x \xrightarrow{\tau} f \in L^1(\mathrm{d}\mu)$$

is continuous and injective.

Now, consider again the relation $(\lambda - S)^2\, x_\lambda = x$. As $x_\lambda \in \mathfrak{M}(x)$, we have $x_\lambda = S(f_\lambda)\, x$, where $f_\lambda \in L^1(\mathrm{d}\mu)$, for any $\lambda \in \mathbf{C}$. We see that

$$S((\lambda - z)^2 f_\lambda)\, x = (\lambda - S)^2\, S(f_\lambda)\, x = x .$$

As $\tau(x) = 1$, we get $(\lambda - z)^2 f_\lambda(z) = 1$, hence $f_\lambda(z) = (\lambda - z)^{-2} \in L^1(\mathrm{d}\mu)$. But setting $g_\lambda(z) = (\lambda - z)^{-1}$, then $g_\lambda \in L^2(\mathrm{d}\mu)$ and $(\lambda - Z)\, g_\lambda = 1$, where $(Zf)(z) = z\, f(z)$; since Z is a normal operator on $L^2(\mathrm{d}\mu)$, the property $\bigcap_\lambda (\lambda - Z)\, L^2(\mathrm{d}\mu) \neq 0$ is impossible, according to [7]. Consequently $x = 0$ and the proof is complete.

4.2. Corollary. *Let T be a spectral operator of type m such that the Boolean algebra corresponding to its spectral measure is complete. Then for any $q \geqq m + 2$*

$$\bigcap_{\lambda \in \mathbf{C}} (\lambda - T)^q\, X = \{0\} .$$

4.3. Corollary. *With the conditions of the previous corollary, if E is the spectral measure of T then for any closed F we have*

$$E(F)\, X = \bigcap_{\lambda \notin F} (\lambda - T)^q\, X ,$$

for any $q \geqq m + 2$.

4.4. Remark. The minimal index given by Theorem 4.1 is the best possible, as shown by our Example 2.10.

4.5. Corollary. *Let X be a separable reflexive Banach space.*

1) *If S is a scalar operator on X then for any integer $q \geqq 2$*

$$\bigcap_{\lambda \in \mathbf{C}} (\lambda - S)^q\, X = \{0\} .$$

2) *If T is a spectral operator of type m then for any integer $q \geqq m + 2$*

$$\bigcap_{\lambda \in \mathbf{C}} (\lambda - T)^q\, X = \{0\} .$$

These facts are direct consequences of Theorem 4.1, Corollary 4.2 and of the following

4.6. Proposition. *Let X be a separable reflexive Banach space and $\mathscr{E}$ the Boolean algebra associated to a countably additive spectral measure E on X. Then $\mathscr{E}$ is complete.*

It is very plausible that this proposition is known. In the sequel we shall give its proof, since we have not found a published one.

Proof. Let $M > 0$ be a constant such that $\|E(\sigma)\| \leqq M$, for any Borel set σ in $\mathbf{C}$. Since X is separable and reflexive then its dual is also separable, hence the weak operator topology on

$$\mathscr{S}_M = \{T \in \mathscr{L}(X);\ \|T\| \leqq M\}$$

may be given by a distance d. Moreover, the space $(\mathscr{S}_M, d)$ is compact.

Let $\{E_\alpha\}_{\alpha \in A}$ be a monotone increasing generalized sequence in $\mathscr{E}$ and denote by $\mathscr{E}_\alpha$ the closure in $(\mathscr{S}_M, d)$ of the set $\{E_\beta;\ \beta \geqq \alpha\}$. Let F be an element of $\bigcap_\alpha \mathscr{E}_\alpha$, which does exist on account of the compactness of $(\mathscr{S}_M, d)$. It is easy to choose a sequence $\{E_{\alpha_j}\}_{j=1}^\infty$ such that $\alpha_1 \leqq \alpha_2 \leqq \ldots$, $E_{\alpha_j} \in \mathscr{E}_{\alpha_j}$ $(j = 1, 2, \ldots)$ and $d(E_{\alpha_j}, F) \to 0$ as $j \to \infty$. Let σ_j be a Borel set such that $E(\sigma_j) = E_{\alpha_j}$ $(j = 1, 2, \ldots)$. Since

$$E(\sigma_{j+1})\, E(\sigma_j) - E(\sigma_j) = E_{\alpha_{j+1}} E_{\alpha_j} - E_{\alpha_j} = 0\,,$$

we have $E(\sigma_j \setminus (\sigma_j \cap \sigma_{j+1})) = 0$ $(j \geqq 1)$, therefore, by neglecting eventually null-sets, we may suppose that the sequence of Borel sets $\{\sigma_j\}_{j=1}^\infty$ is itself increasing. Denote by σ the union $\bigcup_j \sigma_j$. Since E is a spectral measure, the sequence $E(\sigma_j)$ is strongly convergent to $E(\sigma)$, hence $E(\sigma) = F$. Let α be an arbitrary index in A and σ_α a Borel set such that $E_\alpha = E(\sigma_\alpha)$. Assume $E(\sigma_\alpha \setminus \sigma) \neq 0$ and take $x = E(\sigma_\alpha \setminus \sigma)\, x \neq 0$. Then $Fx = E(\sigma)\, x = 0$ and $E_\beta x = E_\beta E_\alpha x = x$, for any $\beta \geqq \alpha$. If x^* is a continuous linear functional on X such that $x^*(x) \neq 0$ then

$$|x^*(E_\beta x) - x^*(Fx)| = |x^*(x)| > 0\,,$$

hence $F \notin \mathscr{E}_\alpha$, which is a contradiction. Consequently $E(\sigma_\alpha \setminus \sigma) = 0$, therefore we may suppose $\sigma_\alpha \subset \sigma$ from the beginning. In order to finish our proof, we have only to show that $E_\alpha = E(\sigma_\alpha)$ $(\alpha \in A)$ is strongly convergent to $F = E(\sigma)$. But we have for $\alpha \geqq \alpha_j$

$$\begin{aligned} \|(E_\alpha - F)\, x\| &= \|(E_\alpha - F)\,(E(\sigma_{\alpha_j}) + E(\complement\sigma_{\alpha_j}))\, x\| = \\ &= \|(E(\sigma_\alpha) - E(\sigma))\, E(\complement\sigma_{\alpha_j})\, x\| \leqq \|(E(\sigma_\alpha) - E(\sigma))\,(E(\complement\sigma_{\alpha_j}) - E(\complement\sigma))\, x\| + \\ &+ \|(E(\sigma_\alpha) - E(\sigma))\, E(\complement\sigma)\, x\| \leqq 2M\|(E(\complement\sigma_{\alpha_j}) - E(\complement\sigma))\, x\|\,, \end{aligned}$$

hence the generalized sequence $\{E_\alpha\}$ is strongly convergent to an element $E(\sigma) \in \mathscr{E}$. According to [2, Lemma XVII 3.4], $\mathscr{E}$ is complete.

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References

[1] *Colojoară I., Foiaş C.:* Generalized spectral operators, Gordon and Breach, Science Publishers, New York, 1968.

[2] *Dunford N., Schwartz J.:* Linear operators, Part III: Spectral operators, Wiley-Interscience, New York, 1971.

[3] *Foiaş C.:* Une application des distributions vectorielles à la théorie spectrale, Bull. Sci. Math. (2), *84* (1960), 147–158.

[4] *Foiaş C.:* Spectral maximal spaces and decomposable operators in Banach space, Arch. Math., *14* (1963), 341–349.

[5] *Hörmander L.:* Linear partial differential operators, Springer-Verlag, 1963.

[6] *Johnson B. E.:* Continuity of linear operators commuting with continuous linear operators, Trans. Amer. Math. Soc., *128* (1967), 88–102.

[7] *Pták V., Vrbová P.:* On the spectral function of a normal operator. Czech. Math. J. *23* (*98*), (1973), 615–616.

[8] *Vasilescu F.-H.:* Residually decomposable operators in Banach spaces, Tôhoku Math. Journal, *21* (1969), 509–522.

[9] *Vrbová P.:* The structure of maximal spectral spaces of generalized scalar operators. Czech. Math. J. *23* (*98*), (1973), 493–496.

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