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ON n°-REGULAR SEMIGROUPS

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The following is an investigation of the class of n°-regular semigroups. In particular we show that for 0-simple semigroups the class of n°-regular $(n \geqq 2)$ and 2°-regular semigroups coincide. Indeed 0-simple n°-regular semigroups are completely 0-simple. It is then shown that each non-null principal factor of a n°-regular semigroup is also completely 0-simple. It follows that when $S = S^\circ$ is n°-regular then S is regular if and only if S is 0-semisimple.

1. PRELIMINARIES, n°-REGULAR SEMIGROUPS

Throughout this paper we will consider only semigroups with zero, 0, and at least one additional element. Following [1] we will designate such semigroups by: $S = S^\circ$. We begin by recalling the following definition from [4]:

Definition 1.1. Let m and n be nonnegative integers with $m + n > 1$. A semigroup S will be in the class of $(m, n)^\circ$-semigroups, written $S \in (m, n)^\circ$, if and only if for each $x \in S$ one of the following holds

(1) $m > 0$ and $x^m = 0$,

(2) $n > 0$ and $x^n = 0$,

(3) $x = x^m u x^n$ for some $u \in S$ where x° is suppressed in the equation when necessary.

We will say that S is $(m, n)^\circ$-*regular* whenever $S \in (m, n)^\circ$ and that S is n°-*regular* when $S \in (n, n)^\circ$.

Proposition 1.2. *Let $S = S^\circ$ be an n°-regular $(n \geqq 2)$ semigroup then some power of each $x \in S$ lies in a subgroup of S.*

Proof. Let $x \in S$. If $x^n = 0$ then as $\{0\}$ is a subgroup of S we are done. On the other hand if $x^n \neq 0$ there exists a $u \in S$ such that $x^n u x^n = x$. Since $n \geqq 2$ we have $x^2(x^{n-2}ux^n) = (x^n u x^{n-2})\, x^2 = x$ (suppressing x^{n-2} if $n = 2$) so that $x^2 \mathscr{H} x$ and hence H_x is a subgroup ([1] Theorem 2.16). In either case some power of x belongs to a subgroup of S.

The following corollary is now immediate.

Corollary 1.3. *Let $S = S^\circ$ be a semigroup and $n \geqq 2$. Then S is n°-regular if and only if for $x \in S$ either $x^n = 0$ or H_x is a subgroup.*

We recall that with $n \geqq 2$, n fixed, for any m, $1 \leqq m \leqq n$ the classes of $(m, n)^\circ$, $(n, m)^\circ$ and n°-regular semigroups coincide ([4] Corollary 1.8). We thus defined the class of N°-regular semigroups as follows.

Definition 1.4. A semigroup $S = S^\circ$ is said to be *N°-regular* if $S \in \bigcup_{n \geqq 2} (n, n)^\circ$. We will write $S \in N^\circ$ if S is N°-regular.

We recall from [1] § 1.6 that a *periodic semigroup* S is a semigroup in which each element generates a subsemigroup of finite order, i.e., for $a \in S$, $[a] = \{a, a^2, a^3, \dots\}$ is a finite set.

Definition 1.5. A subset T of a semigroup S is said to be *bounded periodic* if there is an (integral) upper bound on the orders of its elements.

Proposition 1.6. *Let $S = S^\circ$. Then $S \in N^\circ$ if and only if S is the disjoint union of its maximal subgroups and a bounded periodic subset of nilpotent elements.*

Proof. If $S \in N^\circ$ then $S \in (n, n)^\circ$ for some n. Thus by (1.3) for $x \in S$ either $x^n = 0$ or H_x is a group. Since the $\mathscr{H}$-classes of S which contain idempotents are the maximal subgroups of S ([1] p. 61 Ex. 1) the implication in this direction is easily completed.

If there is a bound, n, to the order of each nilpotent element then with the converse assumption either x is nilpotent and $x^n = 0$ or H_x is a group. Thus by (1.3) $S \in (n, n)^\circ \subseteq N^\circ$ and the result follows.

The following corollary is now immediate.

Corollary 1.7. *Let $S = S^\circ$ be a finite semigroup. Then $S \in N^\circ$ if and only if each $x \in S$ is either nilpotent or lies in a subgroup of S.*

If for each $n \geqq 2$ we define $C_n = \{a_n, a_n^2, \dots, a_n^n = 0\}$ where $a_n^k \neq 0$ for $1 \leqq k < n$ and for $m \neq n$ define $C_n C_m = 0$ and take $S = \bigcup_{n \geqq 2} C_n$ then each $x \in S$ is nilpotent but $S \notin N^\circ$. Thus the overall assumption in (1.7) of finiteness is crucial for the converse.

Again we recall from [1] § 1.6 that when $a \in S$ is of finite order and a^s is the smallest positive integral power of a repeating a previous positive integral power a^r, that r is said to be the *index of a*, while $m = s - r$ called the *period of a*. It is easy to verify the following result.

Proposition 1.8. *Let S be a semigroup and suppose $a \in S$ is of finite order. Then a belongs to a subgroup of S if and only if the index of a is* 1, *i.e., $a^n = a$ for some $n > 1$.*

If an element a is nilpotent it clearly is of period 1.

Corollary 1.9. If $S = S^\circ \in N^\circ$ and S is periodic then each $a \in S$ is either of index 1 or period 1. Conversely if $S = S^\circ$ is a bounded periodic semigroup such that each $a \in S$ is either of index 1 or nilpotent then $S \in N^\circ$.

We recall the following definition and remark from [3, 4]:

Definition 1.10. A semigroup $S = S^\circ$ is *absorbent* if either $ab = 0$ or $ab \in R_a \cap L_b$ for any $a, b \in S$.

Remark 1.11. An absorbent semigroup is easily seen to be 2°-regular by taking $x = a = b$ and observing that the equation in Definition 1.1.3 is solvable in $H_x = R_x \cap L_x$ which is a subgroup when $x^2 \neq 0$.

Theorem 1.12. *Let $S = S^\circ$ be a semigroup. Then S is completely* 0-*simple if and only if S is N°-regular and* 0-*simple.*

Proof. Suppose S is completely 0-simple. Then S is regular and absorbent ([3] Theorem 2.4) and hence 2°-regular. Thus $S \in N^\circ$ and S is 0-simple.

Conversely suppose S is N°-regular and 0-simple. Then S is n°-regular for some $n \geqq 2$ and by (1.2) some power of each element lies in a subgroup of S. The result now follows from [1] Theorem 2.55.

Corollary 1.13. *Let $S = S^\circ$ be a* 0-*simple semigroup. Then S is 2°-regular if and only if S is n°-regular* ($n \geqq 2$).

Corollary 1.14. *Let $S = S^\circ$ be a regular* 0-*bisimple semigroup. Then S is completely* 0-*simple if and only if S is n°-regular* ($n \geqq 2$).

Proof. The regularity of S is sufficient for $S^2 \neq \{0\}$ so that S is 0-simple and the result follows immediately.

We conclude this section with a theorem which further illuminates (1.12) and which is analogous to [4] Theorem 2.7.

Theorem 1.15. *Let $S = S^\circ$ be a n°-regular* ($n \geqq 2$) *semigroup. Then if we restrict the usual ordering, $\leqq$, of the idempotents of S by $\leqq \cap \mathscr{D}$ the non-zero idempotents of S are primitive, i.e., if $e \mathscr{D} f$ and $e \leqq f$ then $e = f$.*

Proof. Under the restricted partial ordering suppose $e \leqq f$ and $e \neq 0$ where e, f are idempotents in some D_a, $a \neq 0$ and $ef = fe = e$. We must show that $f = e$.

Let $x \in R_f \cap L_e \neq \emptyset$. Then ([1] Lemma 2.14) since f is idempotent we have $fx = x$ so that ϱ_x is a right translation of L_f onto $L_x = L_e$ ([1] Lemma 2.2) and thus there exists an $x' \in L_f \cap R_e$ such that $x'x = e$. Moreover $xx' \in R_x \cap L_{x'} = R_f \cap L_f = H_f$ since $L_x \cap R_{x'} = H_e$ is a group ([1] Theorem 2.17). One readily checks that xx' is idempotent and it then follows that $xx' = f$.

Since e is a right identity on its $\mathscr{L}$-class and $ef = e$ by hypothesis we have $x = xe = x(ef) = (xe)f = xf$. Hence $x^2x' = x$ and it follows that $x^kx' = x^{k-1}$ for $k \geqq 2$. Thus if $x^n = 0$ it would follow that $x = 0$, a contradiction since $a \in S \setminus \{0\}$ and $D_a \neq \{0\}$. Whence H_x is a subgroup of S by (1.3).

Now since H_x is a group and $L_e = L_x$, $R_f = R_x$ we have $ef \in R_e \cap L_f$ ([1] Theorem 2.17). From $e = ef$ it follows that $e \in L_f$ and thus ([1] Lemma 2.14) $fe = f$. Since we assume $e = ef = fe$ it follows that $e = f$. Thus the idempotents of any non-zero $\mathscr{D}$-class of S under this restricted ordering are primitive.

2. PRINCIPAL FACTORS OF n°-REGULAR SEMIGROUPS

We give here for the reader's convenience the following definition and lemma, modified for $S = S^\circ$, from [1] § 2.6.

Definition 2.1. Let S be a semigroup and $a \in S$. The *principal factor* $P(a)$ of a is the Rees quotient: $P(a) = J(a)/I(a)$, where $J(a) = S^1aS^1$ and $I(a) = J(a) \setminus J_a$.

Lemma 2.2. ([1] Lemma 2.39). *Each principal factor of a semigroup $S = S^\circ$ is either 0-simple or null.*

It is now easy to prove the following results.

Lemma 2.3. *If $S = S^\circ$ is n°-regular $(n \geqq 2)$ then $P(a)$ is n°-regular for each $a \in S$.*

Proof. Let $x \in P(a)$ for $a \neq 0$ and suppose $x^n \neq \bar{0} \in P(a)$, $\bar{0} = I(a)$. Then surely $x^n \neq 0$ so that H_x is a subgroup of S by (1.3). Since $H_x \subseteq J_x = J_a$ we can find a u in J_a, and hence in $P(a)$, such that $x = x^nux^n$. If $x^n = \bar{0}$ there is nothing further to show. Thus in either case $P(a)$ is n°-regular according to the definition (1.1).

We remark that the converse is false. Consider the infinite cyclic semigroup with adjoined zero: $S = S^\circ = \{0, a, a^2, a^3, \ldots\}$. Here for $s \in S \setminus 0$, since $\mathscr{J} = \Delta_S$, each principal factor $P(s)$ is null and of order 2. Thus $P(s)$ is 2°-regular for each $s \in S \setminus 0$ but S is far from being n°-regular for any $n \geqq 2$.

Theorem 2.4. *Let $S = S^\circ$ be n°-regular. Then each non-null principal factor of S is a completely 0-simple semigroup and hence 2°-regular.*

Proof. If $P(a)$ for $a \in S$ is a non-null principal factor of S then $P(a)$ is 0-simple by (2.2). By (2.3) it is also n°-regular. The result now follows from (1.12) and (1.13).

Definition 2.5. A semigroup $S = S^0$ is said to be 0-*semisimple* if each of its non-zero principal factors is 0-simple.

If one adjoins a zero, 0, to a semisimple ([1] p. 74) semigroup T, then $T \cup \{0\}$ is readily seen to be 0-semisimple. Indeed, one sees as in [1] p. 74 that a semigroup $S = S^\circ$ is 0-semisimple precisely when 0 is the only null principal factor of S.

Corollary 2.6. *If $S = S^\circ$ is n°-regular and 0-semisimple then each non-zero principal factor is a completely 0-simple semigroup.*

Corollary 2.7. *Let M be a 0-minimal ideal of a semigroup $S = S^\circ$ which is n°- regular. If $M^2 \neq 0$ then M is itself a completely 0-simple semigroup.*

Proof. Suppose $M^2 \neq 0$. Clearly $M = P(m)$, for each $m \in M \backslash 0$, and the result now follows directly from (2.4).

It is easily shown that a regular semigroup, $S = S^\circ$, is 0-semisimple. There are Baer-Levi semigroups ([1] Chap. 8) which are not regular but left simple and hence 0-semisimple. However with the added assumption of n°-regularity we do have the following result.

Corollary 2.8. *If S is n°-regular and 0-semisimple then $\mathcal{J} = \mathcal{D}$ and S is regular.*

Proof. Suppose $a \mathcal{J} b$ and $a \neq 0$. Then clearly $P(a) = P(b) = P \neq \{0\}$ and $a, b \in P$. Since P is completely 0-simple by (2.6) we have $a \mathcal{D} b$ in P (solvable in $J_a = J_b$) and hence in S. Since $\mathcal{D} \subseteq \mathcal{J}$ we have $\mathcal{J} = \mathcal{D}$.

Now each $a \in S \setminus \{0\}$ belongs to a principal factor, $P(a)$ which is a completely 0-simple semigroup and hence surely regular. But $P(a)$, as a Rees quotient, consists of the individual elements of $J_a = D_a$ and a zero, $I(a)$, so that it readily follows that S is itself regular.

The natural next step in determining the structure of n°-regular semigroups is to examine that subclass consisting of those at are 0-semisimple and have a principal series. By [1] Theorem 2.40 each factor of such a series is isomorphic to a principal factor which when non-zero is completely 0-simple by (2.4). What remains then is an extension problem: namely to characterize n°-regular extensions of one completely 0-simple semigroup by another completely 0-simple semigroup. This will be treated elsewhere.

Bibliography

[1] *A. H. Clifford* and *G. B. Preston,* The Algebraic Theory of Semigroups, vol. 1, 2 Math. Survey 7, Amer. Math. Soc., 1961, 1967.

[2] *M. R. Croisot,* Demi-groups inversif et demi-groups reunions de demi-groups simples, Ann. Sci. Ecole Norm. Sup. (3) *70* (1953), 361–379.

[3] *Kenneth M. Kapp,* Green's relations and quasi-ideals, Czech. Math. Journal, *19* (*94*) 1969, 80–85.

[4] *Kenneth M. Kapp,* On Croisot's Theory of Decompositions, Pacific J. Math. (1) *28* (1969), 105–115.

[5] *E. S. Ljapin,* Semigroups, Amer. Math. Soc. Translation, Vol. *3,* 1963.

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