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# ON INFINITESIMAL ISOMETRIES OF A HYPERSURFACE

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Let $E^n$ be the Euclidean $n$-space and $V(E^n)$ its vector space. Let $M \subset E^n$ be a hypersurface. Consider the system of hypersurfaces $\mu_t : M \to E^n$, $t \in (-\varepsilon, \varepsilon) = J \subset \mathcal{R}$, such that $\mu_0 = \text{id.}$ and the mapping $\mu_t : M \times J \to E^n$ is analytic in $t$. Then, in a suitable neighborhood $(-\varepsilon_1, \varepsilon_1) \subset J$,

$$\mu_t(M) = M + tv_1 + t^2 v_2 + \dots \tag{1}$$

with $v_\alpha : M \to V(E^n)$. The metric on the hypersurface $M_t = \mu_t(M)$ is given by means of the form

$$G_t = \mathrm{d}M_t \,.\, \mathrm{d}M_t = G_0 + \sum_{\alpha=1}^{\infty} t^\alpha \Big(2\, \mathrm{d}M \,.\, \mathrm{d}v_\alpha + \sum_{\beta=1}^{\alpha-1} \mathrm{d}v_\beta \,.\, \mathrm{d}v_{\alpha-\beta}\Big) \,. \tag{2}$$

The surfaces $M_t$ and $M$ being isometric for each $t \in J_1$, we have

$$2\, \mathrm{d}M \,.\, \mathrm{d}v_\alpha + \sum_{\beta=1}^{\alpha-1} \mathrm{d}v_\beta \,.\, \mathrm{d}v_{\alpha-\beta} = 0 \quad \text{for} \quad \alpha = 1, 2, \dots \tag{3}$$

**Definition.** The mapping $v_1 : M \to V(E^n)$ is said to be an *infinitesimal deformation* of $M$ if

$$\mathrm{d}M \,.\, \mathrm{d}v_1 = 0 \,. \tag{4}$$

A formal series of the type (1) is called a *formal deformation* of $M$ if the vector fields $v_\alpha$ satisfy (3).

We are looking for the conditions under which each infinitesimal deformation $v_1$ of $M$ has an extension to a formal deformation (1).

## I.

Let $g_{ij}$ and $h_{ij}$ $(i, j, \dots = 1, \dots, n = \dim M)$ be the fundamental tensors of $M$; further, let $\nabla_k$ be the covariant differentiation with respect to $g_{ij}$. Consider the diagram

$$
(5)\qquad
\begin{array}{ccccccc}
0 & \longrightarrow & \mathscr{A} & \longrightarrow & A & \xrightarrow{d_1} & A_1 \\
 & & & & \big\downarrow{\scriptstyle h_1} & & \\
0 & \longrightarrow & \mathscr{B} & \longrightarrow & B & \xrightarrow{d_2} & B_1 \\
 & & & & \big\downarrow{\scriptstyle h_2} & & \\
 & & & & C & &
\end{array}
$$

where: (i) $A$ is the $\mathscr{R}$-module of symmetric (2, 0)-tensors $a_{ij}$ on $M$, $A_1$ is the $\mathscr{R}$-module of (3, 0)-tensors $a_{ijk}$ on $M$, $B$ is the $\mathscr{R}$-module of (4, 0)-tensors $b_{ijpq}$ on $M$ satisfying $b_{ijpq} = -b_{jipq} = -b_{ijqp}$, $B_1$ is the $\mathscr{R}$-module of (5, 0)-tensors $b'_{rijpq}$ on $M$, $C$ is the $\mathscr{R}$-module of (4, 0)-tensors $c_{ijpq}$ on $M$; (ii) the differential operators $d_1$ and $d_2$ are defined by

$$(6)\qquad d_1(a_{ij}) = \nabla_k a_{ij} - \nabla_j a_{ik},$$

$$(7)\qquad d_2(b_{ijpq}) = \nabla_r b_{ijpq} + \nabla_p b_{ijqr} + \nabla_q b_{ijrp}$$

resp; (iii) the homomorphisms $h_1$ and $h_2$ are given by

$$(8)\qquad h_1(a_{ij}) = a_{ip}h_{jq} - a_{jp}h_{iq} - a_{iq}h_{jp} + a_{jq}h_{ip},$$

$$(9)\qquad h_2(b_{ijpq}) = \delta^{rs}(h_{rj}b_{ispq} + h_{rp}b_{isqj} + h_{rq}b_{isjp})$$

resp. with $\delta^{rs} = 0$ for $r \neq s$ and $\delta^{rr} = 1$; (iv) $\mathscr{A}$ or $\mathscr{B}$ is the sheaf of the solutions of the equation $d_1 a = 0$ or $d_2 b = 0$ resp.

**Proposition 1.** *We have* $h_1(\mathscr{A}) \subset \mathscr{B}$.

**Proposition 2.** (Poincaré lemma.) *Let* $m \in M$, $U \subset M$ *be a neighborhood of* $m$, *and let* $b \in \Gamma(\mathscr{B}, U)$ *satisfy* $h_2(b) = 0$. *Then there is a neighborhood* $U_1 \subset U$ *of* $m$ *and an* $a \in \Gamma(\mathscr{A}, U_1)$ *such that* $h_1(a) = b$ *on* $U_1$.

**Theorem.** *If* $h_1(\Gamma(\mathscr{A}, M)) = \Gamma(\mathscr{B}, M) \cap \operatorname{Ker} h_2$, *then to each infinitesimal deformation* $v_1$ *of* $M$ *there is a formal deformation* $M + tv_1 + t^2v_2 + \ldots$

We are going to prove the propositions and the theorem.

Be given a neighborhood $U \subset M$ such that to each point $m \in U$ there is an ortonormal frame $\sigma_m = \{e_1, \ldots, e_{n+1}\}$ with $e_1, \ldots, e_n \in T_m(M)$, the field of frames $\sigma_m$ being smooth over $U$. Then there are 1-forms $\omega^i$, $\omega_i^j$, $\omega_i^{n+1}$, $\omega_{n+1}^i$ $(i, j, \ldots = 1, \ldots, n)$ over $U$ such that

$$(10)\qquad dM = \omega^i e_i, \quad de_i = \omega_i^j e_j + \omega_i^{n+1} e_{n+1}, \quad de_{n+1} = \omega_{n+1}^i e_i;$$

the summation convention is used throughout. Of course,

$$(11)\qquad \omega_i^j + \omega_j^i = 0\,, \quad \omega_{n+1}^i + \omega_i^{n+1} = 0\,,$$

$$(12)\qquad d\omega^i = \omega^j \wedge \omega_j^i\,, \quad 0 = \omega^i \wedge \omega_i^{n+1}\,,$$

$$d\omega_i^j = \omega_i^k \wedge \omega_k^j + \omega_i^{n+1} \wedge \omega_{n+1}^j\,, \quad d\omega_i^{n+1} = \omega_i^j \wedge \omega_j^{n+1}\,.$$

On the domain $\mu_t(U)$ of the isometric surface $M_t$, we may introduce frames $\{e_1(t), \ldots \ldots, e_{n+1}(t)\}$ such that

$$(13)\qquad dM_t = \omega^i\, e_i(t)\,, \quad de_i(t) = \tau_i^j\, e_j(t) + \tau_i^{n+1}\, e_{n+1}(t)\,, \quad de_{n+1}(t) = \tau_{n+1}^i\, e_i(t)$$

with

$$(14)\qquad \tau_i^j = \omega_i^j + \sum_{\alpha=1}^{\infty} {}^{(\alpha)}\varphi_i^j \,.\, t^\alpha\,, \quad \tau_i^{n+1} = \omega_i^{n+1} + \sum_{\alpha=1}^{\infty} {}^{(\alpha)}\varphi_i^{n+1} \,.\, t^\alpha\,;$$

of course,

$$(15)\qquad \tau_i^j + \tau_j^i = 0\,, \quad \tau_i^{n+1} + \tau_{n+1}^i = 0\,,$$

$$(16)\qquad d\omega^i = \omega^j \wedge \tau_j^i\,, \quad 0 = \omega^i \wedge \tau_i^{n+1}\,,$$

$$d\tau_i^j = \tau_i^k \wedge \tau_k^j + \tau_i^{n+1} \wedge \tau_{n+1}^j\,, \quad d\tau_i^{n+1} = \tau_i^j \wedge \tau_j^{n+1}\,.$$

From $(16_1)$, we get

$$(17)\qquad \tau_i^j = \omega_i^j\,,$$

this being, essentially, the affirmation of the Gauss Theorem. The equations $(16_{2,3,4})$ reduce to

$$(18)\qquad 0 = \omega^i \wedge {}^{(\alpha)}\varphi_i^{n+1}\,, \quad d\,{}^{(\alpha)}\varphi_i^{n+1} = \omega_i^j \wedge {}^{(\alpha)}\varphi_j^{n+1}\,,$$

$$0 = \omega_i^{n+1} \wedge {}^{(\alpha)}\varphi_j^{n+1} + {}^{(\alpha)}\varphi_i^{n+1} \wedge \omega_j^{n+1} + \sum_{\beta=1}^{\alpha-1} {}^{(\beta)}\varphi_i^{n+1} \wedge {}^{(\alpha-\beta)}\varphi_j^{n+1}\,.$$

Let us consider the system

$$(19)\qquad 0 = \omega^i \wedge \varkappa_i\,, \quad d\varkappa_i = \omega_i^j \wedge \varkappa_j\,, \quad \omega_i^{n+1} \wedge \varkappa_j + \varkappa_i \wedge \omega_j^{n+1} = \Omega_{ij}\,,$$

$\Omega_{ij}$ being exterior 2-forms satisfying $\Omega_{ij} + \Omega_{ji} = 0$. The exterior differentiation of (19) yields

$$(20)\qquad \omega_{n+1}^j \wedge \Omega_{ij} = 0\,, \quad d\Omega_{ij} = \omega_i^k \wedge \Omega_{kj} + \omega_j^k \wedge \Omega_{ik}\,;$$

(20) are thus the conditions for the local existence of $\varkappa_i$'s satisfying (19).

From $(19_1)$, we get the existence of a tensor $a_{ij}$ such that

$$(21)\qquad \varkappa_i = a_{ij}\omega^j\,, \quad a_{ij} = a_{ji}\,.$$

The equation $(19_2)$ yields

(22) $$(da_{ij} - a_{ik}\omega_j^k - a_{kj}\omega_i^k) \wedge \omega^j = 0 .$$

Because of the well known relation $da_{ij} - a_{ik}\omega_j^k - a_{kj}\omega_i^k = \nabla_k a_{ij}\omega^k$, (22) reduces to

(23) $$\nabla_k a_{ij} = \nabla_j a_{ik} .$$

Write

(24) $$\omega_i^{n+1} = h_{ij}\omega^j , \quad h_{ij} = h_{ji} ,$$

$h_{ij}$ being the second fundamental tensor of $M$. The forms $\Omega_{ij}$ defined by $(19_3)$ are

(25) $$\Omega_{ij} = (a_{ip}h_{jq} - a_{jp}h_{iq})\,\omega^p \wedge \omega^q .$$

The forms $\varkappa_i$ satisfying $(19_{1,2})$, the form $(19_3)$ satisfies $(20_2)$, and this proves Proposition 1. Write $\Omega_{ij} = b_{ijpq}\omega^p \wedge \omega^q$; it is easy to see that the conditions $(20_1)$ and $(20_2)$ are equivalent to $h_2(b_{ijpq}) = 0$ and $d_2(b_{ijpq}) = 0$ resp. This proves Proposition 2. To prove our Theorem, it is obviously sufficient to prove the following assertion: Let the forms ${}^{(1)}\varphi_i^{n+1}, \ldots, {}^{(a)}\varphi_i^{n+1}$ satisfy (18) for $\alpha = 1, \ldots, a$, then the forms

(26) $$\Omega_{ij} = \sum_{\beta=1}^{a} {}^{(\beta)}\varphi_i^{n+1} \wedge {}^{(a-\beta+1)}\varphi_j^{n+1}$$

satisfy (20). But this is to be seen by a direct calculation.

## II.

In the second part, I propose to work out the formal aspects of an apparatus leading to the solutions of problems analoguous to the problem treated above.

Be given a Lie algebra $G$, its subalgebra $H$ and suppose the existence of a subalgebra $K \subset G$ such that

(27) $$G = H + K , \quad [H, K] \subset K .$$

Further, be given a differentiable manifold $M$ and a $G$-valued 1-form $\varphi$ over $M$ satisfying

(28) $$d\varphi(X, Y) = -[\varphi(X), \varphi(Y)]$$

for any two tangent vector fields $X, Y$ on $M$.

**Definition.** A *formal H-deformation* of the form $\varphi$ is a formal series

(29) $$\omega = \varphi + \omega_1 t + \omega_2 t^2 + \ldots$$

with $\omega_\alpha$ $H$-valued 1-forms on $M$ which formally satisfies the equation of the type (28), i.e.,

$$(30) \quad d\omega_\alpha(X, Y) = -[\varphi(X), \omega_\alpha(Y)] - [\omega_\alpha(X), \varphi(Y)] - \sum_{\beta=1}^{\alpha-1} [\omega_\beta(X), \omega_{\alpha-\beta}(Y)] \quad \text{for} \quad \alpha = 1, 2, \ldots$$

The $H$-valued 1-form $\omega_1$ on $M$ is called an *infinitesimal* H-*deformation* of $\varphi$ if

$$(31) \quad d\omega_1(X, Y) = -[\varphi(X), \omega_1(Y)] - [\omega_1(X), \varphi(Y)] .$$

Our problem is to exhibit conditions under which each infinitesimal $H$-deformation $\omega_1$ of $\varphi$ may be extended to a formal $H$-deformation (29).

Let us write

$$(32) \quad \varphi = \varphi^H + \varphi^K ,$$

$\varphi^H$ being an $H$-valued and $\varphi^K$ a $K$-valued form resp. From (28) and (30), we get

$$(33) \quad \begin{aligned} d\varphi^H(X, Y) &= -[\varphi^H(X), \varphi^H(Y)] , \\ d\varphi^K(X, Y) &= -[\varphi^H(X), \varphi^K(Y)] - [\varphi^K(X), \varphi^H(Y)] - [\varphi^K(X), \varphi^K(Y)] ; \end{aligned}$$

$$(34) \quad \begin{aligned} d\omega_\alpha(X. Y) &= -[\varphi^H(X), \omega_\alpha(Y)] - [\omega_\alpha(X), \varphi^H(Y)] - \sum_{\beta=1}^{\alpha-1} [\omega_\beta(X), \omega_{\alpha-\beta}(Y)] , \\ 0 &= [\varphi^K(X), \omega_\alpha(Y)] + [\omega_\alpha(X), \varphi^K(Y)] . \end{aligned}$$

Notice that the exterior differential $d\tau$ of a $G$-valued $p$-form $\tau$ is to be defined by the formula

$$(35) \quad \begin{aligned} d\tau(X_1, \ldots, X_{p+1}) &= \sum_i (-1)^{i+1} X_i \tau(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) + \\ &+ \sum_{i<j} (-1)^{i+j} \tau([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}) . \end{aligned}$$

**Lemma.** *Let $\varrho, \sigma$ be G-valued 1-forms on M, and let the G-valued 2-form R be defined by*

$$(36) \quad R(X, Y) = [\varrho(X), \sigma(Y)] + [\sigma(X), \varrho(Y)] .$$

*Then*

$$(37) \quad \begin{aligned} dR(X, Y, Z) &= [d\varrho(X, Y), \sigma(Z)] - [d\varrho(X, Z), \sigma(Y)] + [d\varrho(Y, Z), \sigma(X)] - \\ &- [\varrho(X), d\sigma(Y, Z)] + [\varrho(Y), d\sigma(X, Z)] - [\varrho(Z), d\sigma(X, Y)] . \end{aligned}$$

Proof follows by a direct calculation.

**Proposition 3.** *Let $\Omega$, $\Psi$ be $H$-valued 2-forms on $M$. The integrability conditions of the system*

$$(38)\qquad \begin{aligned} d\omega(X, Y) &= -[\varphi^H(X), \omega(Y)] - [\omega(X), \varphi^H(Y)] + \Omega(X, Y), \\ \Psi(X, Y) &= [\varphi^K(X), \omega(Y)] + [\omega(X), \varphi^K(Y)] \end{aligned}$$

*for the $H$-valued 1-form $\omega$ are*

$$(39)\qquad \begin{aligned} d\Omega(X, Y, Z) = &- [\varphi^H(X), \Omega(Y, Z)] + [\varphi^H(Y), \Omega(X, Z)] - \\ &- [\varphi^H(Z), \Omega(X, Y)], \\ d\Psi(X, Y, Z) = &- [\varphi^K(X), \Omega(Y, Z)] + [\varphi^K(Y), \Omega(X, Z)] - \\ &- [\varphi^K(Z), \Omega(X, Y)] - [\varphi(X), \Psi(Y, Z)] + \\ &+ [\varphi(Y), \Psi(X, Z)] - [\varphi(Z), \Psi(X, Y)]. \end{aligned}$$

Proof. By the exterior differentiation of $(12_1)$, we get

$$\begin{aligned} d\Omega(X, Y, Z) = &[d\varphi^H(X, Y), \omega(Z)] - [d\varphi^H(X, Z), \omega(Y)] + [d\varphi^H(Y, Z), \omega(X)] - \\ &- [\varphi^H(X), d\omega(Y, Z)] + [\varphi^H(Y), d\omega(X, Z)] - [\varphi^H(Z), d\omega(X, Y)] = \\ = &- [[\varphi^H(X), \varphi^H(Y)], \omega(Z)] + [[\varphi^H(X), \varphi^H(Z)], \omega(Y)] - \\ &- [[\varphi^H(Y), \varphi^H(Z)], \omega(X)] + [\varphi^H(X), [\varphi^H(Y), \omega(Z)]] + \\ &+ [\varphi^H(X), [\omega(Y), \varphi^H(Z)]] - [\varphi^H(X), \Omega(Y, Z)] - \\ &- [\varphi^H(Y), [\varphi^H(X), \omega(Z)]] - [\varphi^H(Y), [\omega(X), \varphi^H(Z)]] + \\ &+ [\varphi^H(Y), \Omega(X, Z)] + [\varphi^H(Z), [\varphi^H(X), \omega(Y)]] + \\ &+ [\varphi^H(Z), [\omega(X), \varphi^H(Y)]] - [\varphi^H(Z), \Omega(X, Y)] = \\ = &- [\varphi^H(X), \Omega(Y, Z)] + [\varphi^H(Y), \Omega(X, Z)] - [\varphi^H(Z), \Omega(X, Y)]. \end{aligned}$$

Further, from $(12_2)$

$$\begin{aligned} d\Psi(X, Y, Z) = &[d\varphi^K(X, Y), \omega(Z)] - [d\varphi^K(X, Z), \omega(Y)] + [d\varphi^K(Y, Z), \omega(X)] - \\ &- [\varphi^K(X), d\omega(Y, Z)] + [\varphi^K(Y), d\omega(X, Z)] - [\varphi^K(Z), d\omega(X, Y)] = \\ = &- [[\varphi^H(X), \varphi^K(Y)], \omega(Z)] - [[\varphi^K(X), \varphi^H(Y)], \omega(Z)] - \\ &- [[\varphi^K(X), \varphi^K(Y)], \omega(Z)] + [[\varphi^H(X), \varphi^K(Z)], \omega(Y)] + \\ &+ [[\varphi^K(X), \varphi^H(Z)], \omega(Y)] + [[\varphi^K(X), \varphi^K(Z)], \omega(Y)] - \\ &- [[\varphi^H(Y), \varphi^K(Z)], \omega(X)] - [[\varphi^K(Y), \varphi^H(Z)], \omega(X)] - \\ &- [[\varphi^K(Y), \varphi^K(Z)], \omega(X)] + [\varphi^K(X), [\varphi^H(Y), \omega(Z)]] + \end{aligned}$$

$$
\begin{aligned}
&+ [\varphi^K(X), [\omega(Y), \varphi^H(Z)]] - [\varphi^K(X), \Omega(Y, Z)] - \\
&- [\varphi^K(Y), [\varphi^H(X), \omega(Z)]] - [\varphi^K(Y), [\omega(X), \varphi^H(Z)]] + \\
&+ [\varphi^K(Y), \Omega(X, Z)] + [\varphi^K(Z), [\varphi^H(X), \omega(Y)]] + \\
&+ [\varphi^K(Z), [\omega(X), \varphi^H(Z)]] - [\varphi^K(Z), \Omega(X, Y)] = \\
= &- [\varphi^K(X), \Omega(Y, Z)] + [\varphi^K(Y), \Omega(X, Z)] - [\varphi^K(Z), \Omega(X, Y)] - \\
&- [\varphi^H(Z), [\omega(X), \varphi^K(Y)]] - [\varphi^H(Y), [\varphi^K(Z), \omega(X)]] + \\
&+ [[\varphi^K(Z), \omega(X)], \varphi^K(Y)] + [[\omega(X), \varphi^K(Y)], \varphi^K(Z)] - \\
&- [\varphi^H(X), [\omega(Y), \varphi^K(Z)]] - [\varphi^H(Z), [\varphi^K(X), \omega(Y)]] - \\
&- [[\varphi^K(Z), \omega(Y)], \varphi^K(X)] - [[\omega(Y), \varphi^K(X)], \varphi^K(Z)] + \\
&+ [\varphi^H(X), [\omega(Z), \varphi^K(Y)]] - [\varphi^H(Y), [\omega(Z), \varphi^K(X)]] + \\
&+ [[\varphi^K(Y), \omega(Z)], \varphi^K(X)] + [[\omega(Z), \varphi^K(X)], \varphi^K(Y)] = \\
= &- [\varphi^K(X), \Omega(Y, Z)] + [\varphi^K(Y), \Omega(X, Z)] - [\varphi^K(Z), \Omega(X, Y)] - \\
&- [\varphi^H(Z), \Psi(X, Y)] + [\varphi^H(Y), \Psi(X, Z)] - [\varphi^H(X), \Psi(Y, Z)] + \\
&+ [\Psi(Y, Z), \varphi^K(X)] - [\Psi(X, Z), \varphi^K(Y)] + [\Psi(Y, Z), \varphi^K(X)]
\end{aligned}
$$

and $(39_2)$ follows.

**Proposition 4.** *On M, be given H-valued 1-forms* $\omega_1, \ldots, \omega_p$ *satisfying*

(40)
$$
\begin{aligned}
d\omega_\alpha(X, Y) &= - [\varphi^H(X), \omega_\alpha(Y)] - [\omega_\alpha(X), \varphi^H(Y)] - \sum_{\beta=1}^{\alpha-1} [\omega_\beta(X), \omega_{\alpha-\beta}(Y)], \\
0 &= [\varphi^K(X), \omega_\alpha(Y)] + [\omega_\alpha(X), \varphi^K(Y)] \\
&\text{for} \quad \alpha = 1, \ldots, p .
\end{aligned}
$$

*The H-valued 2-form* $\Omega_{p+1}$ *be defined by*

(41)
$$\Omega_{p+1}(X, Y) = - \sum_{\beta=1}^{p} [\omega_\beta(X), \omega_{p-\beta+1}(Y)] .$$

*Then*

(42)
$$
\begin{aligned}
&d\Omega_{p+1}(X, Y, Z) = \\
&= - [\varphi^H(X), \Omega_{p+1}(Y, Z)] + [\varphi^H(Y), \Omega_{p+1}(X, Z)] - [\varphi^H(Z), \Omega_{p+1}(X, Y)], \\
&0 = [\varphi^K(X), \Omega_{p+1}(Y, Z)] - [\varphi^K(Y), \Omega_{p+1}(X, Z)] + [\varphi^K(Z), \Omega_{p+1}(X, Y)] .
\end{aligned}
$$

Proof. Let us prove (42) for $p = 1$, the general proof being then almost obvious. Suppose that the 1-form $\omega_1$ satisfies

$$d\omega_1(X, Y) = - [\varphi^H(X), \omega_1(Y)] - [\omega_1(X), \varphi^H(Y)],$$
$$0 = [\varphi^K(X), \omega_1(Y)] + [\omega_1(X), \varphi^K(Y)]$$

and the 2-form $\Omega_2$ is given by

$$\Omega_2(X, Y) = - [\omega_1(X), \omega_1(Y)].$$

Then

$$[\varphi^K(X), \Omega_2(Y, Z)] - [\varphi^K(Y), \Omega_2(X, Z)] + [\varphi^K(Z), \Omega_2(X, Y)] =$$
$$= [[\omega_1(Y), \omega_1(Z)], \varphi^K(X)] - [[\omega_1(X), \omega_1(Z)], \varphi^K(Y)] + [[\omega_1(X), \omega_1(Y)], \varphi^K(Z)] =$$
$$= - [[\omega_1(Z), \varphi^K(X)], \omega_1(Y)] - [[\varphi^K(X), \omega_1(Y)], \omega_1(Z)] +$$
$$+ [[\omega_1(Z), \varphi^K(Y)], \omega_1(X)] + [[\varphi^K(Y), \omega_1(X)], \omega_1(Z)] -$$
$$- [[\omega_1(Y), \varphi^K(Z)], \omega_1(X)] - [[\varphi^K(Z), \omega_1(X)], \omega_1(Y)] = 0.$$

Further,

$$d\Omega_2(X, Y, Z) = [\omega_1(X), d\omega_1(Y, Z)] - [\omega_1(Y), d\omega_1(X, Z)] + [\omega_1(Z), d\omega_1(X, Y)],$$

i.e.,

$$d\Omega_2(X, Y, Z) + [\varphi^H(X), \Omega_2(Y, Z)] - [\varphi^H(Y), \Omega_2(X, Z)] + [\varphi^H(Z), \Omega_2(X, Y)] =$$
$$= - [\omega_1(X), [\varphi^H(Y), \omega_1(Z)]] - [\omega_1(X), [\omega_1(Y), \varphi^H(Z)]] +$$
$$+ [\omega_1(Y), [\varphi^H(X), \omega_1(Z)]] + [\omega_1(Y), [\omega_1(X), \varphi^H(Z)]] -$$
$$- [\omega_1(Z), [\varphi^H(X), \omega_1(Y)]] - [\omega_1(Z), [\omega_1(X), \varphi^H(Y)]] -$$
$$- [\varphi^H(X), [\omega_1(Y), \omega_1(Z)]] + [\varphi^H(Y), [\omega_1(X), \omega_1(Z)]] -$$
$$- [\varphi^H(Z), [\omega_1(X), \omega_1(Y)]] = 0.$$

From the preceding two propositions, we get

**Theorem 2.** *Let $\omega_1$ be an infinitesimal H-deformation of $\varphi$, and let $m \in M$ be a given point. Then there are neighborhoods $M \supset U_2 \supset U_3 \supset \ldots$ of $m$ and H-valued 1-forms $\omega_2, \omega_3, \ldots, \omega_\alpha$ being defined in $U_\alpha$, such that $\varphi + \omega_1 t + \omega_2 t^2 + \ldots$ is a formal H-deformation of $\varphi$ in $\bigcap_{\alpha=2}^{\infty} U_\alpha$.*

Let us turn our attention to the global problem.

**Definition.** Denote by $\mathscr{A}^p$ ($p = 0, 1, \ldots$) the sheaf of $H$-valued $p$-forms $\tau$ on $M$ having the following properties:

(i) we have

$$\sum_{i=1}^{p+1}(-1)^{i+1}\,[\varphi^K(X_i), \tau(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})] = 0\,, \tag{43}$$

(ii) the form

$$\begin{aligned}\delta\tau(X_1, \ldots, X_{p+1}) &= \\ = \mathrm{d}\tau(X_1, \ldots, X_{p+1}) &+ \sum_{i=1}^{p+1}(-1)^{i+1}\,[\varphi^H(X_i), \tau(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})]\end{aligned} \tag{44}$$

is $H$-valued.

**Proposition 5.** *If* $\tau \in \mathscr{A}^p$, *then* $\delta\tau \in \mathscr{A}^{p+1}$. *Further,* $\delta^2 = 0$.

Proof. Let us restrict ourselves to the case $p = 0$, the general case is to be treated in a similar manner. Thus, let $\tau \in \mathscr{A}^0$, i.e., let the form

$$\delta\tau(X) = X\tau + [\varphi^H(X), \tau] \tag{45}$$

be $H$-valued and satisfy

$$[\varphi^K(X), \tau] = 0\,. \tag{46}$$

From (46),

$$[Y\varphi^K(X), \tau] + [\varphi^K(X), Y\tau] = 0$$

for any vector fields $X$, $Y$ on $M$, and we get

$$[\mathrm{d}\varphi^K(X, Y), \tau] = -\,[\varphi^K(Y), X\tau] + [\varphi^K(X), Y\tau]\,.$$

Thus

$$\begin{aligned}&[\varphi^K(X), \delta\tau(Y)] - [\varphi^K(Y), \delta\tau(X)] = \\ &\quad = [\varphi^K(X), Y\tau + [\varphi^H(Y), \tau]] - [\varphi^K(Y), X\tau + [\varphi^H(X), \tau]] = \\ &\quad = -\,[[\varphi^H(X), \varphi^K(Y)], \tau] - [[\varphi^K(X), \varphi^H(Y)], \tau] - [[\varphi^K(X), \varphi^K(Y)], \tau] + \\ &\qquad + [\varphi^K(X), [\varphi^H(Y), \tau]] - [\varphi^K(Y), [\varphi^H(X), \tau]] = \\ &\quad = [[\varphi^K(Y), \tau], \varphi^K(X)] + [[\tau, \varphi^K(X)], \varphi^K(Y)] + [[\tau, \varphi^K(X)], \varphi^H(Y)] + \\ &\qquad + [[\varphi^K(Y), \tau], \varphi^H(X)] = 0\,.\end{aligned}$$

Let us write $\delta\tau = d\tau + \Omega$ with $\Omega(X) = [\varphi^H(X), \tau]$. Then

$$\begin{aligned}\delta^2\tau(X, Y) &= d\Omega(X, Y) + [\varphi^H(X), \delta\tau(Y)] - [\varphi^H(Y), \delta\tau(X)] = \\ &= [X\varphi^H(Y), \tau] + [\varphi^H(Y), X\tau] - [Y\varphi^H(X), \tau] - [\varphi^H(X), Y\tau] - \\ &\quad - [\varphi^H([X, Y]), \tau] + [\varphi^H(X), Y\tau] + [\varphi^H(X), [\varphi^H(Y), \tau]] - \\ &\quad - [\varphi^H(Y), X\tau] - [\varphi^H(Y), [\varphi^H(X), \tau]] = \\ &= [d\varphi^H(X, Y), \tau] + [[\varphi^H(X), \varphi^H(Y)], \tau] = 0\,.\end{aligned}$$

**Proposition 6.** (Poincaré lemma.) *Let $\sigma \in \mathscr{A}^p$ be defined in a neighborhood $U \subset M$ of the point $m \in M$, and let $\delta\sigma = 0$. Then there is a neighborhood $U_1 \subset U$ of $m$ and a form $\tau \in \mathscr{A}^{p-1}$ defined in $U_1$ such that $\delta\tau = \sigma$.*

Proof. For $p = 2$, see Proposition 3. Let us restrict ourselves to the case $p = 1$. Let $\sigma \in \mathscr{A}^1$ be an $H$-valued 1-form on $U$, and let $\delta\sigma = 0$, i.e.,

$$\begin{aligned}&[\varphi^K(X), \sigma(Y)] - [\varphi^K(Y), \sigma(X)] = 0\,, \\ &d\sigma(X, Y) + [\varphi^H(X), \sigma(Y)] - [\varphi^H(Y), \sigma(X)] = 0\end{aligned} \tag{47}$$

We have to prove that the integrability conditions of the system

$$d\tau(X) + [\varphi^H(X), \tau] = \sigma(X)\,, \quad [\varphi^K(X), \tau] = 0 \tag{48}$$

for the $H$-valued 0-form $\tau$ are satisfied. From $(48_1)$, we get

$$\begin{aligned}d\sigma(X, Y) &= X\,\sigma(Y) - Y\sigma(X) - \sigma([XY]) = \\ &= XY\tau + [X\,\varphi^H(Y), \tau] + [\varphi^H(Y), X\tau] - YX\tau - [Y\varphi^H(X), \tau] - \\ &\quad - [\varphi^H(X), Y\tau] - [X, Y]\,\tau - [\varphi^H([X, Y]), \tau] = \\ &= [d\varphi^H(X, Y), \tau] + [\varphi^H(Y), \sigma(X) - [\varphi^H(X), \tau]] - \\ &\quad - [\varphi^H(X), \sigma(Y) - [\varphi^H(Y), \tau]] = [\varphi^H(Y), \sigma(X)] - [\varphi^H(X), \sigma(Y)]\,,\end{aligned}$$

i.e., $(46_2)$. Let us write $\varrho(X) = [\varphi^K(X), \tau]$. Then

$$\begin{aligned}d\varrho(X, Y) &= [d\varphi^K(X, Y), \tau] + [\varphi^K(Y), \sigma(X) - [\varphi^H(X), \tau]] - \\ &\quad - [\varphi^K(X), \sigma(Y) - [\varphi^H(Y), \tau]] = \\ &= [\varphi^K(Y), \sigma(X)] - [\varphi^K(X), \sigma(Y)] - [[\varphi^H(X), \varphi^K(Y)], \tau] - \\ &\quad - [[\varphi^K(X), \varphi^H(Y)], \tau] - [[\varphi^K(X), \varphi^K(Y)], \tau] - [[\tau, \varphi^K(Y)], \varphi^H(X)] - \\ &\quad - [[\varphi^K(Y), \varphi^H(X)], \tau] + [[\tau, \varphi^K(X)], \varphi^H(Y)] + [[\varphi^K(X), \varphi^H(Y)], \tau] =\end{aligned}$$

$$= [\varphi^K(Y), \sigma(X)] - [\varphi^K(X), \sigma(Y)] - [[\tau, \varphi^K(Y)], \varphi^H(X)] +$$
$$+ [[\tau, \varphi^K(X)], \varphi^H(Y)] + [[\varphi^K(Y), \tau], \varphi^K(X)] + [[\tau, \varphi^K(X)], \varphi^K(Y)],$$

and $d\varrho(X, Y) = 0$ follows from $(47_2)$ and $(46_1)$.

**Theorem 3.** *Let $\mathscr{S} \subset \mathscr{A}^0$ be the sheaf of solutions of the system*

$$(49) \qquad \delta s(X) = Xs + [\varphi^H(X), s] = 0, \quad [\varphi^K(X), s] = 0.$$

*Then*

$$(50) \qquad 0 \to \mathscr{S} \to \mathscr{A}^0 \xrightarrow{\delta} \mathscr{A}^1 \xrightarrow{\delta} \mathscr{A}^2 \to \ldots$$

*is the resolution of $\mathscr{S}$.*

Proof follows from Propositions 5 and 6.

Denote by $\Gamma(\mathscr{A}^p, M)$ the $\mathscr{R}$-module of the sections of $\mathscr{A}^p$ over $M$, and introduce the following notation:

$$(51) \qquad \mathscr{B}^p = \{\delta\tau;\ \tau \in \Gamma(\mathscr{A}^{p-1}, M)\} \quad \text{for} \quad p \geqq 1,$$
$$\mathscr{Z}^p = \{\tau';\ \tau' \in \Gamma(\mathscr{A}^p, M),\ \delta\tau' = 0\} \quad \text{for} \quad p \geqq 0;$$

$$(52) \qquad \mathscr{H}^p = \mathscr{Z}^p/\mathscr{B}^p \quad \text{for} \quad p \geqq 1,$$
$$\mathscr{H}^0 = \mathscr{Z}^0.$$

**Theorem 4.** *Let $\mathscr{H}^2 = 0$. Then to each infinitesimal H-deformation $\omega_1$ of $\varphi$ there is a formal H-deformation $\varphi + t\omega_1 + t^2\omega_2 + \ldots$*

Proof. Suppose the existence of forms $\omega_1, \ldots, \omega_p \in \Gamma(\mathscr{A}^1, M)$ satisfying (40); we have to prove the existence of a form $\omega_{p+1} \in \Gamma(\mathscr{A}^1, M)$ satisfying $\delta\omega_{p+1} = \Omega_{p+1}$, $\Omega_{p+1}$ being given by (41). Proposition 4 says that $\Omega_{p+1} \in \Gamma(\mathscr{A}^2, M)$ and $\delta\Omega_{p+2} = 0$, i.e., $\Omega_{p+1} \in \mathscr{Z}^2$. From $\mathscr{H}^2 = 0$, we get $\Omega_{p+1} \in \mathscr{B}^2$, and the existence of a solution of $\delta\omega_{p+1} = \Omega_{p+1}$ follows.

## III.

It is almost obvious that the suppositions (27) and $[K, K] \subset K$ are superfluous for the proof of Theorem 3. Nevertheless, I have technical difficulties in proving the general result; in this section, I intend to sketch an approach to such a proof. Perhaps new more simple methods are to be developped.

Be given a Lie algebra $G$ and its subalgebra $H$. Choose a complement $K$ of $H$ in $G$, i.e., let $G = H + K$ as vector spaces. Each vector $x \in G$ may now be written in the form

(53) $$x = x^H + x^K\,; \quad x^H \in H\,, \quad x^K \in K\,.$$

Introduce the bilinear mappings

(54) $$A^H : H \times K \to H\,, \quad A^K = H \times K \to K\,, \quad B^H : K \times K \to H\,,$$
$$B^K : K \times K \to K$$

by

(55) $$A^H(x^H, y^K) = [x^H, y^K]^H\,, \quad A^K(x^H, y^K) = [x^H, y^K]^K\,,$$
$$B^H(x^K, y^K) = [x^K, y^K]^H\,, \quad B^K(x^K, y^K) = [x^K, y^K]^K\,;$$

of course, the mapping $B^K$ is skewsymmetric. From the Jacobi identity in $G$, we get: Write

(56) $$\begin{aligned} R^H(x, y, z) = {} & [A^H(x^H, y^K), z^H] - [A^H(y^H, x^K), z^H] + [B^H(x^K, y^K), z^H] + \\ & + A^H([x^H, y^H], z^K) + A^H(A^H(x^H, y^K), z^K) - \\ & - A^H(A^H(y^H, x^K), z^K) + A^H(B^H(x^K, y^K), z^K) - \\ & - A^H(z^H, A^K(x^H, y^K)) + A^H(z^H, A^K(y^H, x^K)) - \\ & - A^H(z^H, B^K(x^K, y^K)) + B^H(A^K(x^H, y^K), z^K) - \\ & - B^H(A^K(y^H, x^K), z^K) + B^H(B^K(x^K, y^K), z^K)\,, \end{aligned}$$

$$\begin{aligned} R^K(x, y, z) = {} & A^K([x^H, y^H], z^K) + A^K(A^H(x^H, y^K), z^K) - A^K(A^H(y^H, x^K), z^K) + \\ & + A^K(B^H(x^K, y^K), z^K) - A^K(z^H, A^K(x^H, y^K)) + \\ & + A^K(z^H, A^K(y^H, x^K)) - A^K(z^H, B^K(x^K, y^K)) + \\ & + B^K(A^K(x^H, y^K), z^K) - B^K(A^K(y^H\; x^K), z^K) + \\ & + B^K(B^K(x^K, y^K), z^K)\,, \end{aligned}$$

then

(57) $$R^H(x, y, z) + R^H(y, z, x) + R^H(z, x, y) = 0\,,$$
$$R^K(x, y, z) + R^K(y, z, x) + R^K(z, x, y) = 0\,.$$

The equation (28) decomposes into

(58) $$\begin{aligned} \mathrm{d}\varphi^H(X, Y) = {} & - [\varphi^H(X), \varphi^H(Y)] - A^H(\varphi^H(X), \varphi^K(Y)) + \\ & + A^H(\varphi^H(Y), \varphi^K(X)) - B^H(\varphi^K(X), \varphi^K(Y))\,, \end{aligned}$$
$$\mathrm{d}\varphi^K(X, Y) = - A^K(\varphi^H(X), \varphi^K(Y)) + A^K(\varphi^H(Y), \varphi^K(X)) - B^K(\varphi^K(X), \varphi^K(Y)).$$

As above, denote by $\mathscr{A}^p$ the sheaf of $H$-valued $p$-forms $\omega$ on $M$ such that $\delta\omega$ is $H$-valued as well, $\delta\omega$ being defined by

$$(59) \qquad \delta\omega(X_1, \ldots, X_{p+1}) = \\ = \mathrm{d}\omega(X_1, \ldots, X_{p+1}) + \Sigma(-1)^{i+1} [\varphi(X_i), \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+})] .$$

For the $H$-valued form $\omega$, it means

$$(60) \qquad \delta\omega(X_1, \ldots, X_{p+1}) = \\ = \mathrm{d}\omega(X_1, \ldots, X_{p+1}) + \Sigma(-1)^{i+1} [\varphi^H(X^i), \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})] - \\ - \Sigma(-1)^{i+1} A^H(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}), \varphi^K(X_i)) ,$$

$$(61) \qquad \Sigma(-1)^{i+1} A^K(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}), \varphi^K(X_i)) = 0 .$$

My claim is that Theorems 3 and 4 remain valid. Let us restrict ourselves just to the proof of the Poincaré lemma on the level $p = 1$. Thus, be given (on a neighborhood $U$ of a point $m \in M$) an $H$-valued 1-form $\omega$ satisfying $\delta\omega = 0$, we have to prove the existence of a neighborhood $U_1 \subset U$ of $m$ and a mapping $v : U_1 \to H$ such that $\delta v = \omega$. Now,

$$(62) \qquad \delta v(X) = \mathrm{d}v(X) + [\varphi^H(X) + \varphi^K(X), v] = \\ = \mathrm{d}v(X) + [\varphi^H(X), v] - A^H(v, \varphi^K(X)) - A^K(v, \varphi^K(X)) .$$

Further,

$$(63) \quad \delta\omega(X, Y) = \mathrm{d}\omega(X, Y) + [\varphi^H(X) + \varphi^K(X), \omega(Y)] + [\omega(X), \varphi^H(Y) + \varphi^K(Y)] = \\ = \mathrm{d}\omega(X, Y) + [\varphi^H(X), \omega(Y)] - A^H(\omega(Y), \varphi^K(X)) - \\ - A^K(\omega(Y), \varphi^K(X)) + [\omega(X), \varphi^H(Y)] + A^H(\omega(X), \varphi^K(Y)) + \\ + A^K(\omega(X), \varphi^K(Y)) .$$

Thus our problem may be formulated as follows: Be given an $H$-valued 1-form $\omega$ satisfying

$$(64) \qquad \mathrm{d}\omega(X, Y) + [\varphi^H(X), \omega(Y)] - A^H(\omega(Y), \varphi^K(X)) + \\ + [\omega(X), \varphi^H(Y)] + A^H(\omega(X), \varphi^K(Y)) = 0 ,$$

$$(65) \qquad A^K(\omega(Y), \varphi^K(X)) - A^K(\omega(X), \varphi^K(Y)) = 0 ;$$

we look for the existence of a mapping $v : U_1 \to H$ such that

$$(66) \qquad \mathrm{d}v(X) + [\varphi^H(X), v] - A^H(v, \varphi^K(X)) = \omega(X) ,$$

$$(67) \qquad A^K(v, \varphi^K(X)) = 0 .$$

To assure the existence of such a mapping, we have to show that the integrability conditions of (66) + (67) are consequences of (64)–(67).

Write

$$(68)\quad \Phi(X) = A^K(v, \varphi^K(X)), \quad \Psi(X) = Xv + [\varphi^H(X), v] - A^H(v, \varphi^K(X)) - \omega(X).$$

Then

$$\begin{aligned} d\Phi(X, Y) &= A^K(Xv, \varphi^K(Y)) - A^K(Yv, \varphi^K(X)) + A^K(v, d\varphi^K(X, Y)) = \\ &= A^K(\Psi(X), \varphi^K(Y)) - A^K(\Psi(Y), \varphi^K(X)) + A^K(\omega(X), \varphi^K(Y)) - \\ &\quad - A^K(\omega(Y), \varphi^K(X)) + \Phi_1(X, Y) \end{aligned}$$

with

$$\begin{aligned} \Phi_1(X, Y) = &- A^K([\varphi^K(X), v], \varphi^K(Y)) + A^K(A^H(v, \varphi^K(X)), \varphi^K(Y)) + \\ &+ A^K([\varphi^H(Y), v], \varphi^K(X)) - A^K(A^H(v, \varphi^K(Y)), \varphi^K(X)) - \\ &- A^K(v, A^K(\varphi^H(X), \varphi^K(Y))) + A^K(v, A^K(\varphi^H(Y), \varphi^K(X))) - \\ &- A^K(v, B^K(\varphi^K(X), \varphi^K(Y))). \end{aligned}$$

From $(57_2)$ for $x = \varphi^H(X) + \varphi^K(X)$, $y = \varphi^H(Y) + \varphi^K(Y)$, $z = v$, we get

$$\begin{aligned} \Phi_1(X, Y) = &B^K(A^K(v, \varphi^K(Y)), \varphi^K(X)) - B^K(A^K(v, \varphi^K(X)), \varphi^K(Y)) + \\ &+ A^K(\varphi^H(Y), A^K(v, \varphi^K(X))), \end{aligned}$$

and $d\Phi(X, Y) = 0$ is the consequence of $\Phi(X) = \Psi(X) = 0$, (65) and (67). Further,

$$\begin{aligned} d\Psi(X, Y) = &[d\varphi^H(X, Y), v] + [\varphi^H(Y), Xv] - [\varphi^H(X), Yv] - A^H(v, d\varphi^K(X, Y)) - \\ &- A^H(Xv, \varphi^K(Y)) + A^H(Yv, \varphi^K(X)) - d\omega(X, Y). \end{aligned}$$

Using, as above, (64)–(67) and $(57_1)$, we get $d\Psi(X, Y) = 0$. This proves the local existence of a solution of (66) + (67).

This paper has been written during my stay at the State University and the Pedagogical Institute at Vilnius, USSR.

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