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NORMAL PRIME FILTERS OF A LATTICE ORDERED GROUP

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The representation of lattice ordered groups as subdirect products of linearly ordered groups was investigated in several papers ([1], [4]–[7], [9]–[13]). Let P be the positive cone of a lattice ordered group G and let W be the union of all normal prime filters of P, $K_0 = \{x \in G : |x| \notin W\}$. Then K_0 is an l-ideal of G and G/K_0 is the largest quotient group of G that can be represented as a subdirect product of linearly ordered groups (cf. [1]).

BANASCHEWSKI [1] remarks that it might be of interest to have a characterization of W and K_0 internally in terms of the elements of G and that it remains as an open question whether W is the set of all elements $0 < a \in G$ such that

$$(1) \qquad (x_1 + a - x_1) \wedge \ldots \wedge (x_k + a - x_k) > 0 \quad \text{for any} \quad x_i \in G\,.$$

Let W_1 be the set of all strictly positive elements a of G satisfying (1). In this paper it will be shown that there exists an l-group G with two generators such that $W \neq W_1$, thus answering in the negative the above mentioned question (§2). An internal characterization of the sets W and K_0 for any l-group G will be given in §3.

Let G be a subdirect product of linearly ordered groups. Then for each $0 < a \in G$ and any finite set $x_1, \ldots, x_n \in G$ the relation (1) is valid. In §4 we show that for an infinite set $\{x_i\}$ $(i \in I)$ such that $\bigwedge_{i\in I}(x_i + a - x_i)$ exists, the relation

$$\bigwedge_{i\in I}(x_i + a - x_i) > 0$$

need not hold.

The standard terminology for lattice ordered groups will be used (cf. Birkhoff [2], Fuchs [3]). The lattice ordered groups will be written additively though they are not assumed to be abelian.

1. PRELIMINARIES

Let us recall some definitions and results that we shall use.

Let G be a lattice ordered group with the positive cone P. A proper subset $\emptyset \neq Q$ of P will be called a prime filter in P if

(i) $x, y \in Q$ implies $x \wedge y \in Q$;

(ii) $x \in Q$ and $z \in P$, $z \geqq x$ implies $z \in Q$;

(iii) $x, y \in P$ and $x + y \in Q$ implies $x \in Q$ or $y \in Q$.

The relation between prime filters of P and homomorphisms of G into totally ordered groups is described by the following proposition:

(*) ([1], Proposition 1.) *For any homomorphism f with $f(G) \neq \{0\}$ of G into a linearly ordered group T, $Q(f) = \{x \in P : f(x) > 0\}$ is normal prime filter and* $\operatorname{Ker}(f) = \{x \in G : |x| \notin Q(f)\}$. *Conversely, for any normal prime filter Q in P there exists an epimorphism f from G into a linearly ordered group such that $Q = Q(f)$, namely the natural homomorphism $G \to G/K$ where K is the l-ideal* $\{x \in G : |x| \in Q\}$.

For any $A \subset G$ we denote

$$A^{\delta} = \{x \in G : |x| \wedge |a| = 0 \text{ for each } a \in A\}.$$

The set A^{δ} will be called a polar of G. Each polar is a convex l-subgroup of G. The following theorem was proven in [13]:

(**) *A lattice ordered group G is a subdirect union of linearly ordered groups if and only if each polar of G is normal.*

2. AN EXAMPLE

Lemma 1. *Let φ be a homomorphism of a lattice ordered group G into a linearly ordered group H. Let $a, x_1, x_2 \in G$ such that*

$$(x_1 + a - x_1) \wedge (x_2 + a - x_2) = 0. \tag{2}$$

Then $\varphi(a) = 0$.

Proof. From (2) it follows

$$\varphi(x_1 + a - x_1) \wedge \varphi(x_2 + a - x_2) = 0$$

and since H is linearly ordered we have either $\varphi(x_1 + a - x_1) = 0$ or $\varphi(x_2 + a - x_2) = 0$; because $\varphi(a)$ is conjugate of $\varphi(x_i + a - x_i)$ we infer that $\varphi(a) = 0$.

Let Z be the additive group of all integers with the natural linear order. Let

$$F = \Pi Z_i \ (i \in I)$$

be the complete direct product of l-groups $Z_i = Z$ for each $i \in I$ where I is the set of all integers. The elements $f \in F$ are written in the form $f = (\ldots, f(i), \ldots)$ $(i \in I)$. For any integer n put

$$p_n f = (\ldots, b(i), \ldots) \ (i \in I)$$

with $b(i) = f(i - n)$. Let m be a fixed positive inger, $m > 1$. We denote by F_m the set of all $f \in F$ such that for any $i, j, k \in I$ satisfying $i - j = km$ we have $f(i) = f(j)$. Let G be the set of all pairs (n, f), $n \in Z$, $f \in F_m$. We put $(n_1, f_1) < (n_2, f_2)$ if either $n_1 < n_2$, or $n_1 = n_2$ and $f_1 < f_2$. We define the operation $+$ in G by the rule

$$(3) \qquad (n_1, f_1) + (n_2, f_2) = (n_1 + n_2, p_{n_2} f_1 + f_2) .$$

Then G is a lattice ordered group that is generated by two elements (cf. [8]). Let $j \in I$, $0 \leqq j < m$. Let $\bar{O}$ and $\bar{\bar{O}}$ be neutral element of F_m and of G, respectively. Further let $f_j \in F_m$ such that $f_j(i) = 1$ when $i - j = km$ for some $k \in I$ and $f_j(i) = 0$ otherwise. Put $a_j = (0, f_j)$, $x_1 = \bar{\bar{O}}$, $x_2 = (1, \bar{O})$. Then we have

$$(x_1 + a_j - x_1) \wedge (x_2 + a_j - x_2) = 0 .$$

According to Lemma 1 and (*) we obtain $a_j \in K_0$. Since K_0 is an l-ideal of G (cf. [1]) we infer that the element $a = \sum a_j$ $(j = 0, 1, \ldots, m - 1)$ belongs to K_0, hence $a \notin W$. For each $x \in G$,

$$x + a - x = a ,$$

thus a fulfils (1) and so $a \in W_1$. Therefore $W_1 \neq W$.

3. CONSTRUCTION OF K_0

Let G be a lattice ordered group. We define by induction subsets K_n and $\bar{K}_n \subset G$ $(n = 1, 2, \ldots)$ as follows. We put $K_1 = \bar{K}_1 = \{0\}$. If K_{n-1} and $\bar{K}_{n-1}$ are already defined we define K_n to be the set of all elements $0 \leqq a \in G$ such that

$$(4) \qquad (x_1 + a - x_1) \wedge (x_2 + a - x_2) \in \bar{K}_{n-1}$$

for some $x_1, x_2 \in G$. Further let $\bar{K}_n$ be the subsemigroup of G generated by K_n; i.e., $\bar{K}_n$ is the set of all elements of G that can be written in the form $b = a_1 + \ldots + a_m$ for some $a_1, \ldots, a_m \in K_n$ and some positive integer m.

Lemma 2. $\bigcup_{n=1}^{\infty} \bar{K}_n \subset K_0$.

Proof. Obviously $\bar{K}_1 \subset K_0$; assume that $\bar{K}_{n-1} \subset K_0$. Let $a \in K_n$. Then

$$(x_1 + a - x_1) \wedge (x_2 + a - x_2) = a_1 + \ldots + a_m$$

for some elements $x_1, x_2 \in G$, $a_1, \ldots, a_m \in \bar{K}_{n-1}$. If $a \in F$ for some normal prime filter F of G, then some a_i belongs to F and hence $a_i \in W$, which is a contradiction. Hence $a \in K_0$. Since K_0 is a subgroup of G we infer that $\bar{K}_n \subset K_0$.

Lemma 3. *For each positive integer* n, K_n *and* $\bar{K}_n$ *are convex subsets of* G *containing* 0.

Proof. The assertion is obviously valid for $n = 1$; assume that the assertion holds for $n - 1$. Let $a, b, x_1, x_2 \in G, 0 \leqq b \leqq a$ and let (4) hold. Then we have

$$0 \leqq (x_1 + b - x_1) \wedge (x_2 + b - x_2) \leqq (x_1 + a - x_1) \wedge (x_2 + a - x_2).$$

Because $\bar{K}_{n-1}$ is convex by the assumption, we obtain

$$(x_1 + b - x_1) \wedge (x_2 + b - x_2) \in \bar{K}_{n-1}$$

and therefore $b \in K_n$. Thus K_n is a convex subset of G. Let $y \in \bar{K}_n$, $z \in G$, $0 \leqq z \leqq y$. Then there are elements $y_1, \ldots, y_n \in K_n$ with $y_1 + \ldots + y_n = y$. Further there are elements $z_1, \ldots, z_n$ with $0 \leqq z_i \leqq y_i$, $z = z_1 + \ldots + z_n$ (cf. [3]). Thus $z_i \in K_n$ and hence $z \in \bar{K}_n$. Therefore $\bar{K}_n$ is convex in G.

Lemma 4. *For each positive integer* n, K_n *and* $\bar{K}_n$ *are normal subsets of* G.

Proof. Obviously $K_1 = \bar{K}_1 = \{0\}$ is normal. Assume that K_{n-1} and $\bar{K}_{n-1}$ are normal for some $n > 1$. Let $a \in K_n$, $x_1, x_2 \in G$ such that (4) holds and let $b \in G$. By putting $x_3 = b + x_1 - b$, $x_4 = b + x_2 - b$, $a' = b + a - b$ we obtain from (4)

$$(x_3 + a' - x_3) \wedge (x_4 + a' - x_4) \in \bar{K}_{n-1},$$

thus $b + a - b \in K_n$. Hence K_n is a normal subset of G. From this it follows immediately that $\bar{K}_n$ is a normal subset of G as well.

From Lemma 2 and Lemma 3 we obtain that $\bar{K} = \bigcup \bar{K}_n$ $(n = 1, 2, 3, \ldots)$ is a convex normal subset of G containing 0; since each $\bar{K}_n$ is a subsemigroup of G, the set $\bar{K}$ is a subsemigroup of G.

It is easy to verify that if A is a convex subsemigroup of G containing 0 then the set

$$B = \{x \in G : -a \leqq x \leqq a \text{ for some } a \in A\} = \{x \in G : |x| \in A\}$$

is a convex l-subgroup of G; if, moreover, A is normal in G, then B is an l-ideal of G. Thus we have the assertion:

Lemma 5. *The set* $K = \{x \in G : |x| \in \bigcup_{n=1}^{\infty} \bar{K}_n\}$ *is an* l*-ideal of* G.

Lemma 6. *Each polar of the factor* l*-group* G/K *is normal.*

Proof. Let

$$C \subset G/K, \quad D = \{Y \in G/K : X \wedge Y = K \text{ for each } X \in C\},$$

$Z \in G/K$. Assume that $Y_1 = Z + Y - Z \notin D$ for some $Y \in D$. Then there is $X \in C$ such that $Y_1 \wedge X = U_1 > K$. Let $x \in X$, $y \in Y$, $z \in Z$. Thus $y_1 = z + y - z \in Y_1$

and $u_1 = y_1 \wedge x \in U_1$, $u_1 \notin K$. Denote $u = -z + u_1 + z$. From

$$0 < u_1 \leqq y_1 = z + y - z$$

we obtain $0 < u \leqq y$ and hence

$$u_1 + K \leqq x + K\,, \quad u + K \leqq y + K\,.$$

Therefore

$$K \leqq (u_1 \wedge u) + K = (u_1 + K) \wedge (u + K) \leqq (x + K) \wedge (y + K) = K$$

and so $u_1 \wedge u \in K$. Since $u_1 \wedge u \geqq 0$, we have $u_1 \wedge u \in \overline{K}_n$ for some positive integer n and hence, according to the definition of K_{n+1}, we obtain $u_1 \in K_{n+1} \subset K$. Thus $U_1 = K$, a contradiction.

Lemma 7. $K_0 \subset K$.

Proof. From Lemma 7 and (**) it follows that G/K is a subdirect product of linearly ordered groups. Since G/K_0 is the largest quotient group of G that can be represented as a subdirect product of linearly ordered groups (cf. [1]) we obtain $K_0 \subset K$.

According to Lemma 2 and Lemma 8, $K_0 = K$. Hence we have the following internal characterization of the sets W and K_0:

Theorem. $K_0 = \{x \in G : |x| \in \bigcup \overline{K}_n \; (n = 1, 2, 3, \ldots)\}$, $W = P - K_0$.

4. MEETS OF CONJUGATE ELEMENTS

Let G be a lattice ordered group that is a subdirect product of linearly ordered groups. From the results of Banaschewski [1] (cf. also the Introduction) it follows that for each $0 < a \in W$ and any finite set $X = \{x_1, \ldots, x_n\} \subset G$,

$$\bigwedge_{x_i \in X} (x_i + a - x_i) > 0\,.$$

We show by an example, that for an infinite set $X \subset G$ we can have

$$\bigwedge_{x_i \in X} (x_i + a - x_i) = 0\,.$$

Let Z, F, and p_n be as in §2. We denote by F_0 the set of all $f \in F$ such that the set

$$s(f) = \{i \in I : f(i) \neq 0\}$$

is finite. Let G be the set of all pairs (n, f) with $n \in Z$, $f \in F_0$. We define the operation $+$ in G by (3), §2. Then G is a group. Further we put $(n, f) > 0$ if either (i) $n > 0$, or (ii) $n = 0$, $s(f) \neq \emptyset$ and $f(i_0) > 0$ where i_0 is the least element of $s(f)$. Then G

turns out to be a linearly ordered group. Let $n \in Z$, $f, f^n \in F_0$ such that $f_0(i) = 0$ for each $i \in I$, $f^n(i) = 0$ for each $i \in I$, $i \neq n$ and $f^n(n) = 1$. Put $a = (0, f^0)$, $x_n = (n, f_0)$. We have $-x_n = (-n, f_0)$ and

$$x_n + a - x_n = (0, f^{-n}),$$

hence

$$\bigwedge(x_i + a - x_i) = 0,$$

where i runs over the set of all positive integers.

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