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Czechoslovak Mathematical Journal, Vol. 24 (1974), No. 1, 40–43

Persistent URL: http://dml.cz/dmlcz/101215

Czechoslovak Mathematical Journal, 24 (99) 1974, Praha

FUNCTIONS WITH CONTINUOUS WALLMAN EXTENSIONS

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(Received October 5, 1972)

It is well known that not all continuous functions between T_1 spaces have continuous extensions to the Wallman compactification. It is also known that the composition of two functions which have unique continuous Wallman extensions need not have a unique continuous extension. There are numerous examples of categories of T_1 spaces, every function in which has a unique continuous Wallman extension. Probably the most important of these is the category of all T_1 spaces and WO functions defined by HARRIS in [3], which includes all Wallman embeddings, all closed functions, and all continuous functions into T_4 spaces. It also has the property that the Wallman extension of a WO function is again a WO function, so that the category of compact T_1 spaces and WO functions is an epireflective subcategory of the category of T_1 spaces and WO functions. In [2] it was shown that any continuous function into a T_3 space which has a continuous Wallman extension must be a WO function. This rather naturally raises the question of whether the category of T_1 spaces and WO functions is maximal with respect to the above mentioned properties, or whether it can be embedded in a larger category which again satisfies all of them. In this paper, besides proving a generalization of the aforementioned result in [2], we construct a category of T_1 spaces which contains the category of T_1 spaces and WO functions as a proper subcategory and each function in which has a unique Wallman extension which is again in the category.

Recall that for any T_1 space X the Wallman compactification of X, $W(X) = \{u : u \text{ is an ultrafilter in the lattice of all closed subsets of } X\}$ with topology generated by $\{C(A) = \{u \in W(X) : A \in u\} : A \text{ a closed subset of } X\}$ as a base for the closed sets, is a compact T_1 space, and that the Wallman embedding ψ_X defined by $\psi_X(x) = \{A : x \in A \text{ and } A \text{ closed in } X\}$ is a dense embedding of X in $W(X)$. If f is a function from a T_1 space X to a T_1 space Y, a function $f^* : W(X) \to W(Y)$ is a Wallman extension of f provided that $\psi_Y \circ f = f^* \circ \psi_X$. One of the results in [3] is that any WO function has exactly one continuous Wallman extension which is, again, a WO function.

Theorem 1. *Let $f : X \to Y$ be a continuous function which has a continuous Wallman extension f^*. If, for every compact subset A of $W(Y)$, $\mathrm{cl}_Y(\psi_Y^{-1}[A] \cap f[X]) \subseteq \psi_Y^{-1}[A]$ then f is a WO function.*

Proof. It is easily proved that f is a WO function if and only if for each element $u \in W(X)$ there is an element $v \in W(Y)$ such that for any open set V in Y containing an element of v there is some open set U of X which contains an element of u and is such that for any closed subset A of X contained in U, $\mathrm{cl}_Y(f[A]) \subseteq V$. If $u \in W(X)$ and V is an open set containing an element of $f^*(u)$, then $C(Y \sim V)$ is a closed subset of $W(Y)$ which does not contain $f^*(u)$; so, since f^* is continuous, $f^{*-1}\,C(Y \sim V)$ is a closed subset of $W(X)$ which does not contain u. Hence, from the way the topology on $W(X)$ is defined, there is some closed set $B \subseteq X$ which is not an element of u such that $f^{*-1}[C(Y \sim V)] \subseteq C(B)$. Clearly $X \sim B$ is an open subset of X which contains an element of u. If D is any closed subset of X contained in $X \sim B$, $C(D)$ is a compact subset of $W(X)$ which is disjoint from $C(B)$, and so $f^*[C(D)]$ is a compact subset of $W(Y)$ contained in $W(Y) \sim C(Y \sim V)$. Hence $\psi_Y^{-1}[f^*[C(D)]] \subseteq V$. Therefore, since

$$f[D] \subseteq f[X] \cap \psi_Y^{-1}[f^*[C(D)]]\,, \quad \mathrm{cl}_Y(f[D]) \subseteq V.$$

It is immediate that the condition of Theorem 1 is satisfied if $\psi_Y^{-1}[A]$ is closed in Y whenever A is a compact set in $W(Y)$. In [1] it was shown that T_3 spaces have this property. Another class of spaces which also have it is the collection of compact T_1 spaces in which every compact subset is closed.

Since for a reasonably large class of functions, the only Wallman extendable ones are WO functions, it might seem reasonable to suspect that the category of T_1 spaces and WO functions is maximal in some sense. We now show that this is not the case by constructing a larger category with essentially the same properties.

Definition 2. A filter $\mathscr{F}$ in the lattice of closed subsets of a T_1 space X will be called indicative provided that $\bigcap\{C(A) : A \in \mathscr{F}\}$ is a singleton in $W(X)$. A continuous function f from a T_1 space X to a T_1 space Y will be called a *WI-function* provided that:

i) f has a continuous Wallman extension.

ii) For every indicative filter $\mathscr{F}$ in the lattice of closed subsets of X, $\{B \subseteq Y :$ B closed in Y and $f[A] \subseteq B$ for some $A \in \mathscr{F}\}$ is indicative.

It is, of course, immediate from the definition that the composition of WI-functions is a WI-function.

Proposition 3. *Every WO-function is a WI-function.*

Proof. Suppose that $\mathscr{F}$ is an indicative filter in X, that u is the element of $\bigcap_{A \in \mathscr{F}} C(A)$, and that $f : X \to Y$ is a WO-function. We know that f has a continuous Wallman extension $f^* : W(X) \to W(Y)$. Let v be any element of $W(Y)$ distinct from $f^*(u)$. There is some element $A \in v$ which is not an element of $f^*(u)$; so $Y \sim A$ is an open subset of Y containing an element of $f^*(u)$. As f is a WO-function there is some open set U in X containing an element of u and such that for any closed set B contained in U, $\mathrm{cl}_Y(f[B]) \subseteq Y \sim A$. Since $\bigcap_{D \in \mathscr{F}} (C(X \sim U) \cap C(D)) = \emptyset$ there is some finite subcollection $\{D_i : i = 1, \ldots, n\} \subseteq \mathscr{F}$ such that $\bigcap_{i=1}^{n} (C(X \sim U) \cap C(D_i)) = \emptyset$. From [1], $\bigcap_{i=1}^{n} C(D_i) = C(\bigcap_{i=1}^{n} D_i)$, and $\bigcap_{i=1}^{n} D_i$ is clearly an element of $\mathscr{F}$ contained in U. Hence $\mathrm{cl}_Y(f[\bigcap_{i=1}^{n} D_i]) \subseteq Y \sim A$ so $C(\mathrm{cl}_Y(f[\bigcap_{i=1}^{n} D_i]))$ is a closed set in $W(Y)$ disjoint from $C(A)$ which contains v, and thus $v \notin \bigcap\{C(E) : E$ closed in Y and $f[F] \subseteq E$ for some $F \in \mathscr{F}\}$. It is, of course, immediate that this intersection is nonempty, and so f must be a WI function.

Proposition 4. *If $f : X \to Y$ is a WI-function, then the continuous Wallman extension $f^* : W(X) \to W(Y)$ is unique.*

Proof. Suppose $g : W(X) \to W(Y)$ is a Wallman extension of f and that for some $u \in W(X)$, $g(u)$ is not the element of $\bigcap\{C(\mathrm{cl}_Y(f[A])) : A \in u\}$, which we will denote by v_u. Then there exists $B \in g(u)$ such that $B \notin v_u$. Since $\{C(B) \cap C(\mathrm{cl}_Y(f[A])) : A \in u\}$ is a collection of closed subsets of the compact space $W(Y)$ which has empty intersection, there is some finite subset $\{A_i : i = 1, 2, \ldots, n\} \subseteq u$ such that $C(B) \cap \cap (\bigcap_{i=1}^{n} C(\mathrm{cl}_Y(f[A_i]))) = \emptyset$, and of course $\bigcap_{i=1}^{n} A_i \in u$. $\mathrm{cl}_Y(f[\bigcap_{i=1}^{n} A_i]) \subseteq \mathrm{cl}_Y(\bigcap_{i=1}^{n} f[A_i]) \subseteq \subseteq \bigcap_{i=1}^{n} \mathrm{cl}_Y(f[A_i])$; so $C(\mathrm{cl}_Y(f[\bigcap_{i=1}^{n} A_i])) \subseteq \bigcap_{i=1}^{n} C(\mathrm{cl}_Y(f[A_i]))$. If g is continuous, $g^{-1}[C(\mathrm{cl}_Y(f[\bigcap_{i=1}^{n} A_i]))]$ is a closed subset of $W(X)$ which contains $\psi_X[\bigcap_{i=1}^{n} A_i]$ but not u. As was stated earlier, the smallest compact subset of $W(X)$ which contains $\psi_X[\bigcap_{i=1}^{n} A_i]$ is $C(\bigcap_{i=1}^{n} A_i)$, which does contain u. Hence g cannot be continuous.

Proposition 5. *If $f : X \to Y$ is a WI-function and if $f^* : W(X) \to W(Y)$ is its continuous Wallman extension, then f^* is a WI-function.*

Proof. Since $\psi_{W(X)}$ and $\psi_{W(Y)}$ are homeomorphisms it is clear that f^* has a (unique) continuous Wallman extension. Suppose that $\mathscr{F}$ is an indicative filter in $W(X)$. Obviously the intersection of the elements of $\mathscr{F}$ is a singleton, and the collection $\{A : A$ closed in X and $B \subseteq C(A)$ some $B \in \mathscr{F}\}$ is indicative in X. Denote by u the element of $\bigcap_{A \in \mathscr{F}} A$ and let v be any element of $W(Y)$ distinct from $f^*(u)$. Since f is a WI-function, there is some closed set B in X such that $C(B)$ contains some element $A \in \mathscr{F}$ and such that $C(\mathrm{cl}_Y(f[B]))$ does not contain v. $f^{*-1}[C(\mathrm{cl}_Y(f[B]))]$ is a closed subset of $W(X)$ containing $\psi_X[B]$ and hence must contain $C(B)$. Therefore $\mathrm{cl}_{W(Y)}(f^*[A])$ does not contain v, and $\{\mathrm{cl}_{W(Y)}(f^*[D]) : D \in \mathscr{F}\}$ is a singleton. Thus f^* must be a WI-function.

Proposition 6. *If $g : W(X) \to W(Y)$ is a WI-Wallman extension of a continuous function $f : X \to Y$, then f is a WI-function.*

Proof. Suppose $\mathscr{F}$ is an indicative filter in X. Then $\{A : A$ closed in $W(X)$ and $C(B) \subseteq A$ for some $B \in \mathscr{F}\}$ is an indicative filter in $W(X)$. Since g is a WI-function $\{D : D$ closed in $W(Y)$ and $\text{cl}_{W(Y)}(g[C(B)]) \subseteq D$ for some $B \in \mathscr{F}\}$ is indicative in $W(Y)$. Clearly $C(\text{cl}_Y(f[B])) \subseteq \text{cl}_{W(Y)}(g[C(B)])$; so, obviously, $\bigcap\{C(\text{cl}_Y(f[B])) : B \in \mathscr{F}\}$ must be a singleton. Hence, since we know f has a continuous Wallman extension, f is a WI-function.

We have shown so far that the category of all T_1 spaces and WI-functions contains all WO-functions and that the Wallman extension of a WI function is unique and WI, so that the subcategory of compact T_1 spaces and WI-functions is epireflective in it. We now construct an example of a WI-function which is not a WO-function.

We will denote by T the space $[0, 1] \cup \{\alpha\}$ where the topology on T is generated by the usual open subsets of $[0, 1]$ and all sets of the form $A \cup \{\alpha\}$ where A is the complement of a finite subset of $[0, 1]$. We will denote by T' the space $T \cup \{\beta\}$ where the topology on T' is generated by the open subsets of T together with all sets of the form $B \cup \{\beta\}$ where B is the complement of a finite subset of T. Since T and T' are compact it is clear that the natural embedding of T in T' has a (unique) continuous Wallman extension. It is also easily seen that all indicative filters in T contain singletons. From this it is immediate that the embedding is a WI function. However, any open set in T containing α contains closed sets (in T) whose closure in T' contains β. Hence the embedding is not a WO-function.

It might be noted that this paper does not assert that the category of WI-functions is maximal with respect to having unique continuous Wallman extensions. The author is inclined to believe it is not, but has been unable to find any functions which can be added.

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