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CONTINUOUS DEPENDENCE OF HOLOMORPHIC FUNCTIONS ON PARTLY GIVEN BOUNDARY VALUES

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Definitions and notation. Let C be the set of complex numbers. Denote by K_ζ^0 the open circle $|z| < \zeta$, by K_ζ the closed circle $|z| \leqq \zeta$ by Γ_ζ the boundary of the circle $|z| = \zeta$. In the case $\zeta = 1$ the index ζ will be omitted: $K_1^0 = K^0$, $K_1 = K$, $\Gamma_1 = \Gamma$. An interval $I(\varphi_0, \delta)$ on the boundary Γ is defined by $I(\varphi_0, \delta) = \{z : z = e^{i\varphi}, |\varphi - \varphi_0| < \delta\}$ where $\varphi_0 \in (-\pi, \pi\rangle$. In the sequel, different types of conditions will be needed.

i) *$f(z)$ is holomorphic on K^0,*

ii) $\limsup_{z \to z_0} |f(z)| \leqq \varepsilon$, $z_0 \in I_1$,

iii) $|f(z)| \leqq M$ *on* K^0,

iv) $\operatorname{Re} f(z)$ *is continuously extensible onto* I_2 *so that* $\operatorname{Re} f(e^{i\varphi})$ *exists for* $e^{i\varphi} \in I_2$, *there exist derivatives* $\mathrm{d}\operatorname{Re} f(e^{i\varphi})/\mathrm{d}\varphi$ *and* $|\mathrm{d}\operatorname{Re} f(e^{i\varphi})/\mathrm{d}\varphi| \leqq N$ *for* $e^{i\varphi} \in I_2$.
(*This condition can be substituted by the weaker codition:* $\operatorname{Re} f(z)$ *is continuously extensible onto* I_2 *and* $\operatorname{Re} f(e^{i\varphi})$ *is Lipschitz continuous as a function of* φ *for* $e^{i\varphi} \in I_2$ *with a coefficient* N),

v) *let* $\mu(x)$ *be a continuous nonnegative function defined for* $x \geqq 0$, $\mu(0) = 0$, *assume that* $\operatorname{Re} f(z)$ *is continuously extensible onto* I_2 *so that* $\operatorname{Re} f(e^{i\varphi})$ *exists for* $e^{i\varphi} \in I_2$ *and* $|\operatorname{Re} f(e^{i\varphi}) - \operatorname{Re} f(e^{i\psi})| \leqq \mu(|\varphi - \psi|)$ *for* $e^{i\varphi} \in I_2$, $e^{i\psi} \in I_2$,

vi) $|\operatorname{Re} f(z)| \leqq M$ *on* K^0,

vii) *$f(z)$ is continuously extensible onto K.*

I_1, I_2 are some intervals on the boundary Γ.

Let $\mathscr{F}_1(\varepsilon, M, I(\varphi_0, \delta))$ *be the class of complex functions which* fulfil i), ii) *and* iii) *where* $I_1 = I(\varphi_0, \delta)$.

First a well-known theorem (the two-constant theorem) (see [1]) is introduced together with a modification which will be useful in our further considerations.

Theorem 1. *Given numbers* $\varphi_0 \in (-\pi, \pi\rangle$, $\delta > 0$, $M > 0$ *and a point* $z_0 \in K^0$, *then*

$$|f(z_0)| \leqq \varepsilon^{1-w(zo)} M^{w(zo)}$$

where $f \in \mathscr{F}_1(\varepsilon, M, I(\varphi_0, \delta))$ *and* $w(z)$ *is the harmonic measure of* I_0 *with respect to* K^0.

This estimate can be generalized to the case that the inaccuracy ε depends on z [2]. This theorem guarantees that the problem to find a value of the holomorphic function $f(z)$ at the inner point of K is well posed if the values of f are known only on I with the possible inaccuracy ε while on the remaining part of Γ only the estimate $|f(z)| \leqq \leqq M$ is known. Frequently the following generalization is needed:

Theorem 1'. *Let numbers* φ_0, δ, M *be the same as in Theorem* 1. *Given a compact set* A, $A \subset K^0 \cup I(\varphi_0, \delta)$, *then*

$$\limsup_{\varepsilon \downarrow 0} \sup_{z \in A} \sup_{f \in \mathscr{F}} |f(z)| = 0 \quad \textit{where} \quad \mathscr{F} = \mathscr{F}_1(\varepsilon, M, I(\varphi_0, \delta))\,.$$

Holomorphic continuation from one part of the boundary to another.

Let I_1, I_2 be intervals on Γ, i.e. $I_k = \{z : z = e^{i\varphi}, |\varphi - \varphi_k| < \delta_k\}$. *Denote by* $\mathscr{F}_2^{(r)}(\varepsilon, M, N, I_1, I_2)$ *the class of complex functions which fulfil* i), ii), iii) *and* iv). Similarly denote by $\mathscr{F}_2^{(i)}(\varepsilon, M, N, I_1, I_2)$ the class of complex functions $f(z)$ such that $if(z) \in \mathscr{F}_2^{(r)}(\varepsilon, M, N, I_1, I_2)$. This means that $f(z) \in \mathscr{F}_2^{(i)}(\varepsilon, M, N, I_1, I_2)$ if the imaginary part $\operatorname{Im} f(z)$ is continuously extensible on I_2 and its derivatives are bounded by N on I_2.

Theorem 2. *Given numbers* φ_1, φ_2, φ_3 *all from* $(-\pi, \pi\rangle$, $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, $M > 0$, $N > 0$ *such that* $\varphi_2 - \delta_2 < \varphi_3 - \delta_3 < \varphi_3 + \delta_3 < \varphi_2 + \delta_2$, *then*

$$\limsup_{\varepsilon \downarrow 0} \sup_{z \in I_3} \sup_{f \in \mathscr{F}} |\operatorname{Im} f(z)| = 0 \quad \textit{where} \quad I_k = I(\varphi_k, \delta_k)\,, \quad k = 1, 2, 3$$

and $\mathscr{F} = \mathscr{F}_2^{(r)}(\varepsilon, M, N, I_1, I_2)$. *If* $I_2 = \Gamma$, *then* $I_3 = \Gamma$ *is possible.*

Remark 1. A similar theorem is valid for $\mathscr{F}_2^{(i)}(\varepsilon, M, N, I_1, I_2)$. If the assumptions of Theorem 2 are fulfilled, then

$$\limsup_{\varepsilon \downarrow 0} \sup_{z \in I_3} \sup_{f \in \mathscr{F}} |\operatorname{Re} f(z)| = 0 \quad \text{where} \quad \mathscr{F} = \mathscr{F}_2^{(i)}(\varepsilon, M, N, I_1, I_2)\,.$$

Proof of Theorem 2. Throughout the first part of the proof it will be assumed that $f(z)$ fulfils vii) so that, evidently, $|f(z)| \leqq M$ on Γ. Under these conditions an auxiliary inequality (2,1) will be proved:

$$(2,1) \qquad \left|\frac{\partial \operatorname{Re} f(\varrho e^{i\varphi})}{\partial \varphi}\right| \leqq \frac{M(1-\varrho^2)}{2\varrho(1-\cos\lambda)^2} + \frac{1}{2\pi}\frac{M(1-\varrho^2)}{\varrho(1-\cos\lambda)} + N$$
$$\text{for} \quad 0 < \varrho < 1\,, \quad |\varphi - \varphi_3| < \delta_3$$

where $\lambda = \min(\varphi_2 + \delta_2 - \varphi_3 - \delta_3, \quad \varphi_3 - \delta_3 - \varphi_2 + \delta_2)$. Since $f_1(\varrho, \varphi) = \operatorname{Re} f(\varrho e^{i\varphi})$ is a harmonic function in K^0 the Poisson formula yields

$$f_1(\varrho, \varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(1, \theta) \frac{1-\varrho^2}{1 - 2\varrho\cos(\varphi-\theta) + \varrho^2}\,d\theta\,.$$

Suppose $0 \leqq \varrho < 1$, then there exists $\partial f_1/\partial\varphi$ and

$$\frac{\partial f_1(\varrho, \varphi)}{\partial\varphi} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(1, \theta)\frac{(1-\varrho^2)\,2\varrho\sin(\varphi-\theta)}{[1 - 2\varrho\cos(\varphi-\theta)+\varrho^2]^2}\,d\theta\,.$$

If this integral is divided into two parts, one over $J = (-\pi, \pi\rangle - \tilde{I}_2$ and the other over $\tilde{I}_2$ where $\tilde{I}_2 = \{\varphi : \varphi \in (-\pi, \pi\rangle,\, e^{i\varphi} \in I_2\}$ and if the latter is integrated by parts, then the formula

$$(2,2) \qquad \frac{\partial f_1(\varrho, \varphi)}{\partial\varphi} = -\frac{1}{2\pi}\int_J f_1(1, \theta)\frac{2\varrho(1-\varrho^2)\sin(\varphi-\theta)}{[1 - 2\varrho\cos(\varphi-\theta)+\varrho^2]^2}\,d\theta -$$
$$-\frac{1}{2\pi}\left[f_1(1, \theta)\frac{1-\varrho^2}{1 - 2\varrho\cos(\varphi-\theta)+\varrho^2}\right]_{\varphi_2-\delta_2}^{\varphi_2+\delta_2} +$$
$$+\frac{1}{2\pi}\int_{\tilde{I}_2}\frac{\partial f_1(1, \theta)}{\partial\theta}\frac{1-\varrho^2}{1 - 2\varrho\cos(\varphi-\theta)+\varrho^2}\,d\theta$$

can be easily obtained (it is assumed $\delta < \pi_2$). As $2\pi - \lambda \geqq |\varphi - \theta| \geqq \lambda$ ($e^{i\varphi} \in I_3$) in the former integral as well as in the expression in the square brackets, it is $1 - 2\varrho\cos(\varphi-\theta) + \varrho^2 > 2\varrho(1-\cos\lambda)$. Obviously $\lambda < \pi$. In virtue of the assumption $|\partial f_1(1, \theta)/\partial\varphi| \leqq N$ on $\tilde{I}_2$ and the additional assumption $|f(z)| \leqq M$ on J the inequality (2,1) follows immediately from (2,2). Denote

$$N^* = \frac{3M}{2(1-\cos\lambda)^2} + \frac{1}{2\pi}\frac{3M}{1-\cos\lambda} + 2N\,.$$

Certainly N^* is an upper bound of the right-hand side of (2,1) divided by ϱ for $\frac{1}{2} \leqq \varrho \leqq 1$. Due to the Cauchy - Riemann relations and (2,1),

$$(2,3) \qquad \left|\frac{\partial \operatorname{Im} f(\varrho e^{i\varphi})}{\partial\varrho}\right| \leqq N^* \quad \text{for} \quad \tfrac{1}{2} \leqq \varrho < 1\,, \quad |\varphi - \varphi_3| < \delta_3$$

is valid. Now let any positive number η be given. Choose ϱ^* first such that $N^*(1-\varrho^*) < \eta/2$ and $\frac{1}{2} \leqq \varrho^* < 1$. Inequality (2,3) implies

$$(2,4) \qquad |\mathrm{Im}\, f(e^{i\varphi}) - \mathrm{Im}\, f(\varrho^* e^{i\varphi})| \leqq N^*(1-\varrho^*) < \eta/2 \quad \text{for} \quad |\varphi - \varphi_3| < \delta_3 \, .$$

According to Theorem 1' where A is the set of points $\varrho^* e^{i\varphi}$, $|\varphi - \varphi_3| \leqq \delta_3$ there exists a number $\varepsilon_0 > 0$ so that

$$(2,5) \qquad |f(\varrho^* e^{i\varphi})| < \eta/2 \quad \text{for} \quad |\varphi - \varphi_3| \leqq \delta_3 \, , \quad 0 \leqq \varepsilon \leqq \varepsilon_0 \, .$$

Inequalities (2,4) and (2,5) give

$$|\mathrm{Im}\, f(e^{i\varphi})| \leqq |f(\varrho^* e^{i\varphi})| + N^*(1-\varrho^*) < \eta \quad \text{for} \quad |\varphi - \varphi_3| < \delta_3 \, , \quad 0 \leqq \varepsilon \leqq \varepsilon_0 \, .$$

Hence the statement of Theorem 2 is proved under the additional assumption that $f(z)$ is continuously extensible onto K.

If the additional assumption is valid only for $\mathrm{Re}\, f(z)$, then instead of (2,4) only

$$|\mathrm{Im}\, f(\varrho_1 e^{i\varphi}) - \mathrm{Im}\, f(\varrho_2 e^{i\varphi})| \leqq N^* |\varrho_1 - \varrho_2| \quad \text{where} \quad \tfrac{1}{2} < \varrho_1 < \varrho_2 < 1 \, ,$$

$|\varphi - \varphi_3| < \delta_3$ is proved. This implies that $\lim_{\varrho \to 1} \mathrm{Im}\, f(\varrho e^{i\varphi})$ exists for $e^{i\varphi} \in I_3$. Put $\mathrm{Im}\, f(e^{i\varphi}) = \lim_{\varrho \to 1} \mathrm{Im}\, f(\varrho e^{i\varphi})$ for $e^{i\varphi} \in I_3$. Since $\mathrm{Im}\, f(\varrho e^{i\varphi})$ are continuous in φ (harmonic functions) and $\mathrm{Im}\, f(\varrho e^{i\varphi})$ converges to $\mathrm{Im}\, f(e^{i\varphi})$ uniformly the function $\mathrm{Im}\, f(e^{i\varphi})$ is continuous on I_3. As $|\mathrm{Im}\, f(e^{i\varphi_1}) - \mathrm{Im}\, f(\varrho e^{i\varphi_2})| \leqq |\mathrm{Im}\, f(e^{i\varphi_1}) - \mathrm{Im}\, f(e^{i\varphi_2})| + |\mathrm{Im}\, f(e^{i\varphi_2}) - \mathrm{Im}\, f(\varrho e^{i\varphi_2})| \leqq |\mathrm{Im}\, f(e^{i\varphi_1}) - \mathrm{Im}\, f(e^{i\varphi_2})| + (1-\varrho) N^*$ the function $\mathrm{Im}\, f(e^{i\varphi})$ is a continuous extension of $\mathrm{Im}\, f(\varrho e^{i\varphi})$ onto I_3.

In what follows, assumption vii) will be removed. With respect to Theorem 1 there exists a region Q_1 in C, $Q_1 \subset K^0$, $\bar{Q}_1 \cap \Gamma = I_1$ ($\bar{Q}_1$ is the closure of Q_1) such that $|f(z)| \leqq \sqrt{(M\varepsilon)}$ for $z \in Q_1$ (uniformly in the class $\mathscr{F}_1$). Let J be the Jordan curve in C such that $J \subset K$, $J \cap \Gamma = I_2$ and $J \cap Q_1 \neq 0$. Denote by Q the interior of J. The region Q can be conformally mapped onto K^0. Since I_2 is the analytic part of J the conformal mapping has continuous nonzero derivatives on I_2 (see Theorem 5, § 3, Chapt. II, [3]). By this transformation the problem is reduced to the previous one.

If $I_2 = \Gamma$, then instead of (2,2) the equation

$$\frac{\partial f_1(\varrho, \varphi)}{\partial \varphi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial f_1(1, \theta)}{\partial \varphi} \, \frac{1 - \varrho^2}{1 - 2\varrho \cos(\varphi - \theta) + \varrho^2} \, d\theta$$

is used so that $|\partial f_1(z)/\partial \varphi| \leqq N$ for $|z| < 1$. The remaining part of the proof is unchanged also in the case $I_3 = \Gamma$ (Q can be chosen as K_ζ).

If the function $f(z)$ describes some physical phenomena it is sometimes better to consider the mean values $f_\alpha(\varrho e^{i\varphi}) = \int_{\varphi-\alpha}^{\varphi+\alpha} f(\varrho e^{i\psi}) \, d\psi/(2\alpha)$ than the values $f(z)$ themselves. The number α is supposed to be sufficiently small positive which is connected with the inaccuracy of measurements. The following theorem for the mean values $f_\alpha(z)$ is sometimes quoted but without proof. Another reason why to treat the problem here is that the proof makes it possible to reduce an estimate of the error of the mean values $f_\alpha(z)$ to the estimate given by Theorem 2 (see Remark 2).

Let $\mathscr{F}_3(\varepsilon, M, I)$, $I = I(\varphi_0, \delta)$ *be the class of complex functions fulfilling* i), ii) *and* iii).

Theorem 3. *Given numbers* $\varphi_0 \in (-\pi, \pi\rangle$, $\delta > 0$, $M > 0$ *and* $\alpha > 0$, *then for every function* $f(z) \in \mathscr{F}_3(\varepsilon, M, I)$ *the corresponding mean value function* $f_\alpha(z)$ *is continuously extensible onto* K *and*

$$\limsup_{\varepsilon \downarrow 0} \sup_{z \in \Gamma} \sup_{f \in \mathscr{F}} |f_\alpha(z)| = 0 \quad \textit{where} \quad I = I(\varphi_0, \delta)\,, \quad \mathscr{F} = \mathscr{F}_3(\varepsilon, M, I)\,.$$

Remark 2. Assume that $I_2 = \Gamma$ in Theorem 2, then there exists a function $\chi(\varepsilon, M, N, I_1)$ so that

$$\lim_{\varepsilon \downarrow 0} \chi(\varepsilon, M, N, I_1) = 0 \quad \text{and} \quad |f(z)| \leqq \chi(\varepsilon, M, N, I_1)$$

for

$$f \in \mathscr{F}_2^{(r)}(\varepsilon, M, N, I_1, \Gamma) \cap \mathscr{F}_2^{(i)}(\varepsilon, M, N, I_1, \Gamma)\,.$$

Let s be a positive integer such that $\alpha/s < \delta$, then

$$|f_\alpha(z)| \leqq \chi\left(\varepsilon, M, \frac{s}{\alpha} M, I_{\alpha/s}\right) \quad \text{for} \; f(z) \in \mathscr{F}_3(\varepsilon, M, I) \quad \text{where} \quad I_{\alpha/s} = I\left(\varphi_0, \delta - \frac{\alpha}{s}\right).$$

Proof of Theorem 3. Let $u(\varrho, \varphi) = u(z)$, $v(\varrho, \varphi) = v(z)$ be the real and imaginary parts of $f(z)$, respectively. Let a positive integer s be chosen as in Remark 2. Denote

$$\beta = \alpha/s\,, \quad u_\beta(\varrho, \varphi) = \int_{\varphi-\beta}^{\varphi+\beta} u(\varrho, \psi)\, \mathrm{d}\psi/(2\beta)\,, \quad v_\beta(\varrho, \varphi) = \int_{\varphi-\beta}^{\varphi+\beta} v(\varrho, \psi)\, \mathrm{d}\psi/(2\beta)\,.$$

Since u_β, v_β fulfil the Cauchy - Riemann relations in K^0 the function $f_\beta(z) = u_\beta + iv_\beta$ is a holomorphic function in K^0. Evidently

$$|f_\beta(z)| \leqq \int_{\varphi-\beta}^{\varphi+\beta} |f(\varrho, \psi)|\, \mathrm{d}\psi/(2\beta) \leqq M \quad \text{for} \quad \varrho < 1\,.$$

The derivatives fulfil $|\partial f_\beta(\varrho, \varphi)/\partial\varphi| = |f(\varrho, \varphi + \beta) - f(\varrho, \varphi - \beta)|/(2\beta) \leqq M/\beta$ for $\varrho < 1$. Using the Cauchy - Riemann relations once more the inequality $|\mathrm{d}f_\beta(z)/\mathrm{d}z| \leqq M/(\beta\varrho)$ for $|z| < 1$ is obtained. Obviously the function $f_\beta(z)$ can be continuously extended onto K. This means that $f_\beta(z) \in \mathscr{F}_2^{(r)}(\varepsilon, M, M/\beta, I_\beta, \Gamma) \cap \cap\, \mathscr{F}_2^{(i)}(\varepsilon, M, M/\beta, I_\beta, \Gamma)$ where $I_\beta = I(\varphi_0, \delta - \beta)$. Using the notation from Remark 2 and applying Theorem 2 (Remark 1) the inequality

$$|f_\beta(z)| \leqq \chi(\varepsilon, M, M/\beta, I_\beta)$$

is easily derived. Since

$$f_\alpha(z) = f_\alpha(\varrho, \varphi) = (1/s) \sum_{i=0}^{s-1} f_\beta(\varrho, \varphi - (s - 1 - 2i)\,\beta)$$

the statement of Theorem 3 is proved.

The assumption of Theorem 2 (viz $\operatorname{Re} f(z)$ must have bounded partial derivatives on I_2) is rather strong. There is a possibility to modify this theorem using Theorem 3 in such a manner that a weaker condition is posed of f on I_2. However, this does not mean that Theorem 4 is a generalization of Theorem 2. While Theorem 2 assumes that $\operatorname{Re} f$ has bounded derivatives and states that $\operatorname{Im} f$ converges to 0 for $\varepsilon \to 0$, both the assumption and the statement of Theorem 4 concern the same (real or imaginary) part of f. Only if Theorem 2 and 4 are used for both parts simultaneously (i.e. for absolute values), then Theorem 4 is a generalization of Theorem 2.

Let I_1, I_2 be intervals on Γ and let $\mu(\eta)$ be continuous function defined for $\eta \geqq 0$, $\mu(0) = 0$, $\mu(\eta) \geqq 0$ for $\eta > 0$. *Denote by* $\mathscr{F}_4^{(r)}(\varepsilon, M, \mu(\eta), I_1, I_2)$ *the class of complex functions which fulfil* i), ii), iii) *and* v). The class $\mathscr{F}_4^{(i)}(\varepsilon, M, \mu(\eta), I_1, I_2)$ is defined so that $f \in \mathscr{F}_4^{(i)}(\varepsilon, M, \mu(\eta), I_1, I_2)$ if and only if $if \in \mathscr{F}_4^{(r)}(\varepsilon, M, \mu(\eta), I_1, I_2)$.

Theorem 4. *Given numbers* $\varphi_1, \varphi_2, \varphi_3$ *all from* $\langle -\pi, \pi \rangle$, $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, $M > 0$ *and a continuous function* $\mu(\eta)$ *such that* $\varphi_2 - \delta_2 < \varphi_3 - \delta_3 < \varphi_3 + \delta_3 < \varphi_2 + \delta_2$, $\mu(0) = 0$, $\mu(\eta) \geqq 0$ *for* $\eta > 0$, *then*

$$\limsup_{\varepsilon \downarrow 0} \sup_{z \in I_3} \sup_{f \in \mathscr{F}} |\operatorname{Re} f(z)| = 0 \quad \textit{where} \quad I_k = I(\varphi_k, \delta_k)\,, \quad \mathscr{F} = \mathscr{F}_4^{(r)}(\varepsilon, M, \mu(\eta), I_1, I_2)\,.$$

Proof. Choose an arbitrary number $\eta > 0$. Let α be a positive number such that $\alpha < \delta_1$, $\alpha < \varphi_2 + \delta_2 - \varphi_3 - \delta_3$, $\alpha < \varphi_3 - \delta_3 - \varphi_2 + \delta_2$ and such that

$$\frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \mu(|x|)\, \mathrm{d}x < \frac{\eta}{2}\,. \tag{4,1}$$

By Theorem 3 the mean value function f_α is continuously extensible onto K and $\limsup_{\varepsilon \downarrow 0} \sup_{z \in \Gamma} \sup_{f \in \mathscr{F}} |f_\alpha(z)| = 0$ ($\mathscr{F} = \mathscr{F}_3$). This yields that there exists $\varepsilon_0 > 0$ such that

$$|f_\alpha(z)| < \frac{\eta}{2} \quad \text{for} \quad z \in \Gamma\,, \quad f \in \mathscr{F}_4(\varepsilon_0, M, \mu, I_1, I_2) \subset \mathscr{F}_3(\varepsilon_0, M, I_1)\,. \tag{4,2}$$

According to the definition of the mean value function

$$|\operatorname{Re} f(e^{i\varphi})| \leqq |\operatorname{Re} f_\alpha(e^{i\varphi})| + \frac{1}{2\alpha} \int_{\varphi-\alpha}^{\varphi+\alpha} \mu(|\varphi - \psi|)\, \mathrm{d}\psi \quad \text{for} \quad e^{i\varphi} \in I_3\,. \tag{4,3}$$

Inequalities (4,1), (4,2) imply $|\operatorname{Re} f(e^{i\varphi})| < \eta$ for $e^{i\varphi} \in I_3$, $\varepsilon = \varepsilon_0$. The last inequality proves Theorem 4 as $\mathscr{F}_4^{(r)}(\varepsilon_1, M, \mu, I_1, I_2) \subset \mathscr{F}_4^{(r)}(\varepsilon_2, M, \mu, I_1, I_2)$ for $\varepsilon_1 \leqq \varepsilon_2$ so that the expression $\sup_{z \in I_3} \sup_{f \in \mathscr{F}} |\operatorname{Re} f(z)|$ depends monotonously on ε.

Remark 3. An estimate for the values of $f(z)$, $z \in I_3$ under the assumptions of Theorem 4 can be established on the basis of inequality (4,3) where $f_\alpha(e^{i\varphi})$ can be estimated as in Remark 2.

All previous theorems assume that the functions considered are uniformly bounded by the same constant M. This assumption can be weakened, namely it is sufficient to assume that only the real (or the imaginary) parts are uniformly bounded. It is clear that in that case the estimates will be much worse and also that it will be much more difficult to derive them.

First, Theorem 1 will be generalized in the following way.

Let I be an interval on Γ, $I = I(\varphi_0, \delta)$. *Denote by* $\mathscr{F}_5(\varepsilon, M, I)$ *the class of complex functions fulfilling* i), ii) *and* vi).

Theorem 5. *Given numbers* $\varphi_0 \in (-\pi, \pi\rangle$, $\delta > 0$, $M > 0$ *and a compact subset* A *of* K^0, *then*

$$\limsup_{\varepsilon \downarrow 0} \sup_{z \in A} \sup_{f \in \mathscr{F}} |f(z)| = 0 \quad \textit{where} \quad \mathscr{F} = \mathscr{F}_5(\varepsilon, M, I(\varphi_0, \delta))\,.$$

Proof. Choose $1 > \alpha > 0$, $\alpha < \delta$. Put $f_\beta^*(z) = \beta f_\beta(z)$ for $0 < \beta \leqq \alpha$ where $f_\beta(z)$ is the mean value function defined above by

$$f_\beta(\varrho e^{i\varphi}) = \int_{\varphi-\beta}^{\varphi+\beta} f(\varrho e^{i\psi})\, d\psi/(2\beta)\,.$$

Since $|\operatorname{Re} f(z)| \leqq M$ in K^0 it is $|\operatorname{Re} f_\beta^*(z)| \leqq \beta M$ and

$$\left|\frac{\partial \operatorname{Re} f_\beta^*(\varrho e^{i\varphi})}{\partial \varphi}\right| \leqq M \quad \text{in} \quad K^0\,. \tag{5,1}$$

Choose three numbers $\frac{1}{2} < \zeta < \lambda < \gamma < 1$ such that $A \subset K_\zeta^0$.

Put $S_\alpha = \{z : z = \varrho e^{i\varphi},\ 0 \leqq \varrho < 1,\ |\varphi - \varphi_0| < \delta - \alpha\}$. The Cauchy - Riemann relations and (5,1) imply $|\partial \operatorname{Im} f_\beta^* / \partial \varrho| \leqq M/\varrho$ in K^0 so that

$$\begin{aligned} |\operatorname{Im} f_\beta^*(z)| &= |\operatorname{Im} f_\beta^*(\varrho e^{i\varphi})| \leqq \\ &\leqq \limsup_{\omega \uparrow 1} \left[|\operatorname{Im} f_\beta^*(\omega e^{i\varphi})| + \left| \frac{\partial \operatorname{Im} f_\beta^*(\varrho_\omega^* e^{i\varphi})}{\partial \varrho} \right| (1 - \gamma) \right] \leqq \varepsilon + M(1 - \gamma)/\gamma \\ &\text{where} \quad \varrho < \varrho_\omega^* < 1\,, \quad z \in S_\alpha - K_\gamma^0\,. \end{aligned} \tag{5,2}$$

Since $\operatorname{Re} f_\beta^*(z)$ is a harmonic function,

$$\frac{\partial \operatorname{Re} f_\beta^*(\varrho e^{i\varphi})}{\partial \varrho} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} f_\beta^*(re^{i\theta}) \frac{2r[-2r\varrho + (r^2 + \varrho^2)\cos(\varphi - \theta)]}{[r^2 - 2r\varrho\cos(\varphi - \theta) + \varrho^2]^2}\, d\theta$$

$$\text{where} \quad \varrho \leqq \gamma\,, \quad r = \frac{1 + \gamma}{2}\,.$$

and since $|\mathrm{Re}\, f_\beta^*(z)| \leqq \beta M \leqq \alpha M$ it is

$$|\partial\ \mathrm{Re}\, f_\beta^*(z)/\partial \varrho| \leqq 128\alpha\, M/(1-\gamma)^4 \text{ in } K_\gamma\,.$$

By the Cauchy - Riemann relations again

$$(5,3) \qquad \left|\frac{\partial\ \mathrm{Im}\, f_\beta^*(z)}{\partial \varphi}\right| \leqq 128\alpha\, M/(1-\gamma)^4 \text{ in } K_\gamma\,.$$

Since

$$\mathrm{Im}\, f_\beta^*(\gamma e^{i\varphi}) = \mathrm{Im}\, f_\beta^*(\gamma e^{i\varphi_0}) + \int_{\varphi_0}^{\varphi} \frac{\partial\ \mathrm{Im}\, f_\beta^*(\gamma e^{i\psi})}{\partial \varphi}\, \mathrm{d}\psi\,,$$

inequalities (5,2) and (5,3) yield

$$|\mathrm{Im}\, f_\beta^*(z)| \leqq \varepsilon + M(1-\gamma)/\gamma + 128\alpha\pi\, M/(1-\gamma)^4 = M^+ \text{ on } \Gamma_\gamma\,.$$

Since $\mathrm{Im}\, f_\beta^*(z)$ is a harmonic function in K^0 the last inequality and (5,2) imply $|\mathrm{Im}\, f_\beta^*(z)| \leqq M^+$ on $K_\gamma \cup S_\alpha$. This means that $f_\beta^*(z)$ is bounded by $M\alpha + M^+$ in $K_\gamma \cup S_\alpha$. If the region $K_\gamma \cup S_\alpha$ is conformly mapped onto K^0 and then Theorem 1′ is used we obtain

$$(5,4) \qquad \limsup_{\varepsilon \downarrow 0} \sup_{z \in K_\lambda} \sup_{\beta \leqq \alpha} \sup_{f \in \mathscr{F}} |f_\beta^*(z)| = 0 \quad \text{where} \quad \mathscr{F} = \mathscr{F}_5(\varepsilon, M, I)\,.$$

In virtue of the Cauchy formula

$$\mathrm{d}f_\beta^*(z)/\mathrm{d}z = (2\pi i)^{-1} \int_{\Gamma_\lambda} f_\beta(w)\,(w-z)^{-2}\, \mathrm{d}w$$

so that

$$(5,5) \qquad \limsup_{\varepsilon \downarrow 0} \sup_{\beta \leqq \alpha} \sup_{z \in K_\zeta} \sup_{f \in \mathscr{F}} |\mathrm{d}f_\beta^*(z)/\mathrm{d}z| = 0 \quad \text{where} \quad \mathscr{F} = \mathscr{F}_5(\varepsilon, M, I)\,.$$

Relations (5,4) and (5,5) have the meaning

$$(5,6) \qquad \int_{\varphi-\alpha}^{\varphi+\alpha} f(\varrho e^{i\psi})\, \mathrm{d}\psi \to 0 \quad \text{for} \quad \varepsilon \to 0$$

uniformly with respect to

$$0 \leqq \varrho \leqq \lambda\,, \quad \varphi \in (-\pi, \pi\rangle\,, \quad f \in \mathscr{F} = \mathscr{F}_5(\varepsilon, M, I)$$

and $f(\varrho e^{i(\varphi+\beta)}) - f(\varrho e^{i(\varphi-\beta)}) \to 0$ for $\varepsilon \to 0$ uniformly with respect to $0 < \beta \leqq \alpha$, $0 \leqq \varrho \leqq \zeta$, $\varphi \in (-\pi, \pi\rangle$, $f \in \mathscr{F}_5$.

The last statement can be modified to

$$(5,7) \qquad |f(\varrho e^{i\varphi_1}) - f(\varrho e^{i\varphi_2})| \to 0 \quad \text{for} \quad \varepsilon \to 0$$

uniformly with respect to

$$0 \leqq \varrho \leqq \zeta\,, \quad \varphi_1, \varphi_2 \in (-\pi, \pi\rangle\,, \quad f \in \mathscr{F}_5\,.$$

Since

$$(5,8) \qquad f(\varrho e^{i\varphi}) = \int_{\varphi-\alpha}^{\varphi+\alpha} f(\varrho e^{i\psi})\, \mathrm{d}\psi/(2\alpha) + \int_{\varphi-\alpha}^{\varphi+\alpha} (f(\varrho e^{i\varphi}) - f(\varrho e^{i\psi}))\, \mathrm{d}\psi/(2\alpha)$$

and since with respect to (5,6) and (5,7) both terms of the right-hand side converge to 0 for $\varepsilon \to 0$ uniformly with respect to $0 \leqq \varrho \leqq \zeta$, $\varphi \in (-\pi, \pi)$, $f \in \mathscr{F}_5$ the statement of Theorem 5 is proved.

Remark 4. It may be easily seen by the reader of the proof that $f_\alpha(z)$ is bounded uniformly on every compact subset $A \subset K^0$, $f \in \mathscr{F}_5(\varepsilon, M, I)$ for every ε, i.e., also for $\varepsilon = M$.

The next theorem is a generalization of Theorem 3. In this case the same class of functions is needed as in Theorem 5. Just for the reason of preserving the correspondence between the classes of functions and theorems the notation $\mathscr{F}_6(\varepsilon, M, I) = \mathscr{F}_5(\varepsilon, M, I)$ is used.

Theorem 6. *Let numbers $\varphi_0 \in (-\pi, \pi\rangle$, $\delta > 0$, $M > 0$ and $\alpha > 0$ be given. If $f(z) \in \mathscr{F}_6(\varepsilon, M, I)$, $I = I(\varphi_0, \delta)$, then the mean value function $f_\alpha(z)$ is continuously extensible onto K and*

$$\lim_{\varepsilon \downarrow 0} \sup_{z \in \Gamma} \sup_{f \in \mathscr{F}} |f_\alpha(z)| = 0 \quad \textit{where} \quad \mathscr{F} = \mathscr{F}_6(\varepsilon, M, I)\,.$$

Proof. Choose a number $0 < \zeta < 1$ and a positive integer s such that $\alpha/s < \delta$. Denote $\beta = \alpha/s$. With respect to the definition of $f_\beta(z)$ it is

$$|\mathrm{Re}\, f_\beta(z)| \leqq M\,, \quad |\partial\, \mathrm{Re}\, f_\beta(z)/\partial\varphi| \leqq M/\beta \text{ in } K^0$$

and

$$\limsup_{z \to z_0} |f_\beta(z)| \leqq \varepsilon \quad \text{for} \quad z_0 \in I_\beta = I(\varphi_0, \delta - \beta)\,.$$

The Cauchy - Riemann relations imply

$$(6,1) \qquad \left|\frac{\partial\, \mathrm{Im}\, f_\beta(z)}{\partial \varrho}\right| \leqq \frac{M}{\beta\varrho} \text{ in } K^0\,.$$

On the other hand, applying Theorem 5 to $f_\beta(z)$ (put $A = K_\zeta$ and cf. Remark 4) we obtain that $f_\beta(z)$ is bounded uniformly with respect to $z \in K_\zeta$ and $f \in \mathscr{F}_5$, i.e., there exists a constant M^* such that

$$(6,2) \qquad |f_\beta(z)| \leqq M^* \quad \text{for} \quad z \in K_\zeta\,, \quad f \in \mathscr{F}_5(\varepsilon, M, I)\,.$$

Let $z = \varrho e^{i\varphi}$ be an arbitrary point from K^0. Since

$$|\mathrm{Im}\, f_\beta(z)| \leqq |\mathrm{Im}\, f_\beta(\zeta e^{i\varphi})| + |\varrho - \zeta| \left|\frac{\partial\, \mathrm{Im}\, f_\beta(\varrho^* e^{i\varphi})}{\partial \varrho}\right|$$

where ϱ^* is a number between ϱ and ζ, inequalities (6,1), (6,2) yield $|\mathrm{Im}\, f_\beta(z)| \leqq$ $\leqq M^* + |\varrho - \zeta|\, M/(\beta\zeta)$. Hence the imaginary part of $f_\beta(z)$ is bounded as well.

If $\delta > 0$ is sufficiently small, then $(f_\beta)_\delta$ is continuously extensible onto K (Theorem 3). $\mathrm{Re}\, f_\beta(\omega z)$ fulfils v) with $0 < \omega < 1$, $\mu(x) = Mx/\beta$ and $I_2 = \Gamma$ which follows from

$$\frac{1}{2\beta}\left[\int_{\varphi_2-\beta}^{\varphi_2+\beta} f(\omega e^{i\psi})\, \mathrm{d}\psi - \int_{\varphi_1-\beta}^{\varphi_1+\beta} f(\omega e^{i\psi})\, \mathrm{d}\psi\right] =$$

$$= \frac{1}{2\beta}\left[\int_{\varphi_1+\beta}^{\varphi_2+\beta} f(\omega e^{i\psi})\, \mathrm{d}\psi - \int_{\varphi_1-\beta}^{\varphi_2-\beta} f(\omega e^{i\psi})\, \mathrm{d}\psi\right].$$

With respct to this (5,8) implies that $\mathrm{Re}\, f_\beta(z)$ is continuously extensible onto K. Theorem 6 now follows directly from Theorems 2 and 4 since $\mathrm{Re}\, f_\beta(z)$ is Lipschitz continuous on Γ with the coefficient M/β.

The proof of Theorem 6 suggests that if $\mathrm{Re}\, f(z)$ or $\mathrm{Im}\, f(z)$ are bounded, then $\mathrm{Re}\, f_\alpha(z)$ and $\mathrm{Im}\, f_\alpha(z)$ are bounded simultaneously for every $\alpha > 0$. This can be used for some generalizations of Theorem 4 which will be presented without proof.

Denote by $\mathscr{F}_7(\varepsilon, M, \mu(\eta), I_1, I_2)$ *the class of complex functions which fulfil* i), ii), v) *and* vi).

Theorem 7. *Given numbers* $\varphi_1, \varphi_2, \varphi_3$ *from* $(-\pi, \pi\rangle$, $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, $M > 0$ *and a continuous function* $\mu(\eta)$ *such that* $\varphi_2 - \delta_2 < \varphi_3 - \delta_3 < \varphi_3 + \delta_3 < \varphi_2 + \delta_2$, $\mu(0) = 0$, $\mu(\eta) \geqq 0$ *for* $\eta > 0$, *then*

$$\limsup_{\varepsilon \downarrow 0}\, \sup_{z \in I_3}\, \sup_{f \in \mathscr{F}} |\mathrm{Re}\, f(z)| = 0 \quad \text{where} \quad I_k = I(\varphi_k, \delta_k)\,, \quad k = 1, 2, 3$$

and $\mathscr{F} = \mathscr{F}_7(\varepsilon, M, \mu(\eta), I_1, I_2)$.

There are more possibilities how to generalize Theorem 4. For example in Theorem 7 condition vi) can be changed by substituting $\mathrm{Im}\, f$ for $\mathrm{Re}\, f$. Even in v) $\mathrm{Re}\, f$ can be replaced by $\mathrm{Im}\, f$ but in this case the same must be done also in the statement of Theorem 7.

Theorem 2 can be also generalized. *Denote by* $\mathscr{F}_8(\varepsilon, M, N, I_1, I_2)$ *the class of functions which fulfil* i), ii), iv) *and* vi).

Theorem 8. *Given numbers* $\varphi_1, \varphi_2, \varphi_3$ *from* $(-\pi, \pi\rangle$, $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, $M > 0$, $N > 0$ *so that* $\varphi_2 - \delta_2 < \varphi_3 - \delta_3 < \varphi_3 + \delta_3 < \varphi_2 + \delta_2$ *is fulfilled, then*

$$\limsup_{\varepsilon \downarrow 0}\, \sup_{z \in I_3}\, \sup_{f \in \mathscr{F}} |\mathrm{Im}\, f(z)| = 0 \quad \textit{where} \quad I_k = I(\varphi_k, \delta_k)\,, \quad k = 1, 2, 3$$

and

$$\mathscr{F} = \mathscr{F}_8(\varepsilon, M, N, I_1, I_2)\,.$$

The proof would be the same as that of Theorem 2, only Theorem 5 is applied instead of Theorem 1.

All the previous theorems remain valid if in the definitions of $\mathscr{F}_i$ a weight function $g(z)$ is introduced. The weight function $g(z)$ can be any holomorphic function in K^0 which is continuously extensible onto I_2 and $g(z) \neq 0$ on I_2 (in the case of Theorem 1 and 5 it is sufficient that $g(z)$ is not identically equal to zero).

Conditions i) ... vi) are then changed in the following manner:

ai) $f(z)$ *is holomorphic on* K^0 (this condition is unchanged),

aii) $\limsup_{z \to z_0} |g(z) f(z)| \leqq \varepsilon,\ z_0 \in I_1$,

aiii) $|f(z)\, g(z)| \leqq M$ *on* K^0,

aiv) $\operatorname{Re} g(z) f(z)$ *is continuously extensible onto* I_2 *and*
$|\mathrm{d} \operatorname{Re} g(e^{i\varphi}) f(e^{i\varphi})/\mathrm{d}\varphi| \leqq N$ *on* I_2,

av) $\operatorname{Re} g(z) f(z)$ *is continuously extensible onto* I_2 *and*
$|\operatorname{Re} g(e^{i\varphi}) f(e^{i\varphi}) - \operatorname{Re} g(e^{i\psi}) f(e^{i\psi})| \leqq \mu(|\varphi - \psi|)$ *for* $e^{i\varphi} \in I_2$, $e^{i\psi} \in I_2$,

avi) $|\operatorname{Re} g(z) f(z)| \leqq M$ *on* K^0.

The "exterior" condition ii) (or aii)) can be replaced by the "interior" condition:

There exists an infinite sequence of points z_n, $z_n \in K_\zeta$, $0 < \zeta < 1$ so that $|g(z_n) f(z_n)| \leqq \varepsilon$ for $n = 1, 2, \ldots$

Let $\mathscr{F}'_1(\varepsilon, M)$ *be the class of complex functions which fulfil* i), iii) *and the last "interior" condition.* Evidently Theorem 1′ can be modified:

Theorem 1″. *Let A be a compact subset of K^0, then*

$$\limsup_{\varepsilon \downarrow 0} \sup_{z \in A} \sup_{f \in \mathscr{F}} |f(z)| = 0 \quad \textit{where} \quad \mathscr{F} = \mathscr{F}'_1(\varepsilon, M)\,.$$

By means of Theorem 1″ the conclusion can be derived that all previous theorems remain valid if only the condition ii) (aii)) is replaced by the "interior" condition.

Acknowledgement. The problems considered have their origin in some practical problems of the S-matrix theory of elementary particles. According to this theory the interaction of two elementary particles can be described by means of one analytic multivariable function defined in some domain. Values of the function on the boundary (or on its part) are determined from some measurable quantities and are given with experimental errors. Some theoretical or model-dependent assumptions

impose certain conditions on the values of the function on the remaining part of the boundary.

Many authors [2, 4 to 7] dealt with the problem to determine an analytic (holomorphic) function fulfilling almost exactly the given boundary values and satisfying some other assumptions, especially to determine the value of such analytic function at some inner point of the domain, or to investigate the dependence of the error of this value on the errors of the given boundary values etc.

In the course of discussions with J. FISCHER[1]), J. FORMÁNEK[2]), P. KOLÁŘ[1]), M. BEDNÁŘ[1]) the importance of determining the values of the analytic (holomorphic) function at some point of the boundary (e.g. if the measurements at this point are subject to big errors or are unrealizable) was formulated and pointed out for the first time. This problem and the corresponding results were thoroughly discussed not only from the mathematical but also from the physical point of view.

References

[1] *Nevalinna R.:* Eindeutige analytische Funktionen. Verlag Springer, Berlin, 1936.
[2] *Ciulli S., Fischer J.:* Nuclear Physics B *24* (1970), 537.
[3] *Голузин Г. М.,* Геометрическая теория функций комплексного переменного. Гос. Изд. Тех. Теор. Лит. Москва 1952.
[4] *Cutkosky R. E., Deo B. B.:* Physical Review Letters *22* (1968), 1272 and Physical Review *174* (1968), 1859.
[5] *Ciulli S.:* Nuovo Cimento *61*A (1969), 787 and *62*A (1969), 301.
[6] *Pišút J., Prešnajder P.:* Nuclear Physics B *12* (1969), 110 and B *14* (1969), 489.
[7] *Pišút J., Prešnajder P., Fischer J.:* Nuclear Physics B *12* (1969), 586.

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