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MEROMORPHISMS OF GRAPHS

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Many mathematicians have studied automorphisms of graphs. Here we shall study a more general concept – the concept of meromorphism of a graph.

Let G be a graph; for the sake of simplicity we shall consider here undirected graphs without loops and multiple edges, but the results can be easily generalized to the graphs with loops and multiple edges and also to digraphs.

A meromorphic mapping, or shortly meromorphism, of the graph G is an isomorphic mapping of G into itself.

In other words, it is a one-to-one mapping m which maps the vertex set of G into itself and the edge set of G into itself, such that the vertices $m(u)$ and $m(v)$ are joined by the edge $m(h)$ if and only if the vertices u and v are joined by the edge h.

Evidently automorphisms are particular cases of meromorphisms. They are such meromorphisms which map G onto G (i.e., they map the vertex set and the edge set of G each onto itself). A meromorphism of G which is not an automorphism of G will be called a proper meromorphism of G. Thus a proper meromorphism m of G maps G not onto G, but onto some proper subgraph of G; denote it by $m(G)$. Obviously only infinite graphs can have proper meromorphisms.

All meromorphisms of a given graph G form a semigroup with respect to their product (superposition).

Lemma 1. *Let a be an automorphism of a graph G, let m be a proper meromorphism of G. Then the products am, ma are both proper meromorphisms of G.*

Proof. As m is a proper meromorphism of G, it maps G onto some proper subgraph $m(G)$ of G. Let u be a vertex of G not belonging to $m(G)$; we have $u \neq m(v)$ for any vertex v of G. As a is an automorphism of G, the inverse automorphism a^{-1} exists. Take $u' = a(u)$. We have $u' \neq am(v)$ for any vertex v of G, because $u' = am(v)$ would imply $u = a^{-1}(u') = m(v)$, which would be a contradiction. Thus we have proved that $u' \neq am(G)$, therefore $am(G) \neq G$ and am is a proper meromorphism of G. Now consider again the vertex u. There is $u \neq m(v)$ for any vertex v of G. Therefore also $u \neq ma(w)$ for any vertex w of G, because $v = a(w)$ is a vertex of G and we have $u \neq m(v)$. Therefore also ma is a proper meromorphism of G.

Lemma 2. *The product $m_1 m_2$ of two proper meromorphisms m_1, m_2 of the graph G is a proper meromorphism of G.*

Proof. There exists a vertex u of G such that $u \neq m_1(v)$ for any vertex v of G. Therefore also $u \neq m_1 m_2(w)$ for any vertex w of G (as in the proof of the previous lemma) and $m_1 m_2$ is a proper meromorphism of G.

Lemma 3. *The semigroup $\mathfrak{M}(G)$ of all meromorphisms of a given graph G is left-cancellative.*

Proof. Let a, b, c be meromorphisms of the graph G and let $ab = ac$. Then $ab(u) = ac(u)$ for any vertex u of G. As a is a meromorphism of G, it maps the vertex set of G into itself in a one-to-one manner. Therefore the equality of images $ab(u)$, $ac(u)$ of $b(u)$ and $c(u)$ in the mapping a implies the equality of $b(u)$ and $c(u)$. As $ab(u) = ac(u)$ holds for any vertex u of G, the equality $b(u) = c(u)$ holds also for any vertex u of G and $b = c$. (The isomorphic mapping of a graph without loops and multiple edges is uniquely determined by its restriction on the vertex set of this graph. Therefore isomorphisms of such graphs are often considered only as mappings of the vertex set onto the vertex set of the image graph.)

Remark. We cannot assert in general that the semigroup $\mathfrak{M}(G)$ is also right cancellative. The equality $ba = ca$ does not imply $b = c$, but only the equality of the restrictions of b and c on the subgraph $a(G)$.

From Lemma 2 it follows that the proper meromorphisms of G form a subsemigroup $\mathfrak{M}_0(G)$ of $\mathfrak{M}(G)$; according to Lemma 1 this subsemigroup is an ideal of $\mathfrak{M}(G)$.

Lemma 4. *The semigroup $\mathfrak{M}_0(G)$ of all proper meromorphisms of G is torsion-free.*

Proof. Let $m \in \mathfrak{M}_0(G)$ and assume that $m^k = m^l$ for some non-negative integers k, l, where $k \neq l$. Without the loss of generality we can assume that $k < l$. Cancelling this equality from the left by m^k we obtain $e = m^{l-k}$, where e is the identity mapping of $G(e \notin \mathfrak{M}_0(G)$, but $e \in \mathfrak{M}(G))$. Thus m^{l-k} is the identical mapping of G and m^{l-k-1} is the inverse mapping to m. As m maps G onto some proper subgraph $m(G)$, the inverse mapping m^{l-k-1} must map $m(G)$ onto G and therefore (as it is one-to-one) it is defined only on $m(G)$ and not on whole G. Therefore m^{l-k-1} is not a meromorphism, but it is a power of a meromorphism, which contradicts Lemma 2.

The results from the Lemmas can be summarized in a theorem.

Theorem 1. *Let $\mathfrak{M}(G)$ be the semigroup of all meromorphisms of a graph G. Then $\mathfrak{M}(G)$ is left-cancellative and $\mathfrak{M}(G) = \mathfrak{A}(G) \cup \mathfrak{M}_0(G)$, where $\mathfrak{A}(G) \cap \mathfrak{M}_0(G) = \emptyset$, $\mathfrak{A}(G)$ is a subgroup of $\mathfrak{M}(G)$ and $\mathfrak{M}_0(G)$ is either a torsion-free subsemigroup of $\mathfrak{M}(G)$ which is an ideal of $\mathfrak{M}(G)$, or the empty set.*

Remark. A subgroup of a semigroup is a subsemigroup of this semigroup which is a group.

Now we shall prove another theorem which is a generalization of the well-known Frucht Theorem [1]. It will be proved in a similar way. Let us note that meromorphisms can be defined also for digraphs in the same way as for undirected graphs.

Theorem 2. *Let $\mathfrak{M}$ be an at most countable left-cancellative semigroup p with a unit and $\mathfrak{M} = \mathfrak{A} \cup \mathfrak{M}_0$, where $\mathfrak{A} \cap \mathfrak{M} = \emptyset$, $\mathfrak{A}$ a group, $\mathfrak{M}_0$ either a torsion-free semigroup which is an ideal in $\mathfrak{M}$, or the empty set. Then there exists a graph G whose meromorphism semigroup is isomorphic to $\mathfrak{M}$.*

Proof. First we shall construct an analogon of the Cayley colour graph of a group. To the elements $m \in \mathfrak{M}$ we assign in a one-to-one manner vertices u_m which will form the vertex set of the graph G_0. To the elements $m \in \mathfrak{M}$ we assign also in a one-to-one manner the colours c_m. Now if a, b, x are elements of $\mathfrak{M}$ and $b = ax$, we lead a directed edge from u_a into u_b and colour it by c_x. We make this for all elements of $\mathfrak{M}$; there are no other edges in G_0. Now for any $m \in \mathfrak{M}$ we define the mapping μ_m so that $\mu_m(u_a) = u_{ma}$ for any $a \in \mathfrak{M}$. The mapping μ_m is a meromorphism of G_0 which preserves the colours of edges. If there exists a directed edge from u_a into u_b of the colour c_x, it means that $b = ax$. The images of u_a and u_b in μ_m are u_{ma} and u_{mb} respectively. We have $mb = max$, therefore there exists also an edge of the colour c_x from u_{ma} to u_{mb}. If there exists no directed edge of the colour c_x from u_a into u_b, it means that $b \neq ax$. Therefore also $mb \neq max$ according to the left-cancellativity of $\mathfrak{M}$ and there is no directed edge from u_{ma} into u_{mb} of the colour c_x. From the left-cancellativity of $\mathfrak{M}$ it follows also that it is one-to-one and therefore it is really a meromorphism of G_0 preserving the colours of edges. Now let π be a meromorphism of G_0 preserving the colours of edges. Let e be the unit element of $\mathfrak{A}$. We shall prove that e is the left unit of the whole semigroup $\mathfrak{M}$. If $m \in \mathfrak{M}$, let $n = em$. We have $en = e^2m$, but $e^2 = e$, thus $en = em$ and by cancelling e from the left we obtain $n = m$. Thus from the vertex u_e a directed edge of the colour c_x leads to u_x for any $x \in \mathfrak{M}$. Let $\pi(u_e) = u_m$ for some $m \in \mathfrak{M}$. For any $x \in \mathfrak{M}$ the image $\pi(u_x)$ of the vertex u_x is the terminal vertex of the directed edge of the colour c_x starting at u_m. Such a vertex is exactly one, namely, u_{mx}. Therefore $\pi(u_x) = u_{mx} = \mu_m(u_x)$ for each $x \in \mathfrak{M}$, and hence $\pi = \mu_m$. We have proved that the semigroup of all meromorphisms of G_0 preserving the colours of edges is formed exactly by all μ_m for $m \in \mathfrak{M}$. Evidently $\mu_m\mu_n = \mu_{mn}$, therefore this semigroup is isomorphic to $\mathfrak{M}$. Now let us number all elements of $\mathfrak{M}$ by positive integers in a one-to-one manner; this is possible because $\mathfrak{M}$ is at most countable. If x is the i-th element of $\mathfrak{M}$ in this numbering, then to c_x we assign a graph H_i which consists of an arc of the length 3 consisting of vertices v_1, v_2, v_3, v_4 and edges v_1v_3, v_3v_4, v_4v_2 and of arcs of the lengths $2i$ and $2i + 1$ outgoing from v_3 and v_4 respectively (see Fig. 1). Such graphs are due to Frucht. They are pairwise non-isomorphic and each of them has only an identical automorphism. Now in G_0 we replace any directed edge $\overrightarrow{ab}$ of the colour c_x by the graph H_i corresponding to the colour c_x so that v_1 is identified with a and v_2 with b. We have obtained the desired graph G.

Now we shall add some other results on meromorphisms of graphs.

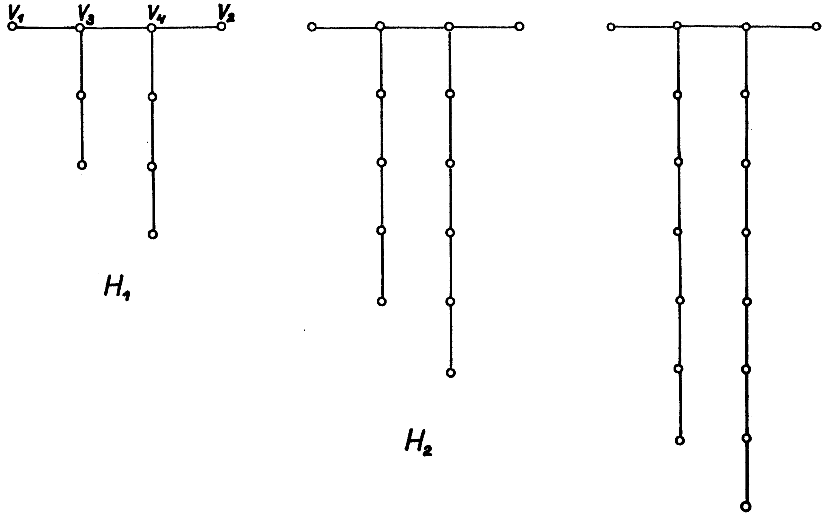

Fig. 1.

Theorem 3. *If no connected component of a graph G contains a subgraph isomorphic to another connected component of G, then any connected component of G is mapped into itself by any meromorphism of G. The assumption of this theorem cannot be weakened to read "no two connected components of G are isomorphic".*

The assertion of this theorem is evident. We shall give only an example of the graph G with two non-isomorphic connected components in which there exist meromorphisms mapping one component into another. The first component C_1 of G consits of the vertices a_i for all positive integers i and b_j for all positive integers $j \geqq 4$ and of the edges a_ia_{i+1} for $i = 1, 2, \ldots$ and a_jb_j for $j = 4, 5, \ldots$ The second component C_2 of G consists of the vertices a'_i for all positive integers i and b'_j for all positive integers $j \geqq 5$ and of the edges $a'_ia'_{i+1}$ for $i = 1, 2, \ldots$ and $a'_jb'_j$ for $j = 5, 6, \ldots$ Let us define a meromorphism m of G. We have $m(a_i) = a'_{i+1}$, $m(b_j) = b'_{j+1}$, $m(a'_i) = a_i$, $m(b'_k) = b_k$ for $i = 1, 2, \ldots$, $j = 4, 5, \ldots$, $k = 5, 6, \ldots$ The graph G is in Fig. 2.

Now we shall investigate a vertex of a graph and its images in different powers of some meromorphism of this graph. Let u be a vertex of the graph G, let m be some meromorphism of G. Let us study the sequence $\{m^k(u)\}_{k=0}^{\infty}$. (By m^0 we understand the identical mapping.) The mapping m is one-to-one, this means that $m^k(u) = m^l(u)$ for some k and l implies $m^{k-1}(u) = m^{l-1}(u)$. Hence, for $k < l$, we can obtain $u = m^{l-k}(u)$. Therefore there are two possibilities: either the terms in the sequence $\{m^k(u)\}_{k=0}^{\infty}$ are pairwise different, or this sequence is periodical without a pre-period. In the first case we are interested in the "backward prolongation" of such a sequence. Consider the sequence $m^{-1}(u)$, $m^{-2}(u)$, ... There are two cases possible. Either this sequence is infinite, or it has some last term, i.e. the term $m^{-l}(u)$ which is not in $m(G)$, i.e. $m^{-l}(u) \neq m(v)$ for all vertices v of G; this means that $m^{-l-1}(u)$ does not exist. In the former case we consider the two-way infinite sequence $\{m^k(u)\}_{k=-\infty}^{\infty}$, in the latter we denote $u_0 = m^{-l}(u)$ and consider the sequence $\{m^k(u_0)\}_{k=0}^{\infty}$. In both cases such a sequence contains the original sequence. Therefore we have three kinds of sequences which will be called cycles: finite (from a periodical sequence we take only one period), one-way infinite, two-way infinite. The sets of elements of cycles form a partition of the vertex set of G.

Proposition 1. *A meromorphism m of a graph G is an automorphism of G, if and only if any of its cycles is either finite, or two-way infinite.*

Proof. It is well-known that any cycle of an automorphism is either finite, or two-way infinite. On the other hand, to any vertex u in a finite or two-way infinite cycle of m there exists a vertex $m^{-1}(u)$, therefore m is a meromorphic mapping onto the vertex set of G, i.e. an automorphism of G.

Theorem 4. *If m is a proper meromorphism of a graph G, then $m^k(G)$ is a proper subgraph of $m^l(G)$ for any two positive integers k, l such that $k > l$.*

Proof. According to Proposition 2 m must contain at least one one-way infinite cycle; this cycle can be chosen so that its first term u is not in $m(G)$. Then for any positive integer k the graph $m^k(G)$ contains all $m^j(u)$ for $j \geqq k$ and does not contain $m^j(u)$ for $j < k$. Therefore if $k > l$, then $m^l(u)$, $m^{l+1}(u)$, ..., $m^{k-1}(u)$ are contained in $m^l(G)$, but not in $m^k(G)$.

Theorem 5. *Denote $m^{\infty}(G) = \bigcap_{k=0}^{\infty} m^k(G)$. If $m^{\infty}(G) \neq \emptyset$, then the restriction of m on $m^{\infty}(G)$ is an automorphism of $m^{\infty}(G)$.*

Proof. Let u be the first term of some one-way infinite cycle of m and let u be not in $m(G)$. Then for any vertex $m^k(u)$ there exists a graph $m^{k+1}(G)$ which does not contain it and therefore no vertex of a one-way infinite cycle belongs to $m^{\infty}(G)$. On the other hand, all vertices of finite and two-way infinite cycles are contained in $m^{\infty}(G)$, because any one of them is an image of some other in any power of m. Thus the vertex set of $m^\infty(G)$ consists exactly of all vertices of G which are contained in finite and two-way infinite cycles. The restriction of m on $m^\infty(G)$ has therefore only finite and two-way infinite cycles and is an automorphism of $m^\infty(G)$.

We shall give an example of a graph G with the meromorphism m such that $m^\infty(G)$ is an empty graph. This graph is a one-way infinite arc consisting of the vertices a_i for all non-negative integers i and of the edges $a_i a_{i+1}$ for $i = 0, 1, \ldots$ The meromorphism m is defined so that $m(a_i) = a_{i+1}$ for $i = 0, 1, \ldots$ This meromorphism

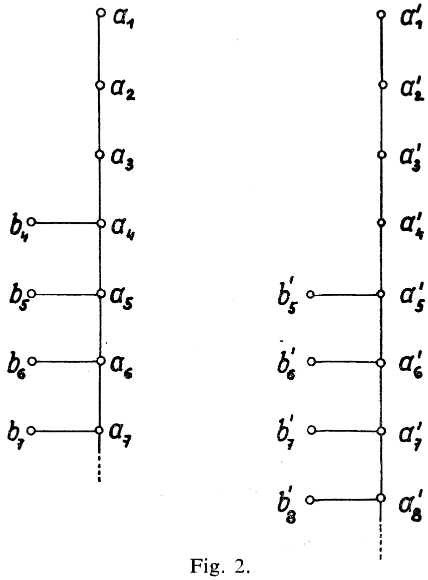

Fig. 2.

has exactly one cycle and this cycle is one-way infinite, therefore $m^\infty(G)$ is empty. Now consider a graph G' consisting of the vertices a_i for all non-negative integers i, the vertices b_j for $j = 1, \ldots, k$ and of the edges $a_i a_{i+1}$ for $i = 0, 1, \ldots$, $b_j b_{j+1}$ for $j = 1, \ldots, k-1$, $b_k b_1$ and all $a_i b_l$, where $i = 0, 1, \ldots$ and $l \equiv i \pmod{k}$, $1 \leqq l \leqq k$. Define m' so that $m'(a_i) = a_{i+1}$, $m'(b_j) = b_{j+1}$ for $i = 0, 1, \ldots$, $j = 1, \ldots, k-1$, $m'(b_k) = b_1$. Here $m'^\infty(G')$ is the circuit consisting of all b_j and the edges joining them. Therefore G' is a graph with a proper meromorphism m' such that $m'^\infty(G')$ is non-empty.

From Theorem 5 a corollary follows.

Corollary. *Let G be a graph, $\mathfrak{M}(G)$ the semigroup of all meromorphisms of G. Let $G_0 = \bigcap_{m \in \mathfrak{M}(G)} m^\infty(G)$. If G_0 is non-empty, then the restrictions of all meromorphisms of G on G_0 are automorphisms of G_0.*

In the end we shall mention the case when $\mathfrak{M}(G)$ is commutative. In this case we can form an analogon of the factor-group of a group. The group $\mathfrak{M}(G)$ can be partitioned into classes according to the group of automorphisms $\mathfrak{A}(G)$. We put $m \sim n$ for two elements m, n of $\mathfrak{M}(G)$, if and only if there exists $a \in \mathfrak{A}(G)$ such that $n = am$. We have $m \sim m$ for each $m \in \mathfrak{M}(G)$, because $m = em$ and $e \in \mathfrak{A}(G)$. Further $m \sim n$ implies $n \sim m$, because $n = am$ implies $m = a^{-1}n$ and $a^{-1} \in \mathfrak{A}(G)$. If $m \sim n, n \sim p$, then $m \sim p$, because $m \sim n$ means $n = am$, $n \sim p$ means $p = bn$, where $a \in \mathfrak{A}(G)$, $b \in \mathfrak{A}(G)$, and $p = abm$, where $ab \in \mathfrak{A}(G)$. Therefore $\sim$ is an equivalence on $\mathfrak{M}(G)$; all elements of $\mathfrak{A}(G)$ belong to the same class of this equivalence. Further $\sim$ is a congruence; if $m \sim n$, $p \sim q$, then $n = am$, $q = bp$, where $a \in \mathfrak{A}(G)$, $b \in \mathfrak{A}(G)$, and $nq = abmp$, where $ab \in \mathfrak{A}(G)$, which means $mp \sim nq$. Therefore the classes of $\sim$ form a semigroup, which may be called the factor-semigroup $\mathfrak{M}(G)/\mathfrak{A}(G)$.

Proposition 2. *All elements of the factor-semigroup $\mathfrak{M}(G)/\mathfrak{A}(G)$ (defined in the case when $\mathfrak{M}(G)$ is commutative) except for the class containing all elements of $\mathfrak{A}(G)$ are of infinite order.*

Proof. Assume that $m^k \sim m^l$, where $k < l$. This means $m^l = am^k$, where $a \in \mathfrak{A}(G)$. The semigroup $\mathfrak{M}(G)$ is left-cancellative according to Lemma 3; if it is also commutative, it is cancellative. Cancelling m^k from the equality $m^l = am^k$ we obtain $m^{l-k} = a$, therefore $m^{l-k} \in \mathfrak{A}(G)$. As $\mathfrak{M}_0(G)$ is an ideal of $\mathfrak{M}(G)$, the power m^{l-k} can be in $\mathfrak{A}(G)$ (therefore not in $\mathfrak{M}_0(G)$) only if $m \in \mathfrak{A}(G)$.

Reference

[1] *O. Ore:* Theory of Graphs. Providence 1962.

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