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DUALITY AND ALGEBRAICALLY IRREDUCIBLE SEMIGROUPS

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1. Definitions and Basic Theorems. In all that follows, a semigroup S is a Hausdorff topological space together with a continuous associative multiplication. We shall use 1 to denote the identity element, K to denote the minimal ideal (which exists if S is compact [13]), E to denote the set of idempotent elements ($e \in E$ if and only if $e = e^2$) and $H(e)$ to denote the maximal subgroup of S with identity $e \in E$. (Each $H(e)$ is a compact topological group if S is compact [13]).

A compact connected semigroup S is algebraically irreducible about $B \subset S$ if S contains no proper closed connected subsemigroup containing B. In particular, a compact connected abelian semigroup with an identity element, 1, algebraically irreducible about $K \cup H(1)$ will be called an $A - I$ semigroup [7]. The left equivalence of GREEN [8] is defined for a semigroup S by $x\mathscr{L}y$ if and only if $\{x\} \cup Sx = \{y\} \cup Sy$, with L_x the equivalence class of those $p \in S$ with $p\mathscr{L}x$. It is known that for S compact and abelian the quotient space, S modulo $\mathscr{L}$, is again a compact abelian semigroup, denoted by S', and that the canonical mapping $\phi : S \to S'$ is a continuous homomorphism.

A standard thread [7], is a compact semigroup S with a total order such that (a) the order topology is the given topology, (b) S is connected in the order topology, (c) S has a maximal element and it is an identity element and (d) S has a minimal element and it is a zero. A nil thread is a standard thread having no interior idempotent element but at least one non-zero nilpotent element. A unit thread is a standard thread with no interior idempotent element and no non-zero nilpotent element.

It is proved in [9] that if S is an $A - I$ semigroup, then S' is a standard thread. Also, it is proved that if S' consists entirely of idempotent elements, then there is an arc subsemigroup $P \subset S$ such that $\phi \mid P$ is an isomorphism onto S' and $\overline{S - K}$ is the union of the orbits of the elements of P under action by $H(1)$. Further, for $A - I$ semigroups, it is proved that if S' is a unit thread, then either

(1) $\overline{S - K}$ is the union of the orbits of a unit thread $p \subset S$ under action by $H(1)$;

or (2) $S - K$ is the union of the orbits of a half open arc $T \subset S$, $T \simeq (0, 1]$ under action by $H(1)$ and T winds on a closed subgroup C of K.

The purpose of this paper is to discuss an analogue of the Pontrjagin duality theorem applicable to $A - I$ semigroups. AUSTIN [1], the BAKERS [2] and BROWN and FRIEDBERG [6] have considered continuous homomorphisms of a topological semigroup. The range semigroup in [1] and [2] is the complex unit disk with complex multiplication, while the range in [6] is quite different. In [4], [10], [11] and [12], the homomorphisms were taken to be measurable (with respect to a fixed "nicely" chosen measure) on the given topological semigroup and the range semigroup the complex unit disk. As is well known a measurable homomorphism of a locally compact topological group to the circle group is of necessity continuous. Thus it seems to us, especially in view of the results of [4], [11] and [12] that the approach of measurable homomorphisms has an advantage.

In [2], the Baker's introduced the concept of an involution in a semigroup and obtained results dealing with duality of semigroups with involution. The main results we give here will concern semigroups with involution and the dual space consisting of those measurable homomorphisms satisfying $\tau(x^*) = \overline{\tau(x)}$, where $x \to x^*$ is the involution on the given semigroup and $\overline{\tau(x)}$ is the complex conjugate of $\tau(x)$.

2. Let S be an $A - I$ semigroup with $K = \{0\}$. In [4], BERGMAN showed that if S were a standard thread containing no nil subthread then the bounded real valued Lebesgue measurable homomorphisms on S as a subsemigroup of the measurable semicharacters S^* were in duality with S via the evaluation mapping $x \to \bar{x}$, $\bar{x}(\chi) = \chi(x)$, where $x \in S$ and $\chi \in S^*$. For $A - I$ semigroups there are natural involutions $(x \to x^*)$ on S such that if S were a standard thread, the measurable semicharacters χ satisfying $\chi(x^*) = \overline{\chi(x)}$ (the bar denotes complex conjugation) would be real valued. These natural involutions will be used in obtaining a dual semigroup for $A - I$ semigroups S for which S' contains no nil thread.

Let S be an $A - I$ semigroup such that S' contains no nil thread. Let $[I_\alpha : \alpha \in A]$ denote the collection of unit threads in S'. For each I_α, there exists a subsemigroup T_α of S contained in $\phi^{-1}(I_\alpha \setminus \{0\})$ such that $\phi \mid T_\alpha$ is an isomorphism onto $I_\alpha \setminus \{0\}$ and $\phi^{-1}(I_\alpha \setminus \{0\}) = T_\alpha H_\alpha$, where H_α is the maximal group of S containing that idempotent mapping to the identity of I_α under ϕ. For each possible choice of T_α, there is a natural involution on $T_\alpha H_\alpha$ namely, $(th)^* = th^{-1}$. It is clear that this involution is continuous on $T_\alpha H_\alpha$. For each nondegenerate component J_β of $S' \setminus \bigcup[I_\alpha : \alpha \in A]$ there is a unique idempotent subsemigroup $P_\beta \subset \phi^{-1}(\bar{J}_\beta) = P_\beta H_\beta$, where H_β is the maximal group containing the maximal element of P_β, and the natural involution $ph \to ph^{-1}$ is continuous on $P_\beta H_\beta$. Thus it is possible to introduce "natural" involutions on $S = (\bigcup[I_\alpha H_\alpha : \alpha \in A]) \cup (\bigcup[P_\beta H_\beta : \beta \in B])$. Note that while the involutions on the $P_\beta H_\beta$ are uniquely chosen, those on the $I_\alpha H_\alpha$ are not necessarily unique.

For each such involution, as above, one can consider those semicharacters τ on S satisfying $\tau(x^*) = \overline{\tau(x)}$. It is easily seen that they form a subsemigroup of the semigroup all semicharacters, $\tau_1\tau_2(x^*) = \tau_1(x^*)\,\tau_2(x^*) = \overline{\tau_1(x)}\,\overline{\tau_2(x)} = \overline{\tau_1(x)}\,\overline{\tau_2(x)} = \overline{\tau_1\tau_2(x)}$.

Definition. An $A - I$ semigroup S is *non nil* if S' contains no nil thread. If S is an $A - I$ semigroup and S^* is the semigroup of measurable semicharacters on S, then it is known [10] that S^* can be endowed with the Gelfand topology of the maximal ideal space of $L^1(S)$.

Let $\tilde{S}$ denote those elements of S^* satisfying the condition $\tau(x^*) = \overline{\tau(x)}$, where some "natural" involution has been chosen for S and let $\tilde{S}$ have the relative Gelfand topology of S^*.

In order to have a consistent notation, we write $N(f)$ for either of T_α or P_β, where f is the idempotent element of S mapping into the identity of I_α or $\overline{J_\beta}$. Instead of H_α or H_β, we can then write $H(f)$, the maximal subgroup of S containing f. We note that the $N(f)$ determine the particular "natural" involution in all cases.

Lemma 1. *Let S be a non nil $A - I$ semigroup and $x \to x^*$ a "natural" involution on S. For any $\tau \in \tilde{S}$, $\tau \mid N(f)$ is real valued for all $f \in E(S)$ such that $N(f)$ exists.*

Proof. Let $\tau \in \tilde{S}$, $t \in N(f)$ for some $f \in E(S)$, then for $h \in H(f)$ $\tau(th) = \overline{\tau(th^{-1})} = \overline{\tau(t)}\,\tau(h)$, that is $\tau(t) = \bar{\tau}(t)$ if $\tau(h) \neq 0$. If $\tau(h) = 0$, then $\tau(f) = 0$ and $\tau(t) = 0$. Thus $\tau(t)$ is real valued for all $t \in N(f)$.

Lemma 2. *$\tilde{S}$ is a closed subsemigroup of S^*.*

Proof. Let $o \neq \tau_0 \in S^*$ and $\{\tau_\alpha\}$ a net in $\tilde{S}$ with $\tau_\alpha \to \tau_0$. Let $x \in S$ such that $\tau_0(x) \neq 0$ and let $\mu \in L^1(S)$ such that $\mu(\tau_0) \neq 0$. Then $(\mu^*x)(\tau_0) = \int \tau_0 \,\mathrm{d}(\mu^*x) = \tau_0(x) \int \tau_0 \,\mathrm{d}\mu$. Now $(\mu^*x^*)(\tau_\alpha) \to (\mu^*x^*)(\tau_0)$, thus $\tau_\alpha(x^*)\,\hat{\mu}(\tau_\alpha) \to \tau_0(x^*)\,\hat{\mu}(\tau_0)$. Since $\hat{\mu}(\tau_\alpha) \to \hat{\mu}(\tau_0)$, we see that $\tau_\alpha(x^*) \to \tau_0(x^*)$, and since $\tau_\alpha(x^*) = \overline{\tau_\alpha(x)} \to \tau_0(x^*)$ and $\tau_\alpha(x) \to \tau_0(x)$ implies $\tau_0(x^*) = \overline{\tau_0(x)}$, $\tilde{S}$ is closed in S^*.

Lemma 3. *Let S be a non nil $A - I$ semigroup and let $x \to x^*$ be a "natural" involution on S. If $\tilde{S}$ separates points of S and e and $f \in E(S)$ are such that $[\phi(e), \phi(f)]$ is a unit thread in S', then there is a unit thread in S from e to f.*

Proof. If $H(e) = e$ the any $\overline{N(f)}$ is a unit thread in S from e to f. If $H(e) \neq \{e\}$ and there is no unit thread in S from e to f, then for any $N(f)$ there is a $t \in N(f)$ such that $te \neq e$. Let $\tau \in \tilde{S}$ such that $\tau(te) \neq \tau(e)$. Then $\tau(t) = \tau(te)$ and $|\tau(t)| = 1$ so that τ is not real valued on $N(f)$, a contradiction. Thus there is a unit thread in S from e to f.

Theorem 4. Let S be a non nil $A - I$ semigroup and $x \to x^*$ a "natural" involution on S. If the map of S into $\tilde{\tilde{S}}$ is one to one then S contains a standard thread from 0 to 1.

Proof. If the map of S into $\tilde{S}$ is one to one then $\tilde{\tilde{S}}$ separates points of S. By the preceding lemma, for each pair of idempotent elements e and f of S such that $e < f$ and $[\phi(e), \phi(f)] \cap E(S') = \{\phi(e), \phi(f)\}$ there is a unit thread $N(e, f)$ from e to f. For each such pair of idempotent elements, fix one such $N(e, f)$. Let $T = E \cup \cup (\cup[N(e, f) : N(e, f)$ a unit thread$])$.

Define: $\theta : S' \to T$ by $\theta(\phi(e)) = e$ if $e \in E(S)$ and $\theta(\phi(x)) = L_x \cap N(e, f)$ if $x \notin E(S)$. Now θ is clearly continuous and one to one, so is a homeomorphism and an algebraic isomorphism. Thus $\theta(S') = T$ is a standard thread in S from 0 to 1.

It is clear that $\tilde{S}$ must separate points of S in order that duality occur. Since each continuous semicharacter is measurable on S, if $\tilde{S}$ separates points then for each unit thread $I_\alpha \subset S'$, $\phi^{-1}(I_\alpha) = N(e_\alpha, f_\alpha) \times H(f_\alpha)$, where $I_\alpha = [\phi(e_\alpha), \phi(f_\alpha)]$, e_α and $f_\alpha \in E(S)$. We use this to prove

Theorem 5. *Let S be a non nil $A - I$ semigroup such that there exists a standard thread N in S from 0 to 1. Then there is a "natural" involution on S such that S and $\tilde{\tilde{S}}$ are isomorphic and homeomorphic via the evaluation mapping if and only if $\tilde{S}$ separates points of S.*

Proof. Since S is an $A - I$ semigroup and N is a standard thread from 0 to 1 in S and $H(1)\,N$ is a compact connected subsemigroup containing 0 and 1, $S = H(1)\,N$.

Define, for $x \in S$ and $x = nh$, $x^* = nh^{-1}$. For this "natural" involution we show duality. Note that x^* is well defined since n is unique and $nh_1 = nh_2$ implies $nh_1^{-1} = nh_2^{-1}$.

Let $\tau \in \tilde{S}$ then $\tau = \phi\gamma$ where $\phi = \tau \mid N$ and $\gamma = \tau \mid H(1)$ and $\tau(x) = \phi(n)\,\gamma(h)$. Now $\tilde{S}$ can be considered as a subset of $\tilde{N} \times \hat{H}(1)$. Let $\alpha : S \to \tilde{\tilde{S}}$ be given by $\alpha(x)(\tau) = \tau(x)$. We first show that α is surjective. Let $\theta \in S$ and let $\varepsilon_0 = \inf[\varepsilon = \varepsilon^2 \in \tilde{S}\ \theta(\varepsilon) = 1]$. Note that the idempotent elements of $\tilde{S}$ can be considered as a subset of $\tilde{N}$ a compact semigroup and hence the infimum exists. We consider two cases. If $\theta(\varepsilon_0) = 1$ then $\theta \mid H(\varepsilon_0)$ is a character on $H(\varepsilon_0)$ and hence there is an $h \in H(1)$ such that $\theta \mid H(\varepsilon_0) = \alpha(h) \mid H(\varepsilon_0)$. Now either $\theta \mid \tilde{S}\varepsilon_0 \setminus H(\varepsilon_0) = 0$ or for some $\tau \in \tilde{S}\varepsilon_0 \setminus H(\varepsilon_0)$, $\theta(\tau) \neq 0$. If $\theta \mid \tilde{S}\varepsilon_0 \setminus H(\varepsilon_0) = 0$ then $\theta(\tau) = \tau(h)$ for all $\tau \in \tilde{S}$. If $\theta \mid \tilde{S}\varepsilon_0 \setminus H(\varepsilon_0) \neq 0$ then ε_0 is the identity of a unit thread $N(\varepsilon_0)$ in $\tilde{S}$ and $\theta \mid N(\varepsilon_0)$ is real valued. Thus $\theta \mid N(\varepsilon_0)$ corresponds to a $t \in N$, i.e. for $\tau \in N(\varepsilon_0)$, $\theta(\tau) = \tau(t)$. For any $\tau \in \tilde{S}$, if $\theta(\tau) = 0$ then $\tau \mid (\tilde{S} \setminus \tilde{S}\varepsilon_0) \cup H(\varepsilon_0)\,N(\varepsilon_0)$ thus $\tau(t)\,\tau(h) = 0$. If $\theta(\tau) \neq 0$ then $\theta(\tau) = \tau(t)\,\tau(h)$ and $\theta \in \alpha(S)$. On the other hand, if $\theta(\varepsilon_0) = 0$ then θ is equivalent to a θ^1 on $H(\varepsilon_0)$, θ^1 corresponding to an $h \in H(1)$ and $\theta^1 \mid \tilde{S}\varepsilon_0 \setminus H(\varepsilon_0) = 0$ and thus α is onto. The mapping α is clearly one to one since S separates points of S.

We now show that α is continuous. Let $\{x_\alpha\}$ be a net in S with $x_\alpha \to x$. Let $\{n_\alpha\} \subset N$ and $\{h_\alpha\} \subset H$ be such that $x_\alpha = n_\alpha h_\alpha$. Then, since $\phi(x_\alpha) \to \phi(x)$, $n_\alpha \to n$. Let h_α^{-1} cluster to $g \in H$, then $x_\alpha h_\alpha^{-1}$, clusters to xg but $x_\alpha h_\alpha^{-1} = n_\alpha$ which converges to n and $x = ng^{-1}$, thus h_α has a subnet converging to g. We pass to subnets and write

$n_\alpha h_\alpha \to nh$ with $n_\alpha \to n$ and $h_\alpha \to h$. Now $\alpha(x_\alpha) \to \alpha(x)$ in $\tilde{S}$ if and only if $\mu \in L^1(\tilde{S})$ implies $\mu(\alpha(x_\alpha)) \to \hat{\mu}(\alpha(x))$. Now

$$|\hat{\mu}(\alpha(x_\alpha)) - \hat{\mu}(\alpha(x))| = \left|\int \tau(n_\alpha h_\alpha) - \tau(nh)\,\mathrm{d}\mu\right| \leqq \int |\tau(n_\alpha h_\alpha) - \tau(nh)|\,\mathrm{d}|\mu| =$$

$$= \int |\phi(n)\,\gamma(h_\alpha) - \phi(n)\,\gamma(h)|\,\mathrm{d}|\mu|\,(\phi, \gamma)$$

where $\phi \in \tilde{N}$ and $\gamma \in \hat{H}$. Since γ is continuous, $\varepsilon > 0$ implies there is an α_0 such that $|\gamma(h_\alpha) - \gamma(h)| < \varepsilon$ for all $\alpha > \alpha_0$. For $\phi \in \tilde{N}$, ϕ is continuous a.e. [4]. By the duality of N with $\tilde{\tilde{N}}$ there is an α^1 such that $\int |\phi(n_\alpha) - \phi(n)|\,\mathrm{d}|\mu|_{\tilde{N}} < \varepsilon\|\mu\|$ for all $\alpha > \alpha^1$. Thus

$$\int |\phi(n_\alpha)\,\gamma(h_\alpha) - \phi(n)\,\gamma(h)|\,\mathrm{d}|\mu| \leqq$$

$$\leqq \int (|\phi(n_\alpha)\,\gamma(h_\alpha) - \phi(n)\,\gamma(h_\alpha)| + |\phi(n)\,\gamma(h_\alpha) - \phi(n)\,\gamma(h)|\,\mathrm{d}|\mu|) \leqq$$

$$\leqq \int |\phi(n_\alpha) - \phi(n)|\,\mathrm{d}|\mu| + \int |\phi(n)|\,|\gamma(h_\alpha) - \gamma(h)|\,\mathrm{d}|\mu| \leqq 2\varepsilon\|\mu\|$$

and $\hat{\mu}(n_\alpha h_\alpha) \to \hat{\mu}(nh)$. Thus the mapping α is a continuous isomorphism onto $\tilde{S}$ and hence, since S is compact, a homeomorphism, too.

In the preceding, since we were interested in duality, we used the separation of points by elements of $\tilde{S}$ to produce a standard thread in S. The existence of such a standard thread clearly implies that the "natural" involution so produced is continuous. We now show that this is precisely the condition even without separation of points, however, the assumption that S is non-nil is necessary.

Theorem 6. *Let S be a non-nil $A - I$ semigroup with zero. A necessary and sufficient condition that S contain a standard thread from* 0 *to* 1 *is that one of the "natural" involutions be continuous.*

Proof. Let $x \to x^*$ be a continuous "natural" involution on S. In order to produce a standard thread in S from 0 to 1, it is clearly sufficient to show that for each unit thread in S' the corresponding $N(f)$ in S is such that $\overline{N(f)} = N(f) \cup \{e\}$, where e is the maximal idempotent element less than f.

Let $f \in N(f)$ such that $N(f)$ maps onto a unit thread minus its zero element in S' and let e be the maximal idempotent element less than f. Let $z \in \overline{N(f)} \setminus N(f)$ and let $\{t_\alpha\}$ be a net in $N(f)$ with $t_\alpha \to z$. Since the involution is continuous, $t_\alpha^* \to z^*$, but $t_\alpha^* = t_\alpha$ and thus $z^* = z$ and $z = z^{-1}$ since $z \in H(e)$. It follows that $\overline{N(f)} \setminus N(f)$ is a compact connected group each of whose elements is of order 2. Thus $\overline{N(f)} \setminus N(f) = \{e\}$ and $\overline{N(f)}$ is a unit thread in S.

The existence of a standard thread from 0 to 1, is what is meant by no essential winding. As an example to show that the condition that S be non-nil is necessary one needs only to look at Clifford's example of a circle with a wisker where the arc meets the circle group at -1. This semigroup can certainly be embedded in an $A - I$ semigroup with zero and the "natural" involution, the identity map on the wisker and inversion on the circle, is continuous, but nil thread exists in the semigroup and hence there is no standard thread in a larger $A - I$ semigroup with zero.

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