Anton Dekrét

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ON CANONICAL FORMS ON NON-HOLONOMIC AND SEMI-HOLONOMIC PROLONGATIONS OF PRINCIPAL FIBRE BUNDLES

ANTON DEKRÉT, Žilina

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Using the theory of jets, KOLÁŘ [3] introduced canonical forms on holonomic prolongations of principal fibre bundles and justified the algorithm for the determination of their structure equations. (This algorithm was also used by LAPTEV [6] and GHEORGHIEV [2].) In the present paper, using Kolář's results, we shall find the structure equations of canonical forms on non-holonomic and semi-holonomic prolongations of principal fibre bundles as well as the Maurer-Cartan equations of the structure groups of these bundles. In particular, we shall show that the structure equations of "the non-holonomic differential group of order r" introduced by LUMISTE in [9] (see also [7]) are the structure equations of the group $\bar{L}^r_n$ of all invertible semi-holonomic r-jets of R^n into R^n with the source and the target O. Further, the semi-holonomic extensions of the canonical form Θ_1 of $W^1(P)$ are introduced and some relations to the theory of linear connections are explained. We shall use the terminology and notation of the theory of jets (see [1]) throughout the paper. Our considerations are in the category C^∞.

1. Let M_1, M_2 be manifolds, $n_i = \dim M_i$, $i = 1, 2$. As usual, $\tilde{J}^r_x(M_1, M_2)$ or $\bar{J}^r_x(M_1, M_2)$ denotes respectively the manifold of all non-holonomic or semi-holonomic r-jets of M_1 into M_2 with source $x \in M_1$. Let U be a coordinate neighbourhood on $\tilde{J}^r_x(M_1, M_2)$ and let

$$c^j_{i_1 \dots i_r}, \quad j = 1, 2, \dots, n_2\,; \quad i_1, \dots, i_r = 0, 1, \dots, n_1$$

be coordinate functions on U, see [12]. Denote by ζ the rule of dropping all the zero components in a multiindex. VIRSÍK [12] deduced the following property of semi-holonomic jets.

Lemma 1. *Let $\tilde{f} \in U \subset \tilde{J}^r_x(M_1, M_2)$. Then $\tilde{f}$ is semi-holonomic if and only if its coordinates $c^j_{i_1 \dots i_r}(\tilde{f})$ satisfy*

$$\zeta(i_1 \dots i_r) = \zeta(k_1 \dots k_r) \Rightarrow c^j_{i_1 \dots i_r}(\tilde{f}) = c^j_{k_1 \dots k_r}(\tilde{f}) .$$

2. In what follows we shall use the following indices: $A, B = 1, \dots, n + m$; $\alpha, \beta, \gamma = n + 1, \dots, n + m$, $h, t, p, q = 1, \dots, n$; $i, j, k = 0, 1, \dots, n$.

Let $P(B, G, \pi)$ be a principal fibre bundle, $n = \dim B$, $m = \dim G$. Let $U \subset R^n$ be an open subset, $0 \in U$ and let V be an open subset of B. A local isomorphism $\Psi_{\varphi,\sigma} : U \times G \to \pi^{-1}(V)$, $\Psi_{\varphi,\sigma}(x, g) = [\sigma(\varphi(x))] \, g$, will be called an *allowable chart* on P, where φ is a diffeomorphism $U \to V$ and σ is a local cross-section $V \to \pi^{-1}(V)$. The first prolongation of P is the set $W^1(P)$ of all 1-jets of allowable charts on P with the source $(0, e) \in R^n \times G$. $W^1(P)$ is a principal fibre bundle over B with the structure group G^1_n of all 1-jets of allowable charts $\Psi_{\varphi,\sigma}$ on $R^n \times G$ with the source $(0, e)$ satisfying $\varphi(0) = 0$. We can identify $W^1(P) \equiv H^1(B) \otimes J^1(P)$ where $H^1(B) \otimes J^1(P)$ means the fibre product over B of $H^1(B)$ and $J^1(P)$ and G^1_n is the semidirect product $L^1_n \mathbin{\overline{\times}} T^1_n(G)$ of the groups L^1_n and $T^1_n(G)$ with respect to the action $S \mapsto SX$ $(S \in T^1_n(G), X \in L^1_n)$ of L^1_n on $T^1_n(G)$. In other words,

$$(1) \qquad (X_2, S_2)(X_1, S_1) = (X_2 X_1, (S_2 X_1) \, S_1)$$

where $X_2 X_1$, $S_2 X_1$ is the composition of jets and $(S_1 X_1) \, S_1$ is the product in the group $T^1_n(G)$. By induction, we define $\tilde{W}^r(P) = W^1(\tilde{W}^{r-1}(P))$, and call it *the r-th non-holonomic prolongation of P*. $\tilde{W}^r(P)$ has a natural structure of a principal fibre bundle $\tilde{W}^r(P)(B, \tilde{G}^r_n)$, where the group $\tilde{G}^r_n$ is determined by the recurrent formula $\tilde{G}^r_n = (\tilde{G}^{r-1}_n)^1_n$.

We can identify

$$\tilde{W}^r(P) = \tilde{H}^r_n(B) \otimes \tilde{J}^r(P) ,$$

$$\tilde{G}^r_n = \tilde{L}^r_n \mathbin{\overline{\times}} \tilde{T}^r_n(G) , \quad \text{see [11] and [4]} .$$

This identification will be denoted by τ. It results from the definition of $\tilde{W}^r(P)$ that $\tilde{W}^r(P)$ is the set of all 1-jets of the local isomorphisms $\Psi_{\varphi,\sigma}$ of $R_n \times \tilde{G}^{r-1}_n$ into $\tilde{W}^{r-1}(P)$ with the source $(0, e_{r-1})$, where e_{r-1} is the unit of the group $\tilde{G}^{r-1}_n$. These local isomorphisms will be called *non-holonomic allowable* $(r - 1)$-*charts* on P.

Let $u \in \tilde{W}^r(P)$, $u = j^1_{(0,e_{r-1})} {}^u\Psi$, where ${}^u\Psi$ is a non-holonomic allowable $(r - 1)$-chart on P. Denote by β the natural projection $\beta u = {}^u\Psi(0, e_{r-1})$. Let $X \in T_u(\tilde{W}^r(P))$, $X = j^1_0 \, \gamma(t)$. Then ${}^u\Psi^{-1}_* \beta_* X = j^1_0 \Psi^{-1}_* [\beta\gamma(t)] \in T_{(0,e_{r-1})}(R^n \times \tilde{G}^{r-1}_n) \equiv \tilde{w}^{r-1}_n(G)$. Thus we get a vector-valued form on $\tilde{W}^r(P)$ with values in $\tilde{w}^{r-1}_n(G)$

$$(2) \qquad \Theta_r(X) = {}^u\Psi^{-1}_* \, \beta_*(X) .$$

Definition 1. The form Θ_r determined by (2) will be said to be the canonical form on $\tilde{W}^r(P)$.

In [4], Kolář introduced an admissible extension of Θ_r in the holonomic case. Put $*\tilde{W}^{r+1}(P) = W^2(\tilde{W}^{r-1}(P))$, i.e., $*\tilde{W}^{r+1}(P)$ is the second holonomic prolongation of the principal fibre bundle $\tilde{W}^{r-1}(P)$. According to Lemma 3 of [4], $*\tilde{W}^{r+1}(P)$ is a principal fibre bundle $*\tilde{W}^{r+1}(P)$ $(\tilde{W}^r(P), {}^0(\tilde{G}_n^{r-1})_n^2)$, where ${}^0(\tilde{G}_n^{r-1})_n^2$ is the kernel of the homomorphism $j_2^1 : (\tilde{G}_n^{r-1})_n^2 \to \tilde{G}_n^r$ (j_2^1 is the natural projection of 2-jets into 1-jets). Since ${}^0(G_n^{r-1})_n^2$ is homeomorphic to a number space, the global sections of $*\tilde{W}^{r+1}(P)$ exist. Let σ be a global section of $*\tilde{W}^{r+1}(P)$; $u \in \tilde{W}^r(P)$, $\sigma(u) = j^2_{(0,e_{r-1})}\Psi$, where Ψ is a non-holonomic allowable $(r-1)$-chart; $X \in T_u(\tilde{W}^r(P))$, $X = j_0^1\,\gamma(t)$. Denote by $\Psi^{-1}\,\gamma(t)$ the image of the jet $\gamma(t)$ by the map Ψ^{-1}. Then $j_0^1\Psi^{-1}\,\gamma(t) \in \tilde{w}_n^r(G)$. The form

$$\Theta_r'(X) = j_0^1\Psi^{-1}\,\gamma(t) \tag{2'}$$

will be said to be *an admissible extension* of the canonical form Θ_r on $\tilde{W}^r(P)$. One can see easily that

$$(j_r^{r-1})_*\,\Theta_r'(X) = \Theta_r(X)\,. \tag{3}$$

3. Let e_α be a basis of $\mathfrak{g}$ (as usual $\mathfrak{g}$ denotes the Lie algebra of G). Let ω^α be the dual basis of $\mathfrak{g}^*$ and let

$$d\omega^\alpha = \tfrac{1}{2}c^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma$$

be the structure equations of G. Let c^α be the canonical coordinates in some neighbourhood of $e \in G$ determined by e_α, and let c^q be canonical coordinates on R^n. Denote by $c^A_{i_1\ldots i_r}$ the corresponding local coordinates on $R^n \times \tilde{G}_n^r$. Let $e_A^{i_1\ldots i_r}$ be the corresponding basis of $\tilde{w}_n^r(G)$. The space $\tilde{w}_n^{r-1}(G)$ is isomorphic to the subspace $\{e_A^{i_1\ldots i_{r-1}0}\} \subset \tilde{w}_n^r(G)$. Taking into account this isomorphism and (3), we can write

$$\Theta_r(X) = \Theta^A_{i_1\ldots i_{r-1}0}(X)\, e^{i_1\ldots i_{r-1}0}\,, \quad \Theta_r'(X) = \Theta_r(X) + \Theta^A_{i_1\ldots i_{r-1}q}(X)\, e_A^{i_1\ldots i_{r-1}q}\,.$$

Kolář, [3], deduced the following structure equations of Θ_1:

$$d\Theta_0^p = \Theta_0^q \wedge \Theta_q^p\,, \quad d\Theta_0 = \tfrac{1}{2}c^\alpha_{\beta\gamma}\Theta_0^\beta \wedge \Theta_0^\gamma + \Theta_0^q \wedge \Theta_q^\alpha\,. \tag{4}$$

The relations (4) do not depend on the choice of an admissible extension of Θ_1 (see [4], Theorem 5). The exterior differentiation of (4) yields

$$d\Theta_q^A = \Theta_q^B \wedge \Theta_B^A + \Theta_0^p \wedge \hat{\Theta}^A_{qp}\,, \tag{4'}$$

where the forms $\hat{\Theta}^A_{qp}$ satisfy $\Theta_0^q \wedge \Theta_0^p \wedge \hat{\Theta}^A_{qp} = 0$, $\Theta_\beta^p = 0$, $\Theta_\beta^\alpha = c^\alpha_{\beta\gamma}\Theta^\gamma$. This implies the following structure equations of G_n^1 (see [4]):

$$d\omega_0^\alpha = \tfrac{1}{2}c^\alpha_{\beta\gamma}\omega_0^\beta \wedge \omega_0^\gamma\,, \quad d\omega_p^A = \omega_p^B \wedge \omega_B^A\,,$$

where $\omega_\beta^\alpha = c^\alpha_{\beta\gamma}\omega_0^\gamma$, $\omega_\beta^q = 0$ and ω_0^α, ω_q^A form the basis of $\mathcal{G}_n^1$ dual to e^0, e_A^q.

Proposition 1. (*Structure equations of the canonical form* Θ_r.) *Let* $\Theta^A_{i_1\ldots i_r}$ *be the components of* Θ'_r. *Then*

$$\text{(5)}\qquad d\Theta^q_{0\ldots0} = \Theta^p_{0\ldots0} \wedge \Theta^q_{0\ldots0p}$$

$$d\Theta^\alpha_{0\ldots0} = \tfrac{1}{2}c^\alpha_{\beta\gamma}\Theta^\beta_{0\ldots0} \wedge \Theta^\gamma_{0\ldots0} + \Theta^p_{0\ldots0} \wedge \Theta^\alpha_{0\ldots0p}\,.$$

Further, if i_p *is the first number different from* 0 *in the sequence* $(i_1, i_2, \ldots, i_r)$, *then*

$$\text{(6)}\qquad d\Theta^A_{i_1\ldots i_p\ldots i_r} = \sum_{(k_{p+1}\ldots k_r)} \Theta^B_{0\ldots0i_p(k_{p+1}\ldots k_r)} \wedge \Theta^A_{0\ldots0B(\hat{k}_{p+1}\ldots\hat{k}_r)} +$$

$$+ \sum_{s=\langle p+1\rangle}^{\langle r\rangle} \sum_{(k_{s+1}\ldots k_r)} \Theta^q_{0\ldots0i_s(k_{s+1}\ldots k_r)} \wedge \Theta^A_{0\ldots0i_p\ldots i_{s-1}q(\hat{k}_{s+1}\ldots\hat{k}_r)} + \Theta^q_{0\ldots0} \wedge \hat{\Theta}^A_{i_1\ldots i_rq}\,,$$

where

$$\Theta^\alpha_{0\ldots0\beta i_q\ldots i_r} = c^\alpha_{\beta\gamma}\Theta^\gamma_{0\ldots0i_q\ldots i_r},\ \Theta^q_{0\ldots0\beta i_t\ldots i_r} = 0\,,$$

$$\hat{\Theta}^A_{i_1\ldots i_{r-1}0q} = \Theta^A_{i_1\ldots i_{r-1}q},\ \Theta^p_{0\ldots0} \wedge \Theta^q_{0\ldots0} \wedge \hat{\Theta}^A_{i_1\ldots i_{r-1}pq} = 0\,;$$

$\sum_{(k_{p+1}\ldots k_r)}$ *denotes the summation over all sequences* $(k_{p+1}, \ldots, k_r)$ *such that* $k_j = i_j$ *or* $k_j = 0$, $\hat{k}_j = i_j - k_j$ *and* $\sum_{s=\langle p+1\rangle}^{\langle r\rangle}$ *means the summation over the integers* $s = p+1, \ldots, r$ *for which* $i_s \neq 0$.

Proof (by induction). For $r = 1$, (5) and (6) are equivalent to (4) and (4′). Assume by induction that the components of Θ'_r satisfy (5) and (6). Then the forms $\omega^\alpha_{0\ldots0}, \omega^\alpha_{i_1\ldots i_r}$ of the dual basis to $e^{0\ldots0}_\alpha, \ldots, e^{i_1\ldots i_r}_\alpha$ satisfy the following structure equations of the group $\tilde{G}^r_n$:

$$d\omega^\alpha_{0\ldots0} = \tfrac{1}{2}c^\alpha_{\beta\gamma}\omega^\beta_{0\ldots0} \wedge \omega^\gamma_{0\ldots0}\,,$$

$$d\omega^A_{i_1\ldots i_p\ldots i_r} = \sum_{(k_{p+1}\ldots k_r)} \omega^B_{0\ldots0i_p(k_{p+1}\ldots k_r)} \wedge \omega^A_{0\ldots0B\hat{k}_{p+1}\ldots\hat{k}_r} +$$

$$+ \sum_{s=\langle p+1\rangle}^{\langle r\rangle} \sum_{(k_{s+1}\ldots k_r)} \omega^q_{0\ldots0i_s(k_{s+1}\ldots k_r)} \wedge \omega^A_{0\ldots0i_p\ldots i_{s-1}q(\hat{k}_{s+1}\ldots\hat{k}_r)}\,,$$

where

$$\omega^\alpha_{0\ldots0\beta i_j\ldots i_r} = c^\alpha_{\beta\gamma}\omega^\gamma_{0\ldots0i_j\ldots i_r}\,,\quad \omega^q_{0\ldots0\beta i_j\ldots i_r} = 0\,.$$

Since $\tilde{W}^{r+1} = W^1(\tilde{W}^r)$, we can use (4). Hence the components of Θ_{r+1} satisfy:

$$d\Theta^q_{0\ldots0} = \Theta^p_{0\ldots0} \wedge \Theta^q_{0\ldots0p}\,,$$

$$d\Theta^\alpha_{0\ldots0} = \tfrac{1}{2}c^\alpha_{\beta\gamma}\Theta^\beta_{0\ldots0} \wedge \Theta^\gamma_{0\ldots0} + \Theta^q_{0\ldots0} \wedge \Theta^\alpha_{0\ldots0q}$$

$$\text{(7)}\qquad d\Theta^A_{i_1\ldots i_p\ldots i_r0} = \Omega\,,$$

where Ω is the form which can be obtained formally from the form on the right side of (6) by adding zero to the end of every multiindex and

$$\hat{\Theta}^A_{i_1\ldots i_rq0} = \Theta^A_{i_1\ldots i_rq}\,.$$

The exterior differentiation of (7) yields

$$\Theta^q_{0\dots0} \wedge \{ \sum_{(k_{p+1}\dots k_r)} [\Theta^B_{0\dots0 i_p(k_{p+1}\dots k_r)q} \wedge \Theta^A_{0\dots0B(\bar{k}_{p+1}\dots\bar{k}_r)0} +$$
$$+ \Theta^B_{0\dots0 i_p(k_{p+1}\dots k_r)0} \wedge \Theta^A_{0\dots0B(\bar{k}_{p+1}\dots\bar{k}_r)q}] +$$
$$+ \sum_{s=\langle p+1\rangle}^{\langle r\rangle} \sum_{(k_{s+1}\dots k_r)} [\Theta^t_{0\dots0 i_s(k_{s+1}\dots k_r)q} \wedge \Theta^A_{0\dots0 i_p\dots i_{s-1}t(\bar{k}_{s+1}\dots\bar{k}_r)0} +$$
$$+ \Theta^t_{0\dots0 i_s(k_{s+1}\dots k_r)0} \wedge \Theta^A_{0\dots0 i_p\dots i_{s-1}t(\bar{k}_{s+1}\dots\bar{k}_r)q}] +$$
$$+ \Theta^t_{0\dots0q} \wedge \Theta^A_{i_1\dots i_r t} - \mathrm{d}\Theta^A_{i_1\dots i_r q}\} = 0 .$$

Using the generalized Cartan's lemma, we get

$$\mathrm{d}\Theta^A_{i_1\dots i_r i_{r+1}} = \sum_{(k_{p+1}k_{r+1})} \Theta^B_{0\dots0 i_p(k_{p+1}\dots k_{r+1})} \wedge \Theta^A_{0\dots0B(\bar{k}_{p+1}\dots\bar{k}_{r+1})} +$$
$$+ \sum_{s=\langle p+1\rangle}^{\langle r+1\rangle} \sum_{(k_{s+1}\dots k_{r+1})} \Theta^t_{0\dots0 i_s(k_{s+1}\dots k_{r+1})} \wedge \Theta^A_{0\dots0 i_p\dots i_{s-1}t(\bar{k}_{s+1}\dots\bar{k}_{r+1})} + \Theta^t_{0\dots0} \wedge \Theta^A_{i_1\dots i_{r+1}t} ,$$

where $i_{r+1} = q$ and $\Theta^q_{0\dots0} \wedge \Theta^t_{0\dots0} \wedge \hat{\Theta}^A_{i_1\dots i_{r+1}t} = 0$. QED.

Remark 1. We have simultaneously proved that the equations (*) are the structure equations of $\tilde{G}^r_n$.

4. The space $W^1(P)$ is the first prolongation of P; $\tau(W^1(P)) = H^1(B) \circledS J^1(P)$. Denote by p_1, p_2 the natural projections

$$p_1 : H^1(B) \circledS J^1(P) \to H^1(B) , \quad p_2 : H^1(B) \circledS J^1(P) \to J^1(P) .$$

Let $\Psi_{\varphi,\sigma}$ be an allowable chart on the principal fibre bundle $W^1(P)(B, G^1_n)$. The chart $\Psi_{\varphi,\sigma}$ will be called a *semi-holonomic allowable chart of the first order on P if*

$$j^1_0[p_1\ \tau(\sigma\varphi)] \in \bar{H}^2(B) , \quad j^1_{\varphi(0)}[p_2\ \tau(\sigma)] \in \bar{J}^2(P) .$$

The set $\bar{W}^2(P) = \bar{W}^1(W^1(P))$ of all 1-jets of all semi-holonomic allowable charts of the first order on P with the source $(0, e_1) \in R_n \times G^1_n$ will be called *the second semi-holonomic prolongation of P*. We can identify

$$\bar{W}^2(P) \equiv \bar{H}^2(B) \circledS \bar{J}^2(P) .$$

By induction, we define $\bar{W}^r(P) = \bar{W}^1(\bar{W}^{r-1}(P))$ and call it, the *r-th semi-holonomic prolongation of P*. It is possible to identify

$$\bar{W}^r(P) \equiv \bar{H}^r(B) \circledS \bar{J}^r(P) .$$

The space $\bar{W}^r(P)$ is a principal fibre bundle $\bar{W}^r(P)$ $(B, \bar{G}^r_n)$, where the group $\bar{G}^r_n$ can be identified with $\bar{L}^r_n \times \bar{T}^r_n(G)$. This identification will be denoted by ξ.

The canonical form Θ_r on $\overline{W}^r(P)$ is defined by (2), where ${}^u\Psi$ is a semi-holonomic admissible chart on $\overline{W}^{r-1}(P)$. Θ_r is a vector-valued form with values in $\overline{w}_n^{r-1}(G) = T_{(0,e_{r-1})}(R^n \times \overline{G}_n^{r-1})$. In particular, we can identify $B \equiv B \times \{e\}$, where $\{e\}$ is the trivial one-element group. Then the canonical form Θ_r on $\overline{W}^r(B \times \{e\})$ coincides with the canonical form of $\overline{H}^r(B)$ introduced in [10].

Let σ be a global section of the principal fibre bundle ${}^*\overline{W}^{r+1}(P)\,(\overline{W}^r(P), {}^*\overline{G}_n^{r+1}) = W^2(\overline{W}^{r-1}(P)) \cap \overline{W}^{r+1}(P)$, where ${}^*G_n^{r+1}$ denotes the kernel of the homomorphism $j_2^1 : (\overline{G}_n^{r-1})_n^2 \cap \overline{G}_n^{r+1} \to \overline{G}_n^r$. Then a form Θ_r' (which will be called an admissible extension of Θ_r on $\overline{W}^r(P)$) is defined analogously to (2'). Θ_r' is a $\overline{w}_n^r(G)$-valued form. Obviously, the diagram

$$\begin{array}{ccc} T(\overline{W}^r(P)) & \longrightarrow & \overline{w}_n^r(G) \\ \big\downarrow{\scriptstyle (\tau^{-1}\xi)_*} & & \big\downarrow{\scriptstyle (\tau^{-1}\xi)_*} \\ T(\widetilde{W}^r(P)) & \longrightarrow & \widetilde{w}_n^r(G) \end{array} \tag{8}$$

is commutative. Diagram (8) implies the identification of Θ_r on $\overline{W}^r(P)$ with the restriction of Θ_r on $\widetilde{W}^r(P)$ to $(\tau^{-1}\xi)_*\,T(\overline{W}^r(P))$. Let

$$\overline{\Theta}_r' = \Theta_r' \,|\, (\tau^{-1}\xi)_*\, T(\overline{W}^r(P))\,, \quad \overline{\Theta}^A_{i_1\ldots i_r} = \Theta^A_{i_1\ldots i_r} \,|\, (\tau^{-1}\xi)_*\, T(\overline{W}^r(P))\,,$$

where $\Theta^A_{i_1\ldots i_r}$ are the components of Θ_r' on $\widetilde{W}^r(P)$. Consequently, writing $\overline{\Theta}^A_{j_1\ldots j_r}$ instead of $\Theta^A_{j_1\ldots j_r}$ in equations (5) and (6) we obtain the structure equations of the canonical form Θ_r on $\overline{W}^r(P)$. Then the structure equations of $\overline{G}_n^r$ have the following form:

$$\mathrm{d}\omega^\alpha_{0\ldots 0} = \tfrac{1}{2} c^\alpha_{\beta\gamma}\,\omega^\beta_{0\ldots 0} \wedge \omega^\gamma_{0\ldots 0} \tag{9}$$

$$\mathrm{d}\overline{\omega}^A_{i_1\ldots i_p\ldots i_r} = \sum_{(k_{p+1}\ldots k_r)} \overline{\omega}^B_{0\ldots 0\, i_p (k_{p+1}\ldots k_r)} \wedge \overline{\omega}^A_{0\ldots 0 B (\bar{k}_{p+1} \bar{k}_r)} + \sum_{s=\langle p+1\rangle}^{\langle r\rangle} \sum_{(k_{s+1}\ldots k_r)} \overline{\omega}^t_{0\ldots 0\, i_s (k_{s+1}\ldots k_r)} \wedge \overline{\omega}^A_{0\ldots 0\, i_p \ldots i_{s-1} t (\bar{k}_{s+1}\ldots \bar{k}_r)}\,,$$

where

$$\overline{\omega}^A_{j_1\ldots j_r} = \omega^A_{j_1\ldots j_r} \,|\, (\tau^{-1}\xi)_*\, T(\overline{G}_n^r)\,,$$

$$\overline{\omega}^\alpha_{0\ldots 0\beta i_j\ldots i_r} = c^\alpha_{\beta\gamma}\overline{\omega}^\gamma_{0\ldots 0 i_j\ldots i_r}\,, \quad \overline{\omega}^q_{0\ldots 0\beta i_j\ldots i_r} = 0\,.$$

Θ' is a vector-valued form with values in $(\tau^{-1}\xi)_*\,\widetilde{w}_n^r(G) \equiv \overline{w}_n^r(G)$. Lemma 1 implies: $\overline{w}_n^r(G)$ is determined by equations

$$\mathrm{d}c^A_{i_1\ldots i_r} = \mathrm{d}c^A_{k_1\ldots k_r}\,, \quad \zeta(i_1 \ldots i_r) = \zeta(k_1 \ldots k_r)\,.$$

For this reason, if $\zeta(i_1 \ldots i_r) = \zeta(k_1 \ldots k_r)$, then

$$\overline{\omega}^A_{i_1\ldots i_r} = \overline{\omega}^A_{k_1\ldots k_r}\,, \quad \overline{\Theta}^A_{i_1\ldots i_r} = \overline{\Theta}^A_{k_1\ldots k_r}\,.$$

Therefore we can drop the zero components in all multiindices in (9). In particular, considering $P = B \times \{e\}$, equations (9) for $A = q$ yield the structure equations of the group $\bar{L}_n^r$. But these equations are identical with the equations (4,36) of (4). Hence the structure equations of the non-holonomic differential group of order r introduced by Lumiste in [5] coincide with the structure equations of $\bar{L}_n^r$.

5. Our previous considerations demonstrate sufficiently the importance of Θ_r' for the determination of the structure equations of Θ_r. Now, we shall show the role of the holonomic prolongations of the second order in the definition of Θ_2'. Let ${}^0\bar{G}_n^2$ be the kernel of the homomorphism $j_2^1 : \bar{G}_n^2 \to G_n^1$. It is easy to see that $\bar{W}^2(P)$ has a structure of a principal fibre bundle $\bar{W}^2(P)\,(W^1(P), {}^0\bar{G}_n^2)$. Let $\bar{\sigma}$ be a global section of $\bar{W}^2(P)\,(W^1(P), {}^0\bar{G}_n^2)$. Let $u \in W^1(P)$, $\bar{\sigma}(u) = j^1_{(0,e_1)}\Psi$, where Ψ is a semi-holonomic allowable chart of the first order on P. Let $X \in T_u(W^1(P))$, $X = j_0^1\,\gamma(t)$. The form

$$^s\Theta_1(X) = j_0^1(\Psi^{-1}\,\gamma(t))$$

will be called a semi-holonomic extension of Θ_1. $^s\Theta_1$ is a vector-valued form on $W^1(P)$ with values in $w_n^1(G)$. Obviously, it holds

$$(j_1^0)_*\, {}^s\Theta_1 = \Theta_1\,.$$

Since our following results are based on direct evaluations we shall consider directly the bundles $W^1(R^n \times G)$ and $\bar{W}^2(R^n \times G)$. Let us identify

$$W^1(R^n \times G) = R^n \times G_n^1 = R^n \times (L_n^1 \mathbin{\bar{\times}} T_n^1(G))\,,$$
$$\bar{W}^2(R^n \times G) = R^n \times \bar{G}_n^2 = R^n \times (\bar{L}_n^2 \mathbin{\bar{\times}} \bar{T}_n^2(G))\,.$$

In what follows, we shall use on $W^1(R^n \times G)$ the local coordinates x^t, a_q^t, b^α, b_q^α where

x^t are the canonical coordinates of $X \in R^n$,

a_q^t are the coordinates of a jet $a \in L_n^1$,

b^α, b_q^α are the coordinates of a jet $b \in T_n^1(G)$.

A non-holonomic allowable 1-chart Ψ on $R^n \times G_n^1$ is given by $\Psi(x, h) = [\varphi(x), \sigma(x)\,h]$, where $\varphi(x)$ is a diffeomorphism $U \to V$, $\sigma(x)$ is a differentiable mapping $U \to G_n^1$ ($U, V \subset R^n$ are open subsets, $0 \in U$) and $\sigma(X)\,h$ is the product in G_n^1. Let us use the following notation: $x' = \varphi(x)$, $h = (a, b)$, $\sigma(x) = ({}^1a, {}^1b) \in L_n^1 \times T_n^1(G)$, $(a', b') = \sigma(x)\,h$, $\Psi(x; a; b) = (x'; a'; b')$. Using (1) we obtain: If Ψ is a semi-holonomic allowable 1-chart on $R^n \times G_n^1$, Ψ is given by the formula

(9)
$$x'^t = {}^0a_q^t x^q + {}^0x^t\,,$$
$$a' = {}^1aa\,,$$
$$b'^\alpha = F^\alpha({}^1ba, b) = ({}^1ba)^\alpha + b^\alpha + \text{terms of higher order}\,,$$
$$b_q'^\alpha = F_q^\alpha({}^1ba, b) = ({}^1ba)_q^\alpha + b_q^\alpha + \text{terms of higher order}\,,$$

where F^α, F^α_q are functions determining the product in G^1_n, ${}^1aa$ or ${}^1ba$ is the composition of jets in L^1_n or $T^1_n(G)$ respectively and

$$ {}^1a^t_q = {}^0a^t_q + {}^0a^t_{qp}x^p\,, \quad {}^1b^\alpha = {}^0b^\alpha + {}^0b^\alpha_p x^p\,, \quad {}^1b^\alpha_q = {}^0b^\alpha_q + {}^0b^\alpha_{qp}x^p\,. $$

Thus, the coordinates of $j^1_0\Psi$ are $({}^0x^t; {}^0a^t_q, {}^0a^t_{qp}; {}^0b^\alpha, {}^0b^\alpha_q, {}^0b^\alpha_{qp})$. Let e_p, e^q_p, e_α, e^q_α be the corresponding basis of $T_{(0,e_1)}(R^n \times G^1_n)$. Then

$$ {}^s\Theta_1 = \Theta^p e_p + \Theta^p_q e^q_p + \Theta^\alpha e_\alpha + \Theta^\alpha_q e^q_\alpha\,, $$

where Θ^p, Θ^p_q, Θ^α, Θ^α_q are some scalar 1-forms on $W^1(R^n \times G^1_n)$. Denote by E_p, E^q_p, E^α, E^α_q the basis of $T(W^1(R^n \times G^1_n))$ dual to Θ^p, Θ^p_q, Θ^α, Θ^α_q.

Let ${}^s\Theta_1$ be determined by a section $\bar\sigma$. Let $\bar\sigma(u) = ({}^0x; a^t_q, {}^0a^t_{qp}; {}^0b^\alpha, {}^0b^\alpha_q, {}^0b^\alpha_{qp})$. Let $\bar\sigma(u) = j^1_0\Psi$. Then Ψ is given by (9). Let $e_t = j^1_0\,\gamma(v)$, $\gamma(v) = (x^p = \delta^p_t v;\ a^p_q = \delta^p_q;\ b^\alpha = 0,\ b^\alpha_q = 0)$. Using (9), we find directly:

$$ E_t(u) = j^1_0\Psi(\gamma(v)) = {}^0a^q_t\frac{\partial}{\partial x^q} + {}^0a^q_{pt}\frac{\partial}{\partial a^q_p} + {}^0b^\alpha_t\frac{\partial}{\partial b^\alpha} + {}^0b^\alpha_{pt}\frac{\partial}{\partial b^\alpha_p} $$

and

$$ [E_t, E_q]_{(u)} = ({}^0a^p_{qt} - {}^0a^p_{tq})\frac{\partial}{\partial x^p} + ({}^0b^\alpha_{qt} - {}^0b^\alpha_{tq})\frac{\partial}{\partial x^\alpha} + 0 \bmod E^p_A\,. $$

But this implies:

$$ {}^s\Theta_1\left[({}^0a^p_{qt} - {}^0a^p_{tq})\frac{\partial}{\partial x^p} + ({}^0b^\alpha_{qt} - {}^0b^\alpha_{tq})\frac{\partial}{\partial x^\alpha}\right] = 0 \bmod e^p_A $$

if and only if ${}^0a^p_{qt} = {}^0a^p_{tq}$, ${}^0b^\alpha_{qt} = {}^0b^\alpha_{tq}$.

Quite analogously to Kolář, [4], we evaluate

$$ [E_q, E^p_t] = -\delta^p_q E_t \bmod E^h_A\,, \quad [E_q, E^p_\alpha] = -\delta^p_q E_\alpha \bmod E^t_A\,, $$
$$ [E_i, E_\alpha] = 0 \bmod E^t_A\,. $$

Now, calculating $\mathrm{d}^s\Theta_1(X, Y)$, we deduce

Proposition 2. *Let ${}^s\Theta_1$ be a semi-holonomic extension of Θ_1 determined by a section $\bar\sigma$. Then the value of $\bar\sigma$ lie in $W^2(P)$ if and only if*

$$ \mathrm{d}\Theta^t = \Theta^p \wedge \Theta^t_p\,, \quad \mathrm{d}\Theta^\alpha = \tfrac{1}{2}c^\alpha_{\beta\gamma}\Theta^\beta \wedge \Theta^\gamma + \Theta^p \wedge \Theta^\alpha_p\,. $$

Corollary. *In particular, if $P = B \times \{e\}$ then it holds: Let ${}^s\Theta_1$ be a semi-holonomic extension of the canonical form Θ_1 of $H^1(B)$ determined by a section $\bar\sigma$. Let Θ^p, Θ^t_p be the components of ${}^s\Theta_1$. Then the values of $\bar\sigma$ lie in $H^2(B)$ if and only if*

$$ \mathrm{d}\Theta^t = \Theta^p \wedge \Theta^t_p\,. $$

Remark 2 (due to Kolář). The preceding corollary is in the following relation to some properties of linear connections. The linear connection without torsion are in a one-to-one correspondence with the reductions of the principal fibre bundle $H^2(B)$ to the subgroup $L_n^1 \subset L_n^2$, see [3]. Libermann, [7], proves that the connections on $H^1(B)$ are in a one-to-one correspondence with the reductions of the principal fibre bundle $\bar{H}^2(B)$ to L_n^1. We can explain this fact in the following way: It is

$$\bar{H}^2(B) = W^1(H^1(B)) = H^1(B) \circledast J^1(H^1(B)) .$$

An element $(u, X) \in H^1(B) \circledast J^1(H^1(B))$ is semi-holonomic if and only if $u = j_1^0 X$. That is why we can identify

$$\bar{H}^2(B) = J^1(H^1(B)) , \quad \text{see also [9]} .$$

Let Γ be a connection on $H^1(B)$ (Γ is an invariant global section $H^1(B) \to J^1(H^1(B))$ see [5]). Hence Γ determines a section $\tilde{\Gamma} : H^1(B) \to \bar{H}^2(B)$. Denote by $R(\Gamma)$ the set $\tilde{\Gamma}(H^1(B))$. One can see easily that $R(\Gamma)$ is the reduction of $\bar{H}^2(B)$ to L_n^1 treated by Libermann. In fact her considerations contain also the assertion that Γ is without torsion if and only if $R(\Gamma) \subset H^2(B)$. We find it remarkable to show that this result follows also from the preceding corollary. Let Θ_2 be the canonical form on $\bar{H}^2(B)$. Denote by Θ_2 its restriction to $R(\Gamma)$. Let ω be the canonical form of Γ and let φ be the canonical form on $H^1(B)$. The diagram

$$\begin{array}{ccc} R^n & \xleftarrow{\ \varphi\ } & T(H^1(B)) \\ \downarrow{\scriptstyle pr_1} & & \downarrow\uparrow \\ R^n \otimes \mathfrak{l}_n^1 & \xleftarrow{\ \tilde{\Theta}_2\ } & T(R(\Gamma)) \\ \downarrow{\scriptstyle pr_2} & & \uparrow\downarrow \\ \mathfrak{l}_n^1 & \xleftarrow{\ \omega\ } & T(H^1(B)) \end{array} \tag{10}$$

is commutative ($\mathfrak{l}_n^1$ denotes the Lie algebra of L_n^1), see [6], Proposition 1. Let φ^p or ω_q^p be the components of φ or ω respectively. Then

$$\mathrm{d}\varphi^p = \varphi^q \wedge \omega_q^p + \mathrm{D}\varphi^p ,$$

where D denotes the absolute differential with respect to the connection Γ and $\mathrm{D}\varphi$ is the torsion form of Γ. Diagram (10) implies that φ^i, ω_j^i are the components of the semi-holonomic extension of ω determined by the section $\tilde{\Gamma}$. The preceding corollary yields: $\mathrm{D}\varphi^p = 0$ (e.i. the connection Γ is without torsion) if and only if $R(\Gamma) \subset H^2(B)$.

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Author's address: Žilina, Marxa-Engelsa 25, ČSSR (Katedra matematiky a deskriptívnej geometrie VŠD).