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Ladislav Bican

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NOTES ON PURITIES

LADISLAV BICAN, Praha

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Throughout this paper, the word "module" always means a unitary Λ-module where Λ is an associative ring with unity. The basic definitions are given in [5] or [1].

1. PROJECTIVELY CLOSED PURITIES

The definitions and notations are given in [5] or [1] and therefore we do not repeat them. The composition of the homomorphisms $\varphi : A \to B$, $\psi : B \to C$ is denoted by $\psi\varphi$.

We start with the following

Lemma 1. *Let U be a submodule of a free module F, $0 \to U \xrightarrow{\chi} F \xrightarrow{\eta} P \to 0$ an exact sequence where χ is the canonical embedding and $i : A \to B$ a monomorphism. Then P is co-projective with respect to i if and only if $i \in \mathfrak{H}_{FU}$.*

Proof. This proof is essentially the same as that of Lemma 1 in [5] and therefore we omit it.

From Theorem 1 and Lemma 3 from [5] it follows that the three following properties of a purity ω are equivalent:

a) ω is projectively closed,

b) ω is a Γ-purity for some class Γ,

c) ω is of the form $\omega^{\mathfrak{M}}$ for some class of modules $\mathfrak{M}$.

Definition 1. Let ω be a projectively closed purity. An arbitrary class Γ of couples (F, U) where U is a submodule of a free module F satisfying $\mathfrak{H}_\omega = \mathfrak{H}_\Gamma$ will be called a basis of ω. Similarly, an arbitrary class $\mathfrak{M}$ of modules satisfying $\omega = \omega^{\mathfrak{M}}$ will be called a $\mathfrak{P}$-basis of ω.

The following simple lemma will be useful in the sequel.

Lemma 2. *If $\Gamma = \{(F, U)\}$ is a basis of a projectively closed purity ω then the class $\mathfrak{M} = \{P, P = F/U, (F, U) \in \Gamma\}$ is a $\mathfrak{P}$-basis of ω. Conversely, if the class $\mathfrak{M}$ is a $\mathfrak{P}$-basis of a projectively closed purity ω then taking to any $P \in \mathfrak{M}$ an exact sequence $0 \to U \xrightarrow{\chi} F \xrightarrow{\eta} P \to 0$ where U is a submodule of a free module F and χ is the canonical embedding we obtain that the class Γ of all such couples (F, U) is a basis of ω.*

Proof follows easily from Lemma 1.

Recall that a family A_α, $\alpha \in \Omega$ of submodules of A is called a covering of A if A_α, $\alpha \in \Omega$ generate A. Further, a module A is called compact if its any countable covering has a finite subcovering.

Theorem 1. *If a projectively closed purity ω has a $\mathfrak{P}$-basis $\mathfrak{M}$ such that any module from $\mathfrak{M}$ is compact then the class $\mathfrak{H}_\omega$ is closed under taking direct sums.*

Proof. Let $0 \to A_\alpha \xrightarrow{i_\alpha} B_\alpha \xrightarrow{\pi_\alpha} C_\alpha \to 0$, $\alpha \in \Omega$ be any set of short exact sequences with $i_\alpha \in \mathfrak{H}_\omega$. Let us put $A = \dot{\sum}_{\alpha\in\Omega} A_\alpha$, $B = \dot{\sum}_{\alpha\in\Omega} B_\alpha$, $C = \dot{\sum}_{\alpha\in\Omega} C_\alpha$, $i = \dot{\sum}_{\alpha\in\Omega} i_\alpha$, $\pi = \dot{\sum}_{\alpha\in\Omega} \pi_\alpha$ and let $P \in \mathfrak{M}$ be an arbitrary module. Then the sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ is exact and the compactness of P guarantees that the image of any homomorphism $f : P \to C$ is contained in the direct sum of a finite number of C_α's (see [1], p. 47). The class $\mathfrak{H}_\omega$ is closed under taking finite direct sums (see (1,1) in [1]) which guarantees the existence of the homomorphism $\varphi : P \to B$ with $\pi\varphi = f$. Therefore $i \in \mathfrak{H}_\omega$ and the proof is complete.

Theorem 2. *If a projectively closed purity ω has a $\mathfrak{P}$-basis $\mathfrak{M}$ such that any module from $\mathfrak{M}$ is compact then the direct sum $A = \dot{\sum}_{\alpha\in\Omega} A_\alpha$ is ω-flat if and only if any A_α, $\alpha \in \Omega$ is ω-flat.*

Proof. Recall that a module A is ω-flat if $i \in \mathfrak{H}_\omega$ for any exact sequence $0 \to K \xrightarrow{i} B \to A \to 0$. If A is ω-flat then any A_α, $\alpha \in \Omega$ is ω-flat by (1,15) from [1]. Conversely, let us suppose that any A_α, $\alpha \in \Omega$ is ω-flat. Taking for any A_α, $\alpha \in \Omega$ an exact sequence $0 \to U_\alpha \xrightarrow{i_\alpha} F_\alpha \to A_\alpha \to 0$ with a free module F_α we have $i_\alpha \in \mathfrak{H}_\omega$. Then the sequence $0 \to \dot{\sum}_{\alpha\in\Omega} U_\alpha \xrightarrow{\sum_{\alpha\in\Omega} i_\alpha} \dot{\sum}_{\alpha\in\Omega} F_\alpha \to A \to 0$ is exact and $\dot{\sum}_{\alpha\in\Omega} i_\alpha \in \mathfrak{H}_\omega$ by Theorem 1. Hence A is ω-flat by (1,12) from [1].

Theorem 3. *If a projectively closed purity ω has a basis Γ where Γ is a set then there exists a free module F and its submodule U such that $\mathfrak{H}_\omega = \mathfrak{H}_{FU}$.*

Proof. By Lemma 2, ω has a $\mathfrak{P}$-basis $\mathfrak{M}$ where $\mathfrak{M}$ is a set. By (1,5) from [1] the module $P = \dot{\sum}_{P'\in\mathfrak{M}} P'$ is ω-projective, $P \in \mathfrak{P}_\omega$, so that $\mathfrak{H}_\omega \subseteq \mathfrak{H}^{\{P\}}$. Conversely, again by (1,5) from [1] we have $\mathfrak{M} \subseteq \mathfrak{P}_{\omega\{P\}}$ hence $\mathfrak{H}^{\{P\}} \subseteq \mathfrak{H}_\omega$ and Lemma 1 completes the proof.

Theorem 4. *If a projectively closed purity ω has a $\mathfrak{P}$-basis $\mathfrak{M}$ (or a basis Γ) which is a set, then the purity ω is projective.*

Proof. In view of Lemma 2 and Theorem 3 we can assume that the module P' is a $\mathfrak{P}$-basis of ω. Let A be an arbitrary module and $\pi' : F \to A$ an epimorphism of some free module F onto A. By (1,5) from [1] the module $P = F \dotplus \sum_{f \in \mathrm{Hom}(P',A)} P'_f$ where $P'_f = P'$ for any $f \in \mathrm{Hom}(P', A)$ is ω-projective. For an element $(q, (p'_f)) \in P$, $q \in F$, $p'_f \in P'_f$, let us put $\pi((q, (p'_f))) = \pi'(q) + \sum_{f \in \mathrm{Hom}(P',A)} f(p'_f)$ (this can be made since only a finite number of p'_f's is non-zero). Here $\pi : P \to A$ is an epimorphism since π' is. If we denote $K = \mathrm{Ker}\, \pi$ and i the corresponding canonical embedding we get an exact sequence $0 \to K \xrightarrow{i} P \xrightarrow{\pi} A \to 0$. It remains to show that $i \in \mathfrak{H}_\omega$. However, taking $f \in \mathrm{Hom}(P', A)$ arbitrarily and denoting by ι_f the canonical embedding of $P' = P'_f$ into P we obviously have $\pi\iota_f = f$ and the proof is complete.

Remark. We have just proved something more, namely: *If a projectively closed purity ω has a set as a $\mathfrak{P}$-basis then there exists a module $P' \in \mathfrak{P}_\omega$ such that to any module A there is an exact sequence $0 \to K \xrightarrow{i} P \to A \to 0$ with $i \in \mathfrak{H}_\omega$ and $P = F \dotplus \sum_{f \in \mathrm{Hom}(P',A)} P'$ where F is free and $P'_f = P'$ for any $f \in \mathrm{Hom}(P', A)$.*

Theorem 5. *The following two conditions are logically equivalent:*

a) *Any projectively closed purity has a set as a $\mathfrak{P}$-basis,*

b) *there exists a cardinal number m such that any module of power at least m is a direct sum of modules of powers less than m.*

Proof. First, let us show that a) $\Rightarrow$ b). For this purpose, let us assume the purity ω to have the class of all modules as a $\mathfrak{P}$-basis. By hypothesis, Theorem 3 and Lemma 2, there exists a module P' which is a $\mathfrak{P}$-basis of ω. Let m be the first uncountable cardinal greater than $\max(|P'|, |\Lambda|)$ [1]. By the remark preceding this theorem, to any module A of power at least m there exists an exact sequence $0 \to K \xrightarrow{i} P \to A \to 0$ with $i \in \mathfrak{H}_\omega$ and $P = F \dotplus \sum_{f \in \mathrm{Hom}(P',A)} P'_f$ where F is free and $P'_f = P'$ for any $f \in \mathrm{Hom}(P', A)$. Hence A is isomorphic to a direct summand of P since $A \in \mathfrak{P}_\omega$ by hypothesis. Therefore A is a direct sum of modules of powers less than m owing to Theorem 4.3 from [4].

Conversely, let us suppose b) and let ω be any projectively closed purity. Let $\mathfrak{M}$ be the set of all pair-wise non-isomorphic modules from $\mathfrak{P}_\omega$ the powers of which are

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[1]) $|M|$ denotes the power of the set M.

less than m. Clearly, $\mathfrak{H}_\omega \subseteq \mathfrak{H}^{\mathfrak{M}}$. On the other hand, any module $P \in \mathfrak{P}_\omega$ of power at least m is, by hypothesis, a direct sum $P = \dot{\sum}_{\alpha \in \Omega} P_\alpha$ of modules of powers less than m. By (1.5) from [1] any P_α lies in $\mathfrak{P}_\omega$ and hence it is isomorphic to an element from $\mathfrak{M}$. Using (1.5) from [1] again, we get $P \in \mathfrak{P}_{\omega\mathfrak{M}}$, hence $\mathfrak{P}_\omega \subseteq \mathfrak{P}_{\omega\mathfrak{M}}$ and $\mathfrak{H}^{\mathfrak{M}} \subseteq \mathfrak{H}_\omega$.

Definition 2. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two sets of modules containing Λ. We say that $\mathfrak{N}$ depends on $\mathfrak{M}$ if any module from $\mathfrak{N}$ is isomorphic to a direct summand of a direct sum of modules from $\mathfrak{M}$. Further, we say that $\mathfrak{M}$ and $\mathfrak{N}$ are equivalent if $\mathfrak{M}$ depends on $\mathfrak{N}$ and conversely, $\mathfrak{N}$ depends on $\mathfrak{M}$.

Theorem 6. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two sets of modules containing Λ. Then $\mathfrak{H}^{\mathfrak{M}} \subseteq \mathfrak{H}^{\mathfrak{N}}$ if and only if $\mathfrak{N}$ depends on $\mathfrak{M}$.*

Particularly, $\mathfrak{H}^{\mathfrak{M}} = \mathfrak{H}^{\mathfrak{N}}$ if and only if $\mathfrak{M}$ and $\mathfrak{N}$ are equivalent.

Proof. The special assertion is a trivial consequence of the general one. First, let us suppose that $\mathfrak{H}^{\mathfrak{M}} \subseteq \mathfrak{H}^{\mathfrak{N}}$ and let $N \in \mathfrak{N}$ be an arbitrary module. By hypothesis, the proof of Theorem 3 and the remark after Theorem 4 there exists an exact sequence $0 \to K \xrightarrow{i} P \to N \to 0$ where $i \in \mathfrak{H}^{\mathfrak{M}}$ and P is a direct sum of modules from $\mathfrak{M}$. Since $\mathfrak{H}^{\mathfrak{M}} \subseteq \mathfrak{H}^{\mathfrak{N}}$, N is co-projective with respect to i and therefore it is isomorphic to a direct summand of P. Hence $\mathfrak{N}$ depends on $\mathfrak{M}$.

Conversely, if $\mathfrak{N}$ depends on $\mathfrak{M}$ then (1,5) from [1] yields $\mathfrak{N} \subseteq \mathfrak{P}_{\omega\mathfrak{M}}$ and therefore $\mathfrak{H}^{\mathfrak{M}} \subseteq \mathfrak{H}^{\mathfrak{N}}$.

Theorem 7. *The intersection of any set of projective purities is a projective purity.*

Proof. Let ω_α, $\alpha \in M$ be a set of projective purities and let us put $\omega = \bigcap_{\alpha \in M} \omega_\alpha$. It is clear that $i \in \mathfrak{H}_\omega$ if and only if any module from $\bigcup_{\alpha \in M} \mathfrak{P}_{\omega_\alpha}$ is co-projective with respect to i so that $\bigcup_{\alpha \in M} \mathfrak{P}_{\omega_\alpha}$ is a $\mathfrak{P}$-basis of ω. Let A be an arbitrary module. The projectivity of ω_α, $\alpha \in M$ implies the existence of exact sequences $0 \to K_\alpha \xrightarrow{i_\alpha} P_\alpha \xrightarrow{\pi_\alpha} A \to 0$ with $i_\alpha \in \mathfrak{H}_{\omega_\alpha}$ and $P_\alpha \in \mathfrak{P}_{\omega_\alpha}$. For $P = \dot{\sum}_{\alpha \in M} P_\alpha$ let us define a mapping $\pi : P \to A$ by the formula $\pi(\{p_\alpha\}_{\alpha \in M}) = \sum_{\alpha \in M} \pi_\alpha(p_\alpha)$ (this can be done since only a finite number of p_α''s is non-zero). It is not too hard to show that π is a homomorphism and, moreover, it is an epimorphism since π_α, $\alpha \in M$ are. Let us introduce the following notation: ι_α is the canonical embedding of P_α into P, $K = \operatorname{Ker} \pi$ and i is the natural embedding of K into P. Since $\bigcup_{\alpha \in M} \mathfrak{P}_{\omega_\alpha}$ is a $\mathfrak{P}$-basis of ω we have $P \in \mathfrak{P}_\omega$ by (1,5) from [1] so that it suffices to show that $i \in \mathfrak{H}_\omega$. Let $P' \in \bigcup_{\alpha \in M} \mathfrak{P}_{\omega_\alpha}$ be an arbitrary module and $\varphi \in \operatorname{Hom}(P', A)$ an arbitrary element. Then $P' \in \mathfrak{P}_{\omega_\alpha}$ for some $\alpha \in M$ so that there exists a homomorphism

$\psi' : P' \to P_\alpha$ with $\pi_\alpha \psi' = \varphi$ (since $i_\alpha \in \mathfrak{H}_{\omega_\alpha}$). Putting $\psi = \iota_\alpha \psi' : P' \to P$ we have $\pi\psi = \pi\iota_\alpha \psi' = \pi_\alpha \psi' = \varphi$, which completes the proof.

Now we shall present two theorems concerning $\mathscr{E}$-purity.

Definition 3. We shall say that a projectively closed purity ω is $\mathscr{E}$-purity if it has a $\mathfrak{P}$-basis $\mathfrak{M}$ containing only cyclical modules.

Recall that a purity ω is called cyclically projective if to any module A there exists an exact sequence $0 \to K \xrightarrow{i} P \to A \to 0$ where $i \in \mathfrak{H}_\omega$ and P is a direct sum of cyclic ω-projective modules.

Theorem 8. *A projectively closed purity ω is $\mathscr{E}$-purity if and only if it is cyclically projective.*

Proof. Firstly, let ω be an $\mathscr{E}$-purity and let $\mathfrak{M}$ be its $\mathfrak{P}$-basis containing only cyclical modules. Without loss of generality we can assume that $\mathfrak{M}$ is a set (in the opposite case take pair-wise non-isomorphic modules from $\mathfrak{M}$). Now the proof runs on the same lines as that of Theorem 3,4 and therefore we omit it.

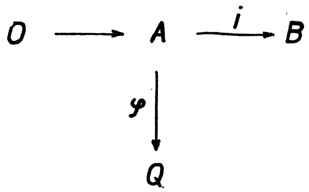
Conversely, let us suppose that ω is cyclically projective and let us denote by $\mathfrak{M}$ the class of all cyclic modules from $\mathfrak{P}'_\omega$. The inclusion $\mathfrak{M} \subseteq \mathfrak{P}_\omega$ gives $\mathfrak{H}_\omega \subseteq \mathfrak{H}^{\mathfrak{M}}$. On the other hand, to any module $A \in \mathfrak{P}_\omega$ there exists an exact sequence $0 \to K \xrightarrow{i} P \to$ $\to A \to 0$ where $i \in \mathfrak{H}_\omega$ and P is a direct sum of cyclic modules from $\mathfrak{P}_\omega$, i.e. the modules from $\mathfrak{M}$. From $i \in \mathfrak{H}_\omega$ it follows that A is isomorphic to a direct summand of P so that $A \in \mathfrak{P}_{\omega\mathfrak{M}}$ by (1,5) from [1]. Hence $\mathfrak{P}_\omega \subseteq \mathfrak{P}_{\omega\mathfrak{M}}$ from which $\mathfrak{H}^{\mathfrak{M}} \subseteq \mathfrak{H}_\omega$ and the proof is complete.

Theorem 9. *For any $\mathscr{E}$-purity, the direct sum $A = \dot{\sum}_{\alpha\in\Omega} A_\alpha$ is $\mathscr{E}$-flat if and only if any module A_α, $\alpha \in \Omega$ is $\mathscr{E}$-flat.*

Proof. It suffices to use Theorem 2 since any cyclic module is compact.

2. INJECTIVELY CLOSED PURITIES

First of all we shall repeat some definitions. A module Q is called injective with respect to a monomorphism $i : A \to B$ if for any diagram



there exists a homomorphism $\psi : B - Q$ making the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B \\ & & \varphi \downarrow & \swarrow \psi & \\ & & Q & & \end{array}$$

commutative. For a purity ω let us call a module Q ω-injective if it is injective with respect to any $i \in \mathfrak{H}_\omega$. The class of all ω-injective modules is denoted by $\mathfrak{Q}_\omega$. If $\mathfrak{M}$ is an arbitrary class of modules then the class $\mathfrak{H}_{\mathfrak{M}}$ of all monomorphisms i such that any $M \in \mathfrak{M}$ is injective with respect to i, defines a purity (see (1,16) in [1]), which we denote by $\omega_{\mathfrak{M}}$. The purity $\underline{\omega} = \omega_{\mathfrak{Q}_\omega}$ is called the injective closure of ω. Finally, a purity ω is called injectively closed, if $\omega = \underline{\omega}$, and a purity ω is called injective if to any module A there exists an exact sequence $0 \to A \xrightarrow{i} Q$ with $i \in \mathfrak{H}_\omega$ and $Q \in \mathfrak{Q}_\omega$.

Theorem 10. *A purity ω is injectively closed if and only if it is of the from $\omega_{\mathfrak{M}}$ for some $\mathfrak{M}$ of modules.*

Proof. For an injectively closed purity ω we have $\omega = \omega_{\mathfrak{Q}_\omega}$. On the other hand we clearly have $\mathfrak{H}_{\mathfrak{M}} \subseteq \mathfrak{H}_{\mathfrak{Q}_{\omega_{\mathfrak{M}}}}$ while the converse inclusion follows at once from $\mathfrak{M} \subseteq \mathfrak{Q}_{\omega_{\mathfrak{M}}}$.

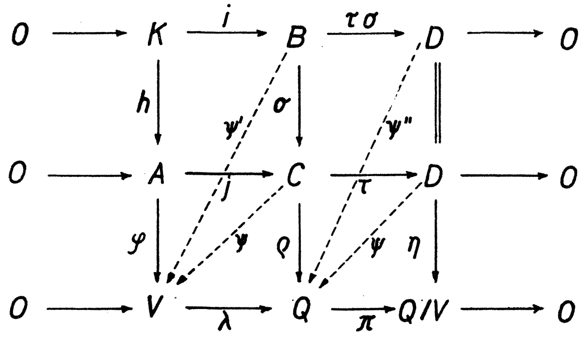
Theorem 11. *Any injectively closed purity ω is bi-triangular.*

Proof. In view of Theorem 10 and (1,16) from [1] it suffices to show that ω is co-triangular. Let us consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & B & \xrightarrow{\tau\sigma} & D & \longrightarrow & 0 \\ & & h \downarrow & & \sigma \downarrow & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow[j]{} & C & \xrightarrow[\tau]{} & D & \longrightarrow & 0 \end{array}$$

with exact rows where τ and σ are given homomorphisms and $i \in \mathfrak{H}_\omega$. The existence of h is guaranteed by $A \cong \operatorname{Ker} \tau$ and $\tau(\sigma i) = 0$. We are going to show that $j \in \mathfrak{H}_\omega$. Let $V \in \mathfrak{Q}_\omega$ be an arbitrary module, Q an arbitrary injective module containing V,

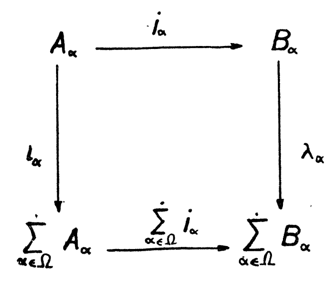
$\lambda : V \to Q$ the canonical embedding, $\pi : Q \to Q/V$ the canonical projection and $\varphi : A \to V$ an arbitrary homomorphism. Let us consider the following diagram:



The existence of ϱ satisfying $\varrho j = \lambda\varphi$ follows from the injectivity of Q while the existence of η satisfying $\eta\tau = \pi\varrho$ follows from $\pi\varrho j = \pi\lambda\varphi = 0$. Therefore the diagram (4) (with full lines only) is a commutative diagram with exact rows. From $i \in \mathfrak{H}_\omega$ the existence of a homomorphism $\psi' : B \to V$ with $\varphi h = \psi' i$ follows. Further, $(\varrho\sigma - \lambda\psi')i = \varrho\sigma i - \lambda\varphi h = 0$ implies the existence of a homomorphism $\psi'' : D \to Q$ with $\psi''\tau\sigma = \varrho\sigma - \lambda\psi'$. We have $\pi\psi''\tau\sigma = \pi\varrho\sigma - \pi\lambda\psi' = \eta\tau\sigma$, so that $\pi\psi'' = \eta$ since $\tau\sigma$ is an epimorphism. Further, from $\pi(\varrho - \psi''\tau) = \pi\varrho - \eta\tau = 0$ we obtain $\lambda\psi = \varrho - \psi''\tau$ for a homomorphism $\psi : C \to V$. Finally $\lambda\psi j = \varrho j - \tau j = \lambda\varphi$ yields $\psi j = \varphi$ since λ is a monomorphism and the proof is therefore complete.

Theorem 12. *For an injectively closed purity ω the class $\mathfrak{H}_\omega$ is closed under taking direct sums.*

Proof. Let $i_\alpha : A_\alpha \to B_\alpha$, $\alpha \in \Omega$ be an arbitrary set of elements of $\mathfrak{H}_\omega$. For any $\alpha \in \Omega$ we have a commutative diagram



where ι_α, λ_α are canonical embeddings. For any $V \in \mathfrak{Q}_\omega$ and any $\varphi : \dot{\sum}_{\alpha\in\Omega} A_\alpha \to V$ there exist homomorphisms $\psi_\alpha : B_\alpha \to V$ with $\psi_\alpha i_\alpha = \varphi\iota_\alpha$ (since $i_\alpha \in \mathfrak{H}_\omega$). The universality of direct sums yields the homomorphism $\psi : \sum_{\alpha\in\Omega} B_\alpha \to V$ with $\psi\lambda_\alpha = \psi_\alpha$. Finally, from $\varphi\iota_\alpha = \psi_\alpha i_\alpha = \psi\lambda_\alpha i_\alpha = \psi(\sum_{\alpha\in\Omega} i_\alpha)\, \iota_\alpha$ and from the universality of direct sums (for $\dot{\sum}_{\alpha\in\Omega} A_\alpha$) we get $\psi(\dot{\sum}_{\alpha\in\Omega} i_\alpha) = \varphi$ and the proof is complete.

Theorem 13. *Let ω be an injectively closed purity. Then the direct sum $A = \dot{\sum}_{\alpha\in\Omega} A_\alpha$ is ω-flat if and only if any A_α, $\alpha \in \Omega$ is ω-flat.*

Proof. Theorem 13 follows from Theorem 12 in a similar way as Theorem 2 follows from Theorem 1.

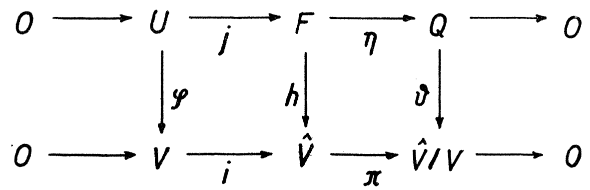
Theorem 14. *Let ω be an injectively closed purity. Then the following three properties of a module Q are equivalent:*

1) *Q is ω-flat,*
2) $\operatorname{Ext}(Q, V) = 0$ [2]) *for any $V \in \mathfrak{Q}_\omega$,*
3) *for any $V \in \mathfrak{Q}_\omega$, Q is co-projective with respect to the canonical embedding $V \to \hat{V}$* [3]).

Proof. 1) $\Rightarrow$ 2): If Q is ω-flat then $\omega \operatorname{Ext}(Q, X) =$[4]) $\operatorname{Ext}(Q, X)$ for any module X. For any $V \in \mathfrak{Q}_\omega$ we have $\omega \operatorname{Ext}(Q, V) = 0$ so that 2) is true.

2) $\Rightarrow$ 3): From the exact sequence $0 \to V \to \hat{V} \to \hat{V}/V \to 0$ we obtain the exact sequence $\operatorname{Hom}(Q, \hat{V}) \to \operatorname{Hom}(Q, \hat{V}/V) \to \operatorname{Ext}(Q, V)$ which yields 3).

3) $\Rightarrow$ 1): To the module Q let us choose an exact sequence $0 \to U \xrightarrow{j} F \xrightarrow{\eta} Q \to 0$ where F is free and let us consider the diagram



where $\varphi : U \to V$ is a given homomorphism. Since $\hat{V}$ is injective, there exists $h : F \to \hat{V}$ with $hj = i\varphi$. From $\pi hj = \pi i\varphi = 0$ it follows that $\vartheta\eta = \pi h$ for some $\vartheta : Q \to \hat{V}/V$ so that the diagram (6) is a commutative diagram with exact rows. By hypothesis 3) there exists a homomorphism $\psi' : Q \to \hat{V}$ with $\pi\psi' = \vartheta$. From $\pi(h - \psi'\eta) = \pi h - \vartheta\eta = 0$ we get $h - \psi'\eta = i\psi$ for some $\psi : F \to V$. Finally, $\psi j = \varphi$ since i is a monomorphism and $i\psi j = hj - \psi'\eta j = i\varphi$.

[2]) In this paper we shall write simply Ext (B, A) instead of $\operatorname{Ext}^1_\Lambda(B, A)$.

[3]) $\hat{V}$ denotes the injective closure of V.

[4]) $\omega \operatorname{Ext}(B, A)$ is the subset of $\operatorname{Ext}(B, A)$ formed by all the sequences $0 \to A \xrightarrow{i} X \to B \to 0$ with $i \in \mathfrak{H}_\omega$.

Definition 4. We shall say that a class $\mathfrak{M}$ of modules is a basis of an injectively closed purity ω, if $\omega = \omega_{\mathfrak{M}}$.

Now we shall formulate three theorems without proofs since they are dual to those of Theorems 3,4 and 7 respectively.

Theorem 15. *If an injectively closed purity ω has a set as a basis then it also has a basis containing exactly one element.*

Theorem 16. *If an injectively closed purity ω has a set as a basis then it is injective.*

Theorem 17. *The intersection of any set of injective purities is an injective purity.*

3. $\mathscr{E}$-DIVISIBLE MODULES

It is a well-known fact in the Abelian groups theory that a group D is divisible if and only if it contains no maximal proper subgroups. This section is devoted to a generalization of this fact.

Recall that a module D is ω-divisible (ω is any purity) if it is ω-pure in any its extension (A is ω-pure in B if the canonical embedding $i : A \to B$ lies in $\mathfrak{H}_\omega$).

Throughout this section let $\mathscr{E} = \{\Lambda\mu, \mu \in M\}$ with $M \subseteq \Lambda$ be any set of maximal principal left ideals of Λ satisfying $\Lambda\mu \subseteq \mu\Lambda$.

Definition 5. We shall say that a submodule B of a module A is an $\mathscr{E}$-submodule if the order of any non-zero element of A/B belongs to $\mathscr{E}$. Further, we shall say that B is an $\mathscr{E}$-maximal submodule of A if B is an $\mathscr{E}$-submodule of A and it is maximal in A.

Theorem 18. *If a module D contains no proper $\mathscr{E}$-maximal submodule then D is $\mathscr{E}$-divisible.*

Proof. Let us suppose to the contrary that D is not $\mathscr{E}$-divisible. By (1.53) from [1] there exist $\mu \in M$ and $d \in D$ such that $d \notin \mu D$. From this and from $\Lambda\mu \subseteq \mu\Lambda$ it follows $d \notin \Lambda\mu D$ and hence $D/\Lambda\mu D \neq 0$. Further, from the inclusion $\Lambda\mu \subseteq \mu\Lambda$ it easily follows that $\mu\Lambda$ is a left ideal of Λ and therefore $\Lambda\mu = \mu\Lambda$, $\Lambda\mu$ being maximal. It is easy to see that $\Lambda/\Lambda\mu$ is a division ring (= non-commutative field). The factor-module $D/\Lambda\mu D$ can be considered as a $\Lambda/\Lambda\mu$-module by defining $(\lambda + \Lambda\mu)(d + \Lambda\mu D) = \lambda d + \Lambda\mu D$. By the well-known theorem on modules over a division ring (see e.g. [7]) the $\Lambda/\Lambda\mu$-module $D/\Lambda\mu D$ is completely decomposable. Therefore it contains a $\Lambda/\Lambda\mu$-submodule $D'/\Lambda\mu D$ with $D/\Lambda\mu D/D'/\Lambda\mu D \cong \Lambda/\Lambda\mu$. It is not too hard to show that D' is Λ-submodule of D. Considering D, D', $\Lambda\mu D$ as Λ-modules, we have $D/\Lambda\mu D/D'/\Lambda\mu D \cong D/D' \cong \Lambda/\Lambda\mu$. This implies that D' is an $\mathscr{E}$-maximal submodule of D — a contradiction proving our theorem.

The following example shows that the converse of the preceding theorem does not hold in general.

Example. As the ring Λ we take the direct sum $\Lambda = C_2 \dotplus C_2 \dotplus C_3$ where C_2 and C_3 are prime fields of the characteristic 2 and 3 respectively. The ideal $C_2 \dotplus C_3$ generated by $\mu = (0, 1, 1)$ satisfies all the conditions for the system $\mathscr{E}$. Direct calculation gives that $d = (0, 1, 0)$ is the only element from $D = C_2 \dotplus C_2$ satisfying $(0 : \mu) \subseteq (0 : d)$ [5]. By (1.53) from [1] D is $\mathscr{E}$-divisible since $d = \mu d$. On the other hand, the second direct summand C_2 is the maximal submodule of D and it is easy to see that the order of the only non-zero element of D/C_2 is just $\Lambda\mu$.

Let us denote by $N = \bigvee_{\mu \in M} (0 : \mu)$ the left ideal of Λ generated by all the ideals $(0 : \mu)$, $\mu \in M$.

Theorem 19. *Let D be an $\mathscr{E}$-divisible module satisfying $N \subseteq (0 : d)$ for any $d \in D$. Then D contains no proper $\mathscr{E}$-maximal submodules.*

Proof. Let us suppose to the contrary that there exists a proper $\mathscr{E}$-maximal submodule H of D. If $d \in D \dot{-} H$ is an arbitrary element then by Definition 5 there exists $\mu \in M$ with $(H : d) = \Lambda\mu$. The $\mathscr{E}$-divisibility of D, the hypothesis of our theorem and (1.53) from [1] imply the existence of $d' \in D$ with $d = \mu d'$. Here $d' \notin H$ since $d \notin H$. On the other hand, $d' = h + \lambda d$, $\lambda \in \Lambda$, $h \in H$, H being maximal in D. Therefore, $d = \mu d' = \mu h + \mu\lambda q = \mu h + \lambda'\mu d \in H$, (since $\Lambda\mu = \mu\Lambda$) which is a contradiction proving our theorem.

Theorem 20. *Let D be an $\mathscr{E}$-divisible module satisfying $N \subseteq (0 : d)$ for any $d \in D$. Then any epimorphic image of D is $\mathscr{E}$-divisible.*

Proof. Let $\varphi : D \to D'$ be an arbitrary epimorphism, $d' \in D'$ an arbitrary element and d any inverse image of d' under φ. Then $(0 : d') = (\operatorname{Ker} \varphi : d) \supseteq (0 : d) \supseteq N$. By (1.53) from [1] we have $d \in \mu D$ for any $\mu \in M$, hence $d' \in \mu D'$ for any $\mu \in M$ and D' is $\mathscr{E}$-divisible by (1.53) from [1] again.

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Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).

[5]) Recall that if B is a submodule and $\mathfrak{M}$ a subset of a module A then $(B : \mathfrak{M}) = \{\lambda \in \Lambda;\ \lambda x \in B$ for any $x \in \mathfrak{M}\}$.