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ON THE CHARACTERIZATION OF WEAK CLOSURE
IN HILBERT SPACE*)

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The purpose of this note is an attempt to characterize the weak closure of bounded sets in a separable Hilbert space. The problem was motivated by a recent paper of G. E. ŠILOV [3] in which he studies the Lévy-Laplace operator on a particular class of norm-closed domains in Hilbert space. In § 7 of [3] it was shown that these domains are weakly compact and that the nullspace of the Lévy-Laplacian is a dense algebra in the algebra of all weakly continuous functions on the given domain. It turns out that the (sup-norm closure of the) algebra considered by Šilov, or equivalently the algebra generated by all continuous linear functionals, on a given bounded (not weakly closed) set B is precisely that algebra of bounded weakly continuous functions which corresponds to the compactification given by the weak closure of B . It would be natural to expect this compactification of B to be "distinguished" in some way; examples suggest, however, that the weak closure may not enjoy any special properties not shared by other compactifications. We conclude the note with a few open problems.

Throughout this paper, \mathcal{H} will denote an infinite dimensional separable Hilbert space over the reals, with real valued inner product. If B is a bounded subset of \mathcal{H} , we write B_w for the topological space $\{B, \tau\}$, where τ is the weak topology of \mathcal{H} restricted to B , and \bar{B}^w for the weak closure of B together with the weak topology. It is well known that \bar{B}^w is compact and metrizable. If $q(x, y)$ is any admissible metric on \bar{B}^w (they are all equivalent there) then \bar{B}^w is isometric to the completion of B_w with respect to q . From now on let K be a closed ball in \mathcal{H} , containing \bar{B}^w , and let q be an admissible metric for the weak topology on K , with $q(x, y) \leq 1$ for all $x, y \in K$.

If S is a set, denote by $F(S)$ the Banach algebra of all bounded (real or complex valued) functions on S , endowed with the sup-norm. If $S = T$ is a topological space, $C(T)$ will denote the subalgebra of all bounded continuous functions. If $\{X, \mathcal{U}\}$ is

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a uniform space, we denote by $\mathcal{U}(X)$ the closed subalgebra of $F(X)$ which consists of all bounded \mathcal{U} - \mathcal{V} -uniformly continuous functions, where \mathcal{V} is the usual uniformity on the space of real or complex numbers.

Definition. A real or complex valued function f defined on the bounded subset B of \mathcal{H} is said to be *simple* [3] if there are (i) a finite number of vectors a_1, \dots, a_n in \mathcal{H} , (ii) a closed bounded domain Q in Euclidean \mathbf{R}^n , and (iii) a continuous function \tilde{f} on Q such that $1^\circ x \in B$ implies $((a_1, x), \dots, (a_n, x)) \in Q$ and $2^\circ f(x) = \tilde{f}((a_1, x), \dots, (a_n, x))$.

The class $D(B)$ of simple functions forms an algebra (operations pointwise) – the arguments of § 1 in [3] remain valid for any bounded $B \subset \mathcal{H}$; and in the case when B is a norm-closed domain in \mathcal{H} , Šilov showed that $D(B)$ is equal to the nullspace of the Lévy-Laplace operator L (the latter is an unbounded operator defined in the larger algebra of “regular” functions; cf. [3]). Let $\bar{D}(B)$ denote the sup-norm closure of $D(B)$.

If \mathcal{U}_I denotes the metric uniformity induced on B_w by that of \bar{B}^w , then $D(B) \subset \mathcal{U}_I(B)$, and as $\mathcal{U}_I(B)$ is complete, also $\bar{D}(B) \subset \mathcal{U}_I(B)$. On the other hand, since the functions \hat{a} given by $\hat{a}(x) = (a, x)$ ($a \in \mathcal{H}$) separate points of \mathcal{H} and by the Stone-Weierstrass theorem form a system of generators for $\mathcal{U}_I(\bar{B}^w) = C(\bar{B}^w)$ – and hence also for the restriction of this algebra to B_w , namely $\mathcal{U}_I(B_w)$ – we obtain $\bar{D}(B) \supset \mathcal{U}_I(B_w)$. We have proved the following.

Proposition 1. *The compactification \bar{B}^w of B_w corresponds to the closed subalgebra $\mathcal{U}_I(B_w) = \bar{D}(B)$ of $C(B_w)$. This algebra is generated by the continuous linear functionals on \mathcal{H} .*

Proposition 2. *Let $\{X, \mathcal{U}\}$ be any uniform Hausdorff space. If $f \in \mathcal{U}(X)$ and $f^{-1} \in F(X)$, then $f^{-1} \in \mathcal{U}(X)$.*

Proof. If $f \in F(X)$, then $f^{-1} \in F(X)$ if and only if $\inf \{|f(x)| : x \in X\} = c > 0$. Suppose $f \in \mathcal{U}(X)$ and $f^{-1} \in F(X)$ now, and set $d = \sup \{|f(x)| : x \in X\}$. Then $f^{-1}(x) = (f(x))^{-1} = (r_{c,d} \circ f)(x)$ where $r_{c,d}(z) = z^{-1}$ and $r_{c,d}$ is uniformly continuous on the annulus $c \leq |z| \leq d$. Consequently the composition $f^{-1} = r_{c,d} \circ f$ is \mathcal{U} -uniformly continuous. (In \mathbf{C} we always use the usual uniformity given by the metric $p(z_1, z_2) = |z_2 - z_1|$.)

Proposition 3. *If $f \in D(B)$ [$f \in \bar{D}(B)$] and $f^{-1} \in F(B)$, then $f^{-1} \in D(B)$ [$f^{-1} \in \bar{D}(B)$].*

The case of $D(B)$ was proved directly in [2]; the case for $\bar{D}(B) = \mathcal{U}_I(B_w)$ follows from the preceding propositions.

Now let $D_0(B)$ denote the set of simple functions which are representable in the form $x \mapsto g((a_1, x), \dots, (a_m, x))$ where $a_1, \dots, a_m \in B$, and denote by $\bar{D}_0(B)$ its sup-norm closure. Then $D_0(B)$ is an algebra, and we have the following result.

Proposition 4. *If the functions $\hat{a}(x) = (a, x)$, $a \in B$, separate the points of B (in particular, if the norm interior of B is not empty) then $\mathcal{U}_1(B_w) = \bar{D}_0(B)$.*

Let us take a look at other possible compactifications of B_w .

Lemma 1. *Let $\{X, p\}$ be a metric space and a, b two distinct points of X . Write $X \setminus \{a, b\} = Y$ and set*

$$(1) \quad r(x, y) = \min [p(x, y), p(x, a) + p(y, b), p(x, b) + p(y, a)].$$

Then r is a metric on Y , and both r and p determine the same topology on that space. Moreover r is strictly coarser than p ; if $\{X, p\}$ is precompact, then so is $\{Y, r\}$.

The verification is straightforward.

Proposition 5. *Let a, b be two distinct points of $\bar{B}^w \setminus B_w$, $p = q$, and r as defined in (1). Then at least one of the functions $\hat{a}(x) = (a, x)$ and $\hat{b}(x) = (b, x)$ is not r -uniformly continuous on B_w , and the compactification of B_w corresponding to r is obtained from \bar{B}^w by identifying a and b .*

This is immediate from Lemma 1.

Remark. If $a \neq 0 \neq b$ in Proposition 5, then both of \hat{a} and \hat{b} are not r -uniformly continuous.

Lemma 2. *Let $\{X, p\}$ be a metric space with $p(x, y) \leq 1$ for all $x, y \in X$, and $a \in X$ a non-isolated point. Set*

$$(2) \quad d'(x, y) = \log \left(1 + \frac{4p(x, y)}{\min [p(x, a), p(y, a)]^2} \right).$$

Then d' is a metric on $X \setminus \{a\} = Y$ which is strictly finer than p , and $\{Y, d'\}$ is not precompact. Moreover, the function g given by $g(x) = \sin(1/p(x, a))$ is d' -uniformly continuous on Y , but not p -uniformly continuous.

Proof. A straightforward calculation shows that d' is indeed a metric on Y , and that $\{Y, d'\}$ is not precompact. We check d' -uniform continuity of g . The function sine is uniformly continuous on \mathbf{R} ; so there is a function δ from $]0, 2]$ to $]0, 2\pi]$ such that $|\xi - \eta| < \delta(\varepsilon)$ implies $|\sin \xi - \sin \eta| < \varepsilon$. Then $d'(x, y) < \log(1 + 4\delta(\varepsilon))$ yields

$$\begin{aligned} \delta(\varepsilon) &> \frac{p(x, y)}{\min [p(x, a), p(y, a)]^2} \geq \frac{p(x, y)}{p(x, a) p(y, a)} \geq \\ &\geq \frac{|p(x, a) - p(y, a)|}{p(x, a) p(y, a)} = \left| \frac{1}{p(x, a)} - \frac{1}{p(y, a)} \right| \end{aligned}$$

and hence

$$\left| \sin \frac{1}{p(x, a)} - \sin \frac{1}{p(y, a)} \right| < \varepsilon.$$

Thus g is d' -uniformly continuous, but it clearly is not p -uniformly continuous.

Lemma 3. Let $\{X, d\}$ be a metric space which is not precompact, and $d(x, y) \leq 1$ for all $x, y \in X$. Let \mathcal{D}^* be the coarsest uniformity on X which leaves all bounded d -uniformly continuous functions still \mathcal{D}^* -uniformly continuous. Then \mathcal{D}^* is also the finest uniformity on X which (i) is Hausdorff, (ii) determines the same topology as d , and (iii) makes $\{X, \mathcal{D}^*\}$ precompact. The uniformity \mathcal{D}^* is generated by the family of semi-metrics $\hat{b}(x, y) = |d(b, x) - d(b, y)|$, $b \in X$ (the \hat{b} form a subbase). If X is separable in the topology given by d , and $\{b_n : n \in \mathbb{N}\}$ is a countable dense set in X , then \mathcal{D}^* is determined by the metric

$$d^*(x, y) = \sum_{n \in \mathbb{N}} 2^{-n} \hat{b}_n(x, y).$$

Proof. The first part is an exercise in § 1 of [1]. The verification that \mathcal{D}^* is generated by the \hat{b} , or by d^* , respectively, is left to the reader.

Proposition 6. Let $a \in \bar{B}^w \setminus B_w$. Then there exists a metric r which is strictly finer than q on B but which determines the same topology as does q (namely the weak topology on B) such that $\{B, r\}$ is precompact, and the function $g(x) = \sin(1/q(x, a))$ is r -uniformly continuous.

Proof. We give two examples of such a metric. (i) Let $X = \bar{B}^w$ and $p = q$ in Lemma 2 (a is not isolated in \bar{B}^w), and $d = \min\{d', 1\}$ in Lemma 3. Then $r = d^*$ has the desired properties. (ii) Let f be any bounded continuous function on B_w which is not q -uniformly continuous, for instance $f(x) = \sin(1/q(x, a))$. Then $r(x, y) = q(x, y) + |f(x) - f(y)|$ is a metric on B for which $\{B, r\}$ is precompact, r determines the weak topology on B , and f is r -uniformly continuous.

The above results are summarized as follows.

Theorem. The weak closure \bar{B} of a bounded set B in a separable, real Hilbert space \mathcal{H} is the compactification determined by the algebra generated by all continuous functionals on \mathcal{H} (or equivalently, by the simple functions). If B is not weakly closed, it has metrizable compactifications which are “larger” than \bar{B} , and if $\bar{B} \setminus B$ contains at least two points, \bar{B} also has metrizable compactifications which are “smaller” than B .

We conclude with some unanswered questions.

Problem 1. (a) Characterize all metrics on B which determine the weak topology, in particular those arising from compactifications. (b) Give an example of a bounded

set B in \mathcal{H} which is not weakly closed, of a nonmetrizable uniformity \mathcal{U} on B determining the weak topology – perhaps so that $\{B, \mathcal{U}\}$ is precompact, and of a \mathcal{U} -uniformly continuous function which is not q -uniformly continuous.

Problem 2. Does every \mathcal{U}_r -uniformly continuous function on B have a \mathcal{U}_r -uniformly continuous extension to the closed convex hull of B (perhaps even to K)?

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