Ján Jakubík

Homogeneous lattice ordered groups

Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 2, 325–337

Persistent URL: http://dml.cz/dmlcz/101101

Czechoslovak Mathematical Journal, 22 (1972), Praha

HOMOGENEOUS LATTICE ORDERED GROUPS

JÁN JAKUBÍK, Košice

(Received June 23, 1971)

Let G be an l-group. We denote by vG the least cardinal α such that card $A \leqq \alpha$ for each bounded disjoint subset of G. The case when vG is finite has been extensively studied (CONRAD and CLIFFORD [3], CONRAD [2], KOKORIN and CHISAMIEV [7], KOKORIN and KOZLOV [8]). G will be said to be v-homogeneous if $vH = vG$ for any convex l-subgroup $H \neq \{0\}$ of the l-group G. In this note we show that any complete l-group G can be represented as a complete subdirect product of v-homogeneous l-groups.

PIERCE [9] studied some types of homogeneous Boolean algebras. A Boolean algebra B is called homogeneous if it satisfies one of the following equivalent conditions: (i) for any $0 \neq b_i \in B$ $(i = 1, 2)$ the convex sublattices B_i of B generated by b_i $(i = 1, 2)$ are isomorphic; (ii) if B_1 is a convex sublattice of B such that B_1 is a Boolean algebra then B_1 is isomorphic to B. Let us consider analogous conditions (i_1) and (ii_1) for a lattice ordered group G:

(i_1) For any $0 \neq g_i \in G$ $(i = 1, 2)$ the convex l-subgroups of G generated by g_i $(i = 1, 2)$ are isomorphic.

(ii_1) If $G_1 \neq \{0\}$ is a convex l-subgroup of G, then G_1 is isomorphic to G.

If G satisfies (i_1) or (ii_1), then it will be called respectively homogeneous or strongly homogeneous. We prove that $v\,G = 1$ for any strongly homogeneous l-group $G \neq \{0\}$ and that $v\,G = 1$ or $v\,G \geqq \aleph_0$ for any homogeneous l-group $G \neq \{0\}$. Moreover, for any infinite cardinal α there exists a homogeneous l-group G with $v\,G = \alpha$.

Let H be a convex l-subgroup of G such that $\sup X \in H$ whenever $X \subset H$ and $\sup X$ does exist in G. Then H is said to be a c-subgroup of G. The closure $c\,A$ of a subset $A \subset G$ is the intersection of all c-subgroups B of G with $A \subset B$. An l-group G_1 is called totally inhomogeneous if for any $0 < g_1 \in G_1$ there is $0 < g_2 \in G_1$ such that (a) g_2 belongs to the convex l-subgroup A_1 of G that is generated by g_1, and (b) the convex l-subgroup A_1 of G generated by g_2 is not isomorphic to A_1. The zero l-group $\{0\}$ is homogeneous and, at the same time, totally inhomogeneous. In each l-group G there exists a greatest convex totally inhomogeneous l-subgroup. Let G be a complete

l-group. We prove that there is a system $\{A_0, A_i\}$ $(i \in I)$ of convex l-subgroups of G such that (i) A_0 is totally inhomogeneous, (ii) each A_i is homogeneous, and (iii) G is a complete subdirect product of l-groups A_0, cA_i $(i \in I)$.

1. PRELIMINARIES

We use the standard notation for lattices and lattice ordered groups, cf. [1], [4]. The lattice operations are denoted by $\wedge$, $\vee$. The group operation is written additively (though it need not be commutative). Let P be a partially ordered set, $a, b \in P$, $a \leqq b$; the interval $[a, b]$ is the set $\{x \in P : a \leqq x \leqq b\}$. A subset $Q \subset P$ is convex if $[a, b] \subset Q$ whenever $a, b \in Q$ and $a \leqq b$.

Let A be a sublattice of a lattice L such that $\sup a_n \in A$ whenever $\{a_n\} \subset A$ and $\sup a_n$ does exist in L, and dually; then A is said to be a σ-sublattice of L. Isomorphisms of lattices and l-groups are denoted by $\sim$ and $\approx$, respectively. Let L be a lattice, $\emptyset \neq Q \subset L$. A set Q is said to be a d-set if there is $x \in L$ such that $q_1 \wedge q_2 = x$ for any pair of distinct elements of Q and $q > x$ for each $q \in Q$. For any interval $[a, b]$ of L, we denote by $w[a, b]$ the least cardinal α such that card $Q \leqq \alpha$ for each d-set Q of $[a, b]$; further we put $w_0[a, b] = \max\{\aleph_0, w[a, b]\}$.

Throughout the whole paper G is an l-group, $G \neq \{0\}$. A subset $Q \subset G$, $Q \neq \emptyset$ is disjoint if Q is a d-set and $q_1 \wedge q_2 = 0$ for any pair of distinct elements q_1, q_2 of Q. Let A be a subgroup of G, $x \in G$. The element x is said to be disjoint to A if $|x| \wedge |a| = 0$ for each $a \in A$. For any $X \subset G$ we denote $X^\delta = \{g \in G : |g| \wedge |x| = 0$ for each $x \in X\}$. For $g \in G$, $[g]$ is the convex l-subgroup of G that is generated by g. We denote by $C(G)$ the system of all convex l-subgroups of G; $C(G)$ is partially ordered by inclusion. An element $0 < e \in G$ is a weak unit in G if $e \wedge x > 0$ for each $0 < x \in G$.

Let $I \neq \emptyset$ be a set and for each $i \in I$ let A_i be a lattice ordered group. The complete direct product of l-groups A_i will be denoted by ΠA_i $(i \in I)$. Let A be an l-subgroup of ΠA_i $(i \in I)$ with the property that for each $i_0 \in I$ and each $x \in A_{i_0}$ there is $a \in A$ such that $a(i_0) = x$ and $a(i) = 0$ for each $i \in I \setminus \{i_0\}$. Then A is said to be a complete subdirect product of l-groups A_i (cf. [10]). If I is a linearly ordered set, we denote by ΓA_i $(i \in I)$ the lexicographic product of l-groups A_i (cf. [4]).

We denote respectively by E or R the additive l-group of all integers (all reals) with the natural order.

2. INTERVALS IN DISTRIBUTIVE LATTICES

Let L be a distributive lattice and let $[a, b]$ be a nontrivial interval of L (an interval is nontrivial if it has more than one element). Obviously w is increasing on L in the following sense: if $[a, b] \subset [c, d] \subset L$, then $w[a, b] \leqq w[c, d]$.

2.1. *Let $a, b, c \in L$, $a < b < c$. Then $w[a, c] \leqq w[a, b] + w[b, c]$.*

Proof. If $w[a, c] = 1$ (i.e., if $[a, c]$ is linearly ordered), then the assertion is obvious. Assume that $w[a, c] > 1$; hence there is a d-set $D \subset [a, c]$ with card $D > 1$. Denote inf $D = d$. For any $x \in [a, c]$ let $x_1 = x \wedge b$, $x_2 = x \vee b$. Further put

$$D_1 = \{d_1^i : d^i \in D,\ d_1 < d_1^i\}, \quad D_2 = \{d_2^i : d^i \in D \setminus D_1\}.$$

For any $d_2^i \in D_2$ we have $d_2 < d_2^i$ because in the opposite case we should have

$$b \wedge d = b \wedge d^i, \quad b \vee d = b \vee d^i,$$

thus $d^i = d$, which is impossible. If x and y are distinct elements of the set D_i, then $x \wedge y = d_i$, therefore either $D_i = \emptyset$ or D_i is a d-set $(i = 1, 2)$. We have $w[a, b] \geqq$ $\geqq$ card D_1, $w[b, c] \geqq$ card D_2 and card D = card D_1 + card D_2; thus $w[a, c] \leqq$ $\leqq w[a, b] + w[b, c]$.

As a corollary, we obtain:

2.2. *Let a, b, c be the same as in 2.1. If $w[a, b]$ and $w[b, c]$ are finite, then $w[a, c]$ is finite as well. Moreover, $w_0[a, c] = w_0[a, b] + w_0[b, c]$.*

2.3. *Let $a, b \in L$. Then $w[a \wedge b, a \vee b] \leqq w[a \wedge b, a] + w[a \wedge b, b]$ and $w_0[a \wedge b, a \vee b] = w_0[a \wedge b, a] + w_0[a \wedge b, b]$.*

Proof. The interval $[a, a \vee b]$ being isomorphic to $[a \wedge b, b]$ we have $w[a, a \vee b] = w[a \wedge b, b]$. Now it suffices to apply 2.1 and 2.2.

Let α be an infinite cardinal, $x \in L$. Denote

$$V(x, \alpha) = \{y \in L : w[x \wedge y, x \vee y] \leqq \alpha\},$$
$$V_0(x, \alpha) = \{y \in L : w[x \wedge y, x \vee y] < \alpha\}.$$

2.4. *$V(x, \alpha)$ is a convex sublattice of L.*

Proof. Let $y_1, y_2 \in V(x, \alpha)$. Denote

$$t_1 = x \vee y_1 \vee y_2, \quad t_2 = (x \vee y_1) \wedge (x \vee y_2).$$

According to the assumption, all cardinals

$$w[x, t_2], \quad w[t_2, x \vee y_1], \quad w[t_2, x \vee y_2]$$

are equal or less than α, thus by 2.3 $w[t_2, t_1] \leqq \alpha$ and so by 2.1 $w[x, t_1] \leqq \alpha$. Dually we can prove that $w[t_3, x] \leqq \alpha$ where $t_3 = x \wedge y_1 \wedge y_2$. By 2.1, $w[t_3, t_1] \leqq \alpha$. Since

$$[x \wedge (y_1 \vee y_2), x \vee (y_1 \vee y_2)] \subset [t_3, t_1],$$

the element $y_1 \vee y_2$ belongs to $V(x, \alpha)$. In a dual way we show that $y_1 \wedge y_2$ belongs to $V(x, \alpha)$, Thus $V(x, \alpha)$ is a sublattice of L. If $y_1 \leqq z \leqq y_2$, then $x \wedge y_1$ and $x \vee y_2$ are elements of $V(x, \alpha)$, thus $w[x \wedge y_1, x \vee y_2] \leqq \alpha$ and clearly $[z \wedge x, z \vee x] \subset \subset [x \wedge y_1, x \vee y_2]$. Therefore $w[z \wedge x, z \vee x] \leqq \alpha$ and so $z \in V(x, \alpha)$.

2.5. $V_0(x, \alpha)$ *is a convex sublattice of* L.

The proof is analogous to that of 2.4.

2.6. *If* $x, y \in L$, $V(x, \alpha) \cap V(y, \alpha) \neq \emptyset$, *then* $V(x, \alpha) = V(y, \alpha)$.

Proof. Let $t \in V(x, \alpha) \cap V(y, \alpha)$ and $z \in V(t, \alpha)$. According to the definition of $V(t, \alpha)$ we have $x \in V(t, \alpha)$; hence by 2.4 $[x \wedge z, x \vee z] \subset V(t, \alpha)$. As a consequence we easily get $w[x \wedge z, x \vee z] \leqq \alpha$, thus $z \in V(x, \alpha)$. Therefore $t \in V(x, \alpha)$ implies $V(t, \alpha) \subset V(x, \alpha)$. Since $x \in V(t, \alpha)$, we have $V(x, \alpha) \subset V(t, \alpha)$ and so $V(x, \alpha) = V(t, \alpha)$. Similarly $V(t, \alpha) = V(y, \alpha)$ and consequently $V(x, \alpha) = V(y, \alpha)$.

Since $x \in V(x, \alpha)$, we obtain:

2.7. *The system* $\{V(x, \alpha)\}$ $(x \in L)$ *is a partition of the set* L.

The equivalence relation on L corresponding to this partition will be denoted by $R(\alpha)$. Analogously we define the equivalence $R_0(\alpha)$ by taking the sets $V_0(x, \alpha)$ instead of $V(x, \alpha)$.

2.8. $R(\alpha)$ *and* $R_0(\alpha)$ *are congruence relations on the lattice* L.

Proof. Let $x, y, z \in L$, $x \equiv y(R(\alpha))$. By 2.5 $x \wedge y \equiv x \vee y(R(\alpha))$. Put $x \wedge y = u$, $x \vee y = v$. The interval $[u \vee z, v \vee z]$ is transposed to the interval $[(u \vee z) \wedge v, v] \subset [u, v]$. Therefore the intervals $[u \vee z, v \vee z]$ and $[(u \vee z) \wedge v, v]$ are isomorphic, hence $w[u \vee z, v \vee z] \leqq \alpha$. Clearly $x \vee z$, $y \vee z$ belong to $[u \vee z, v \vee z]$, thus $w[(x \vee z) \wedge (y \vee z), (x \vee z) \vee (y \vee z)] \leqq \alpha$. Hence we obtain $x \vee z \equiv y \vee z(R(\alpha))$. The relation $x \wedge z \equiv y \wedge z(R(\alpha))$ can be proved dually. Hence $R(\alpha)$ is a congruence relation on L. The proof for $R_0(\alpha)$ is analogous.

2.9. *Let* $\{x_n\} \subset L(n = 0, 1, 2, \ldots)$, $x_0 \leqq x_1 \leqq x_2 \leqq \ldots$, $\bigvee x_n = y$, $w_0[x_{i-1}, x_i] \leqq \alpha$ $(i = 1, 2, \ldots)$. *Assume that the lattice* L *is infinitely distributive. Then* $w_0[x_0, y] \leqq \alpha$.

Proof. If the interval $[x_0, y]$ is linearly ordered, then the assertion is obvious. Assume that $[x_0, y]$ is not linearly ordered; then there is a d-set $D \subset [x_0, y]$ with card $D > 1$. Denote $\inf D = d$. For $z \in [x_0, y]$ and $i = 1, 2, \ldots$ put $z^i = z \wedge x_i$, $D^i = \{z^i : z \in D, d^i < z^i\}$. For each $z \in D$ there is $i \in \{1, 2, \ldots\}$ such that $z^i \in D^i$. For, if not, then

$$d = d \wedge y = d \wedge (\bigvee x_i) = \bigvee(d \wedge x_i) = \bigvee(z \wedge x_i) = z \wedge (\bigvee x_i) = z\,,$$

a contradiction. Let $D_0^i = \{z \in D : z^i \in D^i\}$. We have $D = \bigcup D_0^i$ $(i = 1, 2, \ldots)$ and for each $i \in \{1, 2, \ldots\}$ either card $D^i \leqq 1$ or D^i is a d-set and $D^i \subset [x_0, x_i]$. From 2.1 we obtain by induction card $D^i \leqq \alpha$. If $z, t \in D_0^i$, then $z^i \wedge t^i = d^i \neq z^i$, $d^i \neq t^i$, hence $z^i \neq t^i$; therefore card D^i = card D_0^i and it follows card $D \leqq \alpha$. Therefore $w_0[x_0, y] \leqq \alpha$.

2.10. *Let $x \in L$ and let A be a convex sublattice of L such that $x \in A$ and $w[a_1, a_2] \leqq \alpha$ whenever $a_1, a_2 \in A$, $a_1 \leqq a_2$. Then $A \subset V[x, \alpha]$.*

Proof. Let $y \in A$. According to the assumption we have $w[x \wedge y, x \vee y] \leqq \alpha$, hence $y \in V(x, \alpha)$.

A similar assertion is valid for $V_0(x, \alpha)$.

Summarizing, we have the following result:

2.11. Theorem. *Let L be a distributive lattice and let α be an infinite cardinal. Then for each $x \in L$ there are convex sublattices $V(x, \alpha)$ and $V_0(x, \alpha)$ of L such that $x \in V_0(x, \alpha) \subset V(x, \alpha)$ and*

(i) *if I is an interval of $V(x, \alpha)$ $(V_0(x, \alpha))$, then $wI \leqq \alpha (wI < \alpha)$,*

(ii) *if A is a convex sublattice of L fulfilling $wI \leqq \alpha (wI < \alpha)$ for each interval $I \subset A$ and $x \in A$, then $A \subset V(x, \alpha)$ $(A \subset V_0(x, \alpha))$,*

(iii) *the systems $\{V(x, \alpha)\}$ $(x \in L)$ and $\{V_0(x, \alpha)\}$ $(x \in L)$ are partitions of L and the corresponding equivalences $R(\alpha)$, $R_0(\alpha)$ are congruence relations on L;*

(iv) *if L is infinitely distributive, then each set $V(x, \alpha)$ is a σ-sublattice of L.*

3. w-HOMOGENEOUS LATTICE ORDERED GROUPS

A cardinal property f on the class of all lattices is a rule that assigns to each bounded lattice A a cardinal $f\,A$ such that $f\,B = f\,A$ whenever B is isomorphic to A. A cardinal property is increasing if $f\,C \leqq f\,A$ for any lattices A and C such that A is bounded and C is isomorphic to an interval of the lattice A (cf. [7]). A lattice L is f-homogeneous if $f\,B_1 = f\,B_2$ for any two nontrivial intervals B_1, B_2 of the lattice L.

Let G be a lattice ordered group and let f be a cardinal property on the class of all lattices. The following conditions on f were considered in [6]:

(c_1) If $0 < t_i \in G$ $(i = 1, 2)$, $f[0, t_1] = f[0, t_2]$ and if $[0, t_1]$ and $[0, t_2]$ are f-homogeneous, then $f[0, t_1 + t_2] = f[0, t_1]$.

(c_2) If $t_i \in G$, $0 < t_1 \leqq t_2 \leqq \ldots$, $f[0, t_1] = f[0, t_i]$, $\bigvee t_i = t$ and if the intervals $[0, t_i]$ are f-homogeneous $(i = 1, 2, \ldots)$, then $f[0, t] = f[0, t_1]$.

3.1. *The cardinal property w_0 fulfils (c_1) and (c_2).*

Proof. Since $0 < t_1 < t_1 + t_2$ and the interval $[t_1, t_1 + t_2]$ is isomorphic to

$[0, t_2]$, it follows form 2.2 that (c_1) is valid. It is known that any lattice ordered group is infinitely distributive. Since w_0 is increasing, 2.9 implies that (c_2) holds.

3.2. *The sets $V(0, \alpha)$ and $V_0(0, \alpha)$ are l-ideals of G and for any $x \in G$, $V(x, \alpha) = V(0, \alpha) + x$, $V_0(x, \alpha) = V_0(0, \alpha) + x$.*

Proof. Let $x \in G$. Since the mapping $\varphi(g) = g + x$ is an automorphism on the lattice G, from the definition of $V(g, \alpha)$ it follows $V(g + x, \alpha) = V(g, \alpha) + x$. In particular, $V(x, \alpha) = V(0, \alpha) + x$. Assume that $x, g \in V(0, \alpha)$. Then according to 2.6,

$$V(x + g, \alpha) = V(x, \alpha) + g = V(0, \alpha) + g = V(g, \alpha) = V(0, \alpha),$$
$$V(-x, \alpha) = V(0, \alpha) - x = V(x, \alpha) - x = V(0, \alpha),$$

thus $V(0, \alpha)$ is a subgroup of G. Moreover, for any $y \in G$,

$$-y + V(0, \alpha) + y = V(-y, \alpha) + y = V(0, \alpha),$$

hence $V(0, \alpha)$ is normal. Since $V(0, \alpha)$ is a convex sublattice of G, it is an l-ideal of G. The proof for $V_0(0, \alpha)$ is similar.

We need the following results:

3.3. ([6], Thm. 1.21.) *Let G be a complete l-group and let f be an increasing cardinal property satisfying (c_1) and (c_2). Then G is isomorphic to a complete subdirect product of f-homogeneous l-groups. If G is also laterally complete, then it is isomorphic to a complete direct product of f-homogeneous l-groups.*

3.4. *Let G be a complete lattice ordered group. Then G is isomorphic to a direct product $A \times B$ such that (i) A is isomorphic to a complete subdirect product of linearly ordered groups, and (ii) B has no linearly ordered direct factor $C \neq \{0\}$.*

Proof. Let $\{A_k\}$ $(k \in K)$ be the set of all maximal linearly ordered subgroups of G, $B = \{\bigcup A_k\}^\delta$, $A = B^\delta$. According to the Riesz-Birkhoff Theorem (cf. [1], Chap. XIV) $G = A \times B$ and clearly B has no linearly ordered factor different from $\{0\}$. Thus it remains to show that A is isomorphic to a complete subdirect product of linearly ordered groups. By [5], Thm. 1 each A_k is a direct factor in G. Hence there exist components $x(A_k)$ for each $x \in A$ and $x(A_k) = \sup \{a_k \in A_k : a_k \leqq x\}$ whenever $x \geqq 0$. Consider the mapping $\varphi(x) = (\ldots, x(A_k), \ldots)$ of A into ΠA_k $(k \in K)$. If $\varphi(x) = 0$, then $\varphi(|x|) = 0$ hence x is disjoint with each A_k $(k \in K)$ and so $|x| \in B$; this implies $x = 0$. Hence φ is an isomorphism of A onto $\varphi(A)$. Let $k_0 \in K$, $f \in \Pi A_k$, $f(k) = 0$ for each $k \in K \setminus \{k_0\}$. Put $f(k_0) = x$. Then $x(A_k) = 0$ for each $k \neq k_0$ and $x(A_{k_0}) = x$, hence $\varphi(A)$ is a complete subdirect product of linearly ordered groups $\varphi(A_k)$ $(k \in K)$.

Let B be the same as in 3.4 and assume that $B \neq \{0\}$. Clearly B is a complete l-group and hence B is Archimedean. From [5], Thm. 1′ it follows that B has no basic element.

Hence $w[a, b]$ is infinite for any nontrivial interval of B and so $w[a, b] = w_0[a, b]$. Any linearly ordered group is w-homogeneous, thus by 3.4 A is a complete subdirect product of w-homogeneous l-groups. According to 3.1 and 3.3 B is isomorphic to a complete subdirect product of w_0-homogeneous l-groups B_k $(k \in K)$, $B_k \neq \{0\}$; but B_k are isomorphic to some convex l-subgroups of B and so $w_0 I = wI$ for any nontrivial interval of B_k, therefore B_k are w-homogeneous. We arrive at

3.5. Theorem. *Any complete l-group is a complete subdirect product of w-homogeneous l-groups.*

3.6. *An l-group is v-homogeneous if and only if it is w-homogeneous.*

Proof. If G is linearly ordered, then the assertion is trivial; assume that G is not linearly ordered. Let $[a, b]$ be an interval of G. Since $[a, b]$ is isomorphic to $[0, b - a]$, we have $w[a, b] = w[0, b - a]$. Assume that G is w-homogeneous and that $wI = \alpha$ for any nontrivial interval I of G. Let M be a bounded disjoint subset of G. Since M is a d-set, we have card $M \leqq \alpha$, thus $vG \leqq \alpha$. On the other hand, if M is a bounded d-set of G with card $M > 1$, $\inf M = m$, then the set $M' = \{x - m : x \in M\}$ is disjoint and therefore $vG = \alpha$.

From 3.5 and 3.6 we obtain

3.7. Theorem. *Any complete l-group is a complete subdirect product of v-homogeneous l-groups.*

4. STRONGLY HOMOGENEOUS LATTICE ORDERED GROUPS

Let $G \neq \{0\}$ be a lattice ordered group. The following assertion is easy to verify:

4.1. *For any* $0 < g \in G$, $[g] = \bigcup[-ng, ng]$ $(n = 1, 2, \ldots)$.

From 4.1 we obtain immediately:

4.2. *If* $0 < g \in G$, *then* g *is a strong unit of the lattice ordered group* $[g]$.

4.3. *Let* $0 < g \in G$ *and assume that the interval* $[0, g]$ *is a chain. Then* $[g]$ *is linearly ordered.*

This follows from 4.1 and [5], 17.2 by using induction.

4.4. *Let G be homogeneous and not linearly ordered. Then G contains a bounded infinite disjoint subset.*

Proof. Since G is not linearly ordered there are incomparable elements $a, b \in G$. Put $a_1 = a - (a \wedge b)$, $b_1 = b - (a \wedge b)$, $g = a_1 \vee b_1$. The set $\{a_1, b_1\}$ is disjoint and the l-group $[g]$ is not linearly ordered. Since G is homogeneous, the l-group $[b_1]$ is not linearly ordered, thus by 4.3 $[0, b_1]$ is not a chain. Hence there is a disjoint subset $\{a_2, b_2\} \subset [0, b_1]$ and clearly $\{a_1, a_2\}$ is a disjoint set. Analogously we construct disjoint sets $\{a_1, a_2, \ldots, a_n\}$ $(n = 1, 2, \ldots)$. Then the set $\{a_n\}_{n=1}^{\infty}$ is disjoint as well and it is a subset of $[0, g]$.

4.5. *Let $\{a_1, a_2, \ldots\}$ be a disjoint subset of G and let $A_n = [a_n]$ $(n = 1, 2, \ldots)$. Denote by A the system of all elements $g \in G$ that can be written in the form $g = b_{n_1} + \ldots + b_{n_k}$ with $b_{n_i} \in A_i$. Then A is a convex l-subgroup of G.*

Proof. Since $|b_{n_i}| \wedge |b_{n_j}| = 0$ for $i \neq j$ we infer that the elements b_{n_i} and b_{n_j} are permutable, therefore A is a subgroup of G. Clearly A is a directed subset of G. If $x \in G$, $g \in A$, $0 < x \leqq g$, then there are elements $b_{n_i} > 0$, $b_{n_i} \in A_i$ such that $g = b_{n_1} + \ldots + b_{n_k}$; hence it follows that $x = c_{n_1} + \ldots + c_{n_k}$ for some $0 \leqq c_{n_i} \leqq b_{n_i}$ $(i = 1, \ldots, k)$. Thus A is a convex subgroup of G and, being directed, it is an l-subgroup of G.

4.6. *Let A be the same as in 4.5. Then A has no weak unit.*

Proof. Let g, b_{n_i} $(i = 1, \ldots, k)$ be as in 4.5. Choose $n > \max\{n_1, \ldots, n_k\}$; we have $a_n \wedge b_{n_i} = 0$, therefore $a_n \wedge g = 0$. This shows that A has no weak unit.

4.7. *If G is strongly homogeneous, then G is linearly ordered.*

Proof. Assume on the contrary that G is strongly homogeneous and that it is not linearly ordered. By 4.4, G contains an infinite disjoint subset $\{a_1, a_2, a_3, \ldots\}$. Let A be as in 4.5 and $0 < g \in G$. According to 4.2 $[g]$ has a weak unit and thus by 4.6 the l-subgroups $[g]$ and A of G are not isomorphic, which is a contradiction.

As a corollary, we obtain

4.7.1. *If G is strongly homogeneous, then $C(G)$ is linearly ordered.*

If φ is an isomorphism of a lattice ordered group G_1 onto G_2, then φ induces an isomorphism φ_1 of the partially ordered set $C(G_1)$ onto $C(G_2)$.

4.8. *Let G be strongly homogeneous, $\{0\} \neq A \in C(G)$. Then there is $A_1 \in C(G)$ such that A_1 is covered by A in $C(G)$.*

Proof. Choose $0 < g \in G$. From the Zorn Lemma it follows that there is a convex l-subgroup B of G that is maximal with respect to not containing the element g; since $C(G)$ is linearly ordered by 4.7, the l-group B is uniquely determined. There is an isomorphism φ of $[g]$ onto A; then the l-group $A_1 = \varphi_1(B)$ is covered by A in $C(A)$, thus clearly A_1 is covered by A in $C(G)$.

Denote $A_1 = f(A)$ for any $A \neq \{0\}$ and $\{0\} = f(\{0\})$; further define inductively $f^\lambda(A)$ for any ordinal number λ as follows: for a non-limit ordinal $\lambda = \lambda_1 + 1$ we put $f^\lambda(A) = f(f^{\lambda_1}(A))$ and if λ is a limit ordinal, we set $f^\lambda(A) = \bigcap_{\nu<\lambda} f^\nu(A)$. Then

$$A \supset \ldots \supset f^\nu(A) \supset \ldots \supset f^\lambda(A) \supset \ldots$$

whenever $\nu < \lambda$ and for any λ either $f^\lambda(A) = f^{\lambda+1}(A) = \{0\}$ or $f^{\lambda+1}(A)$ is covered by $f^\lambda(A)$.

In 4.9 – 4.14 we assume that G is strongly homogeneous.

4.9. *For any ordinal λ, $f^\lambda(G)$ is an l-ideal of G.*

Proof. According to 4.8, $\varphi(f(G)) = f(G)$ for any automorphism of the l-group G; by transfinite induction we get $\varphi(f^\lambda(G)) = f^\lambda(G)$. Thus $f^\lambda(G)$ is an l-ideal of G.

4.10. *If $f^\lambda(G) \neq \{0\}$, then the factor l-group $f^\lambda(G)/f^{\lambda+1}(G)$ is isomorphic to an l-subgroup of R.*

Proof. From the assumption it follows that $f^{\lambda+1}(G)$ is covered by $f^\lambda(G)$, the factor l-group $f^\lambda(G)/f^{\lambda+1}(G) = F \neq \{0\}$ has no convex subgroups distinct from $\{0\}$ and F, thus F is Archimedean; being linearly ordered F is isomorphic to an l-subgroup of R (cf. [1], Chap. XIV).

By the definition of f, for any λ either $f^\lambda(G) = \{0\}$ or $f^{\lambda+1}(G)$ is a proper subset of $f^\lambda(G)$; hence we obtain

4.11. *There is an ordinal λ_0 such that $f^\lambda(G) = \{0\}$ if and only if $\lambda \geqq \lambda_0$.*

4.12. *Let A be a convex l-subgroup of G, $\{0\} \neq A \neq G$. Then there is an ordinal $\lambda_1 < \lambda_0$ such that $A = f^{\lambda_1}(G)$.*

Proof. From 4.11 it follows that the set $\Lambda = \{\lambda \leqq \lambda_0 : f^\lambda(G) \subset A\}$ is non-empty; let λ_1 be the first element of the set Λ. If λ_1 is a limit ordinal, then $f^{\lambda_1}(G) = \bigcap f^\lambda(G)$ $(\lambda < \lambda_1)$, and for each such λ we have $f^\lambda(G) \supset A$, therefore $f^{\lambda_1}(G) \supset A$; this implies $f^{\lambda_1}(G) = A$. Assume that λ_1 is nonlimit, $\lambda_1 = \lambda_2 + 1$. Then A is a proper subset of $f^{\lambda_2}(G)$ and since $f^{\lambda_1}(G) \subset A$ is covered by $f^{\lambda_2}(G)$ we obtain $f^{\lambda_1}(G) = A$.

If α, β are ordinals, $\alpha \leqq \beta$, we denote by $[\alpha, \beta]$ the system of all ordinals λ with $\alpha \leqq \lambda \leqq \beta$.

4.13. *For any $\lambda < \lambda_0$, $[1, \lambda_0]$ is isomorphic to $[\lambda, \lambda_0]$.*

Proof. According to 4.11 and 4.12, $[1, \lambda_0]$ and $[\lambda, \lambda_0]$ is the order type of the chain $C(G)$ and $C(f^\lambda(G))$, respectively. Since G is isomorphic to $f^\lambda(G)$, $C(G)$ is isomorphic to $C(f^\lambda(G))$.

4.14. *For any $\lambda < \lambda_0$, the l-groups $G/f(G)$ and $f^\lambda(G)/f^{\lambda+1}(G)$ are isomorphic.*

Proof. There exists an isomorphism φ of G onto $f^\lambda(G)$ and $\varphi(f(G)) = f^{\lambda+1}(G)$; therefore $G/f(G)$ is isomorphic to $f^\lambda(G)/f^{\lambda+1}(G)$.

Denote $h(G) = G/f(G)$. Let us remark that if G_1 and G_2 are strongly homogeneous l-groups such that $C(G_1)$ is isomorphic to $C(G_2)$ and $h(G_1)$ is isomorphic to $h(G_2)$, then G_1 and G_2 need not be isomorphic. Moreover, we have:

4.15. *Let G be strongly homogeneous and assume that* card $C(G) > 2$. *Then there exists a strongly homogeneous l-group G_1 such that $C(G) \sim C(G_1)$, $h(G) \approx \approx h(G_1)$ and G is not isomorphic to G_1.*

Proof. Let I be the order type isomorphic to $C(G)$. For each $i \in I$ let $H_i = h(G)$. Put $H = \Gamma H_i\ (i \in I)$. Let $A \neq \{0\}$ be a convex l-subgroup of H and let i_0 be the least element of I such that there exists $a \in A$ with $a(i_0) \neq 0$. Then $A = \Gamma H_i(i \in I : i \geqq i_0)$. Since according to 4.13 the linearly ordered set $\{i \in I : i \geqq i_0\}$ is isomorphic to I, A is isomorphic to H and therefore H is strongly homogeneous. Clearly $h(H) \approx h(G)$ and $C(H) \sim C(G)$. If H is not isomorphic to G, we put $G_1 = H$. Assume that H is isomorphic to G. For any $x \in H$ let $s(x)$ be the support of x. Let X be the set of all $x \in H$ such that $s(x)$ is finite. It is easy to verify that X is strongly homogeneous, $C(X) \sim C(H)$, $h(X) \approx h(H)$ and X is not isomorphic to G; we put $G_1 = X$.

4.16. *Let α be an infinite cardinal. There exists a strongly homogeneous l-group G with* card $G = \alpha$.

Proof. Let ω_α be the first ordinal such that the power of the set of all ordinals less than ω_α equals α. Let $\lambda < \omega_\alpha$. Since card $[1, \lambda] < \alpha$, we have card $[\lambda, \omega_\alpha] = \alpha$ and so the order type of $[\lambda, \omega_\alpha]$ is isomorphic to $[1, \omega_\alpha]$. Hence it follows that the l-group

$$A = \Gamma A_\lambda\ (\lambda < \omega_\alpha)$$

with $A_\lambda = E$ for each $\lambda < \omega_\alpha$ is strongly homogeneous. Let G be the set of all $a \in A$ with a finite support. Then G is strongly homogeneous as well and card $G = \alpha$.

4.17. An l-group G will be said to be *totally inhomogeneous* if for each $0 < g \in G$ there exists $g_1 \in G$ such that $0 < g_1 \in [g]$ and the l-groups $[g_1]$, $[g]$ are not isomorphic. The following example shows that there exist totally inhomogeneous l-groups: Let $I = \{1, 2, \ldots\}$ and let p be a prime. Put $G_1 = \Gamma A_i\ (i \in I)$, where

$$A_i = E \quad \text{if} \quad i = p^k \quad (k = 0, 1, 2, \ldots),$$

and

$$A_i = R \quad \text{otherwise}.$$

Then it is easy to verify that G is totally inhomogeneous. If p_1, p_2 are distinct primes, then G_{p_1} and G_{p_2} are not isomorphic.

Let G be an l-group.

5.1. If $\{G_i\}$ $(i \in I)$ is a chain of the lattice $C(G)$ such that each G_i is homogeneous, then $H = \bigcup G_i$ is homogeneous.

Proof. If $0 < h_k \in H$ $(k = 1, 2)$, then $h_1, h_2 \in G_i$ for some i, hence $[h_1] \approx [h_2]$.

By using the Zorn Lemma, we obtain from 5.1:

5.2. *If H_0 is a homogeneous convex l-subgroup of G, then there is a maximal convex homogeneous l-subgroup H of G such that $H_0 \subset H$.*

Moreover, from 5.2 and from the Axiom of Choice we infer:

5.3. *There exists a system $\mathscr{A} = \{A_k\}$ $(k \in K)$ of convex l-subgroups of G such that:*

(i) *Each $A_k \in \mathscr{A}$ is a maximal homogeneous l-subgroup of G.*

(ii) *The system $\mathscr{A}$ is disjoint.*

(iii) *If $0 < x \in G$ and x is disjoint with each $A_k \in \mathscr{A}$, then $[x]$ is not homogeneous.*

5.4. *Let $\mathscr{A}$ be the same as in 5.3 and $0 < x \in G$. Then the following conditions are equivalent:* (iii$_1$) *x is disjoint with each $A_k \in \mathscr{A}$;* (iv) *$[x]$ is totally inhomogeneous.*

Proof. Assume that (iii$_1$) holds and let $0 < y \in [x]$. Then y is disjoint with each $A_k \in \mathscr{A}$ and thus by 5.3 the l-group $[y]$ is not homogeneous. Hence there is $0 < z \in$ $\in [y]$ such that $[z]$ is not isomorphic to $[y]$ and so $[x]$ is totally inhomogeneous. Conversely, assume that $[x]$ is totally inhomogeneous. If $x \wedge a_k = y$ for some $0 < a_k \in A_k \in \mathscr{A}$, then the l-group $[y]$ is homogeneous since $y \in A_k$ and at the same time $[y]$ is totally inhomogeneous because $[y] \subset [x]$; thus $[y] = \{0\}$ and therefore (iii$_1$) holds.

5.5. Theorem. *In any l-group G there is a greatest convex totally inhomogeneous l-subgroup.*

Proof. Denote $X = (\bigcup A_k)^\delta$ $(k \in K)$. Then X is a convex l-subgroup of G. From 5.4 it follows that X is totally inhomogeneous and that any totally inhomogeneous convex l-subgroup of G is a subset of X.

If P is a direct factor of G and $g \in G$, then we denote by $g(P)$ the component (= projection) of g in P; for any $0 \leqq g \in G$ we have $0 \leqq g(P) \leqq g$. Each c-subgroup of a complete l-group G is a direct factor of G and for any $Z \subset G$, Z^δ is a closed l-subgroup of G (cf. Riesz-Birkhoff Thm., [1], Chap. XIV).

5.6. *Let X and A_k be the same as in 5.5. Assume that G is a complete l-group, $0 < g \in G$. Then*

$$g = g(X) \vee (\bigvee g(cA_k)) .$$

Proof. Since X and cA_k are c-subgroups of G, the projections $g(X)$, $g(cA_k)$ exist in G and belong to the interval $[0, g]$. Hence $y = \bigvee g(cA_k)$ does exist in G and $0 \leqq y \leqq x$. According to the definition of X we have $g(cA_k) \in X^\delta$, thus $y \in X^\delta$ and so $g(X) \wedge y = 0$, whence $g(X) \vee y = g(X) + y$. Denote $t = -g(X) - y + g$. Then $t(X) = -g(X)(X) - y(X) + g(X) = -g(X) + g(X) = 0$ since $y(X) = 0$, thus t is disjoint to X. Similarly we can show that t is disjoint to each cA_k. According to the definition of X we have $t = 0$, hence $g = g(X) \vee (\bigvee g(cA_k))$.

5.7. Theorem. *Let G be a complete l-group. Then there exists a system of convex l-subgroups $\{X, A_k\}$ $(k \in K)$ in G such that*

(i) *X is the greatest convex l-subgroup of G that is totally inhomogeneous;*

(ii) *each A_k is homogeneous;*

(iii) *the l-group G is isomorphic to the complete subdirect product of the l-groups X, cA_k $(k \in K)$.*

Proof. The assertions (i) and (ii) were already proved. Let $k_0 \notin K$, $K' = K \cup \{k_0\}$, $A_{k_0} = X$ and consider the mapping $\varphi(g) = (\dots, g_k, \dots)_{k \in K'}$ of G into the direct product of l-groups A_{k_0}, cA_k $(k \in K)$ such that $g_{k_0} = g(A_{k_0})$, $g_k = g(cA_k)$ for $k \in K$. Since X and cA_k are direct factors of G the mapping φ is a homomorphism. Denote $\varphi(G) = G_1$. If $g \in X$, then $g_{k_0} = g$ and $g_k = 0$ for each $k \in K$; similarly, if $g \in cA_{k_1}$ for $k_1 \in K$, then $g_{k_1} = g$ and $g_{k_0} = 0$, $g_k = 0$ for each $k \in K \setminus \{k_1\}$. Therefore G_1 is a complete subdirect product of l-groups X and cA_k $(k \in K)$. If $0 \neq g_1 \in G$, $\varphi(g_1) = 0$, then for $g = |g_1|$ we have $g > 0$, $\varphi(g) = 0$, thus $g(X) = 0$ and $g(cA_k) = 0$ for each $k \in A_k$. Hence according to 5.6 $g = 0$, a contradiction. This implies that φ is an isomorphism of G onto G_1.

Let B be a Boolean algebra and let $X(B)$ be the Stone space of B. Then B is isomorphic to the system B^* consisting of the subsets of $X(B)$ that are simultaneously closed and open. Let $F_1(B)$ be the system of all real functions defined on $X(B)$ with the following property: for each $f \in F_1(B)$ there is a system $A_1, \dots, A_n \in B^*$ such that

$$\bigcup A_i = X(B), \quad A_{i_1} \cap A_{i_2} = \emptyset \quad \text{for distinct} \quad i_1, i_2 \in \{1, \dots, n\}$$

and f is a constant on each subset A_i $(i = 1, \dots, n)$. Then $F_1(B)$ is an additive group and it is an l-group if we put $f \leqq g$ whenever $f(x) \leqq g(x)$ for each $x \in X(B)$. It is easy to verify that $v(G) = w(B)$. If $0 < f \in F_1(B)$, let $s(f) = \{x \in X(B) : f(x) \neq 0\}$. The set $S = s(f)$ belongs to B^*. Denote $B_1 = [\emptyset, S] \subset B^*$; then B_1 is a Boolean algebra and $F_1(B_1)$ is isomorphic to $[f]$. Therefore the l-group $F_1(B)$ is homogeneous whenever the Boolean algebra B is homogeneous. For any infinite cardinal α

there is a homogeneous Boolean algebra B with $wB = \alpha$ (cf. [9], Thm. 3.5 and Lemma 3.12). Thus for any infinite cardinal α there exists an l-group $G = F_1(B)$ such that G is homogeneous and $vG = \alpha$.

References

[1] C. *Birkhoff:* Lattice theory, third edition, Providence 1967.

[2] *P. Conrad:* The structure of a lattice-ordered group with a finite number of disjoint elements, Michigan Math. J. *7* (1960), 171—180.

[3] *P. Conrad, A. H. Clifford:* Lattice-ordered groups having at most two disjoint elements, Proc. Glasgow Math. Assoc., *4* (1960), 111—113.

[4] *Л. Фукс:* Частично упорядоченные алгебраически системы, Москва 1965.

[5] *J. Jakubík:* Konvexe Ketten in *l*-Gruppen, Časop. pěst. mat. *84* (1959), 53—63.

[6] *J. Jakubík:* Cardinal properties of lattice-ordered groups, Fundam. math. *74* (1972), 85—98.

[7] *А. И. Кокорин, Н. Г. Хисамиев:* Элементарная классификация структурно упорядоченных абелевых групп с конечным числом нитей, Алгебра и логика, *5* (1966), 41—50.

[8] *А. И. Кокорин, Г. Т. Козлов:* Расширенные элементарная и универсальная теории решеточно упорядоченных групп с коненым числом нитей, Алгебра и логика *7* (1968), 91—103.

[9] *R. S. Pierce:* Some questions on complete Boolean algebras, Proc. Symp. Pure Math. Vol. *2*, Lattice theory, Amer. Math. Soc., 1961, 129—140.

[10] *F. Šik:* Über subdirekte Summen geordneter Gruppen, Czechosl. Math. J. *10* (1960), 400—424.

Author's address: Košice, Zbrojnicka 7, ČSSR (Vysoká škola technická).