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COMPLETE PRIME IDEALS OF BOOLEAN RINGS

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In this paper necessary and sufficient conditions are given for a proper ideal P of a Boolean ring B to be suprema preserving, as well as, for P to be complete. In particular, it is shown that if P is complete then so is B.

We recall that a Boolean ring is a ring B such that $x^2 = x$ for every $x \in B$ (thus, B is commutative and has characteristic 2), and that $\leqq$ is a partial order in B where $\leqq$ is defined by:

(1) $$x \leqq y \quad \text{if and only if} \quad xy = x$$

for every element x and y of B.

In what follows, any reference to order in a Boolean ring B is made in connection with $\leqq$ as given by (1).

A nonzero element a of a Boolean ring B is called an *atom* [1, p. 27] of B if and only if for every element x of B,

(2) $$xa \neq 0 \quad \text{implies} \quad xa = a$$

i.e., if and only if for every element x of B,

(3) $$x < a \quad \text{implies} \quad x = 0\,.$$

A Boolean ring need not have a multiplicative unit. If it does then it is called a *Boolean algebra*.

If H is a subset of a Boolean ring B and m is an element of B, we let mH denote the subset $\{mx \mid x \in H\}$ of B. Clearly, mB is an ideal (and a fortiori, a subring) of B.

In what follows we make use of the fact that a proper prime ideal P of a Boolean ring B is a maximal ideal of B. This is because the quotient B/P is a Boolean ring with more than one element and without a divosor of zero and the only such Boolean ring is the two-element field.

Theorem 1. *Let P be a proper prime ideal of a Boolean ring B. Then* $\sup P$ *exists if and only if B has a unit, i.e., if and only if B is a Boolean algebra.*

Proof. Let $\sup P$ exist. If $x \leqq \sup P$, for every $x \in B$ then clearly, $\sup P$ is the unit of B. If there exists an element m of B such that $m \nleqq \sup P$ then we consider $u = m + \sup P + m \sup P$. Since $m \neq m \sup P$, we see that $u > \sup P$. But then since P is a proper prime ideal of B, the ideal generated by u is equal to B, i.e., $\{x \mid x \leqq u\} = B$. Obviously, this implies that u is the unit of B. Thus, indeed, B has a unit.

Conversely, let 1 be the unit of B. Clearly, 1 is an upper bound of P. If $1 \neq \sup P$ then there exists an upper bound u of P such that $u \neq 1$. But then, the subset uB of B is a proper ideal of B containing the proper prime ideal P and therefore $uB = P$. However, since $u \in B$, we see that $u \in P$ and therefore $u = \sup P$. Thus, $\sup P$ exists.

Theorem 2. *Let P be a proper prime ideal of a Boolean algebra A with unit 1. Then the following statements are pairwise equivalent.*

$$1 \neq \sup P . \tag{4}$$

$$(\sup P) \in P . \tag{5}$$

$$1 + \sup P \quad \text{is an atom of} \quad A . \tag{6}$$

Proof. In view of Theorem 1, we see that $\sup P$ exists. Let $1 \neq \sup P$ and $u = \sup P$. Then $u \in P$, as shown in the second half of the Proof of Theorem 1. This shows that (4) implies (5). Next, let $(\sup P) \in P$. But then $(1 + \sup P) \in (A - P)$. Moreover, by DeMorgan's law $(1 + \sup P) = \inf (A - P)$ and therefore $(1 + \sup P) = \min (A - P)$. However, $A - P$ is a filter (in fact, an ultrafilter) of A and therefore, for every $x \in A$ if $x < \min (A - P)$ then $x = 0$. But this, in view of (3), shows that $\min (A - P) = 1 + \sup P$ is an atom of A. Thus, (5) implies (6). Finally, if $1 + \sup P$ is an atom of A then $(1 + \sup P) \neq 0$ and hence $1 \neq \sup P$. Consequently, (6) implies (4) and the Theorem is proved.

Corollary 1. *Let P be a proper prime ideal of a Boolean algebra A with unit 1. Then* $\sup P = 1$ *if and only if P contains all the atoms of A.*

Proof. Clearly, it is enough to show that $1 \neq \sup P$ if and only if there exists an atom a such that $a \in (A - P)$. But this follows readily from (4), (5) and (6) since $(\sup P) \in P$ implies $(1 + \sup P) \in (A - P)$.

Lemma 1. *Let m and a be elements of a Boolean ring B. Then ma is an atom of the subring mB if and only if ma is an atom of B.*

Proof. Let ma be an atom of the subring mB, and, let $xma \neq 0$ for some element x of B. Hence, $mxma \neq 0$ and since ma is an atom of mB, from (2) it follows that

$mxma = xma = ma$. Thus, again, in view of (2), we see that ma is an atom of B. The converse is obvious.

As expected, a subset H of a Boolean ring is called *suprema preserving* if and only if for every subset S of H, if sup S exists then (sup S) $\in H$.

Theorem 3. *Let P be a proper prime ideal of a Boolean ring B. Then P is suprema preserving if and only if B has an atom a and $a \notin P$.*

Proof. Let P be suprema preserving and $m \in (B - P)$. But then $m \neq \sup P$. Hence, the unit m of the Boolean algebra mB is not the supremum of the proper prime ideal mP of mB. Consequently, by (4) and (6), we see that $m + \sup mP$ which is equal to $m(m + \sup mP)$ is an atom of mB. But then from Lemma 1 it follows that $m + \sup mP$ is an atom of B. Moreover, $(m + \sup mP) \notin P$ since otherwise $m + \sup mP$ would be an element of P contradicting (5).

Conversely, let a be an atom of B and $a \notin P$. Let S be a subset of P and $s = \sup S$. We show that $s \in P$. Assume on the contrary that $s \notin P$. Clearly,

$$aP = \{0\} \quad \text{and} \quad a(B - P) = \{a\}. \tag{7}$$

Hence, $(s + a)\, x = x$ for every $x \in S$ which implies that $s + a$ is an upper bound of S. But this contradicts the hypothesis that $s = \sup S$, since from (7) it follows that $(s + a) < s$.

Thus, Theorem 3 is proved.

Let us recall that a subset H of a Boolean ring is called *complete* if and only if sup S of every subset S of H exists and (sup S) $\in H$. Clearly, if H is complete then it is also suprema preserving.

Theorem 4. *Let P be a proper prime ideal of a Boolean ring B. If P is complete then B is complete.*

Proof. Let P be complete. But then from Theorem 1 it follows that B has a unit 1. Now, let S be a subset of B. We show that sup S exists. Clearly,

$$S = (S \cap P) \cup (S - P).$$

Since P is complete $\inf \{1 + x \mid x \in (S - P)\}$ exists and by DeMorgan's law

$$1 + \inf \{1 + x \mid x \in (S - P)\} = 1 + \sup (S - P).$$

Hence, $\sup (S - P)$ exists. But then clearly,

$$\sup (S \cap P) + \sup (S - P) + (\sup (S \cap P)) (\sup (S - P))$$

is equal to sup S. Hence, sup S exists, as desired.

Theorem 5. *Let P be a proper prime ideal of a Boolean ring B. Then P is complete if and only if B is complete and has an atom a such that $a \in (B - P)$.*

Proof. Let P be complete. Then by Theorem 4 we see that B is complete and by Theorem 3 we see that B has an atom a such that $a \in (B - P)$. Conversely, let B be complete and have an atom a such that $a \in (B - P)$. But then from Theorem 3 it follows that P is suprema preserving. However, since B is complete we see that P is also complete.

Corollary 2. *Let P be a proper prime ideal of a Boolean ring B. Then the following statements are pairwise equivalent.*

(8) *P is complete.*

(9) *P is suprema preserving and B is complete.*

(10) *B is complete and has an atom a such that $a \in (B - P)$.*

Proof. (8) implies (9) by virtue of Theorem 4. Also, (9) implies (10) by virtue of Theorem 3. Finally, (10) implies (8) by virtue of Theorem 5.

Reference

[1] *Sikorski, R.*, Boolean Rings, Springer-Verlag, 1969.

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