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ARCHIMEDEAN EQUIVALENCE ON ORDERED SEMIGROUPS

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Archimedean properties in some special kinds of ordered semigroups have been studied by several authors (for example [1]–[8]). In the book [5], L. FUCHS defined the Archimedean equivalence on a simple ordered semigroup as follows:

$a \sim b$ if and only if one of the four conditions:

$$a \leqq b \leqq a^n\,, \quad b \leqq a \leqq b^n\,, \quad a^n \leqq b \leqq a\,, \quad b^n \leqq a \leqq b$$

holds for some positive integer n.

T. SAITÔ [7] showed that this relation is not an equivalence relation. Then he studied the Archimedean equivalence on nonnegatively simple ordered semigroups. In this paper we shall consider the Archimedean equivalence on a general ordered semigroup. On the other hand, in our paper [9] we studied the equivalence $\overline{\mathbf{K}}$ on a semigroup S: for $a, b \in S$, $a\overline{\mathbf{K}}b$ if and only if there exist positive integers m, n such that $a^m = b^n$. We shall define the Archimedean equivalence on an ordered semigroup S in a similar way.

Let $\mathscr{C}(S)$ denote the set of all $\mathscr{C}$-closure operations for a non-empty set S, i.e.

(0) $$\mathbf{U} \in \mathscr{C}(S) \Leftrightarrow \mathbf{U} : \exp S \to \exp S \quad \textit{and}$$

(1) $$\mathbf{U}(\emptyset) = \emptyset\,,$$

(2) $$A \subset B \subset A \Rightarrow \mathbf{U}(A) \subset \mathbf{U}(B)\,,$$

(3) $$A \subset \mathbf{U}(A) \quad \textit{for each} \quad A \subset S\,,$$

(4) $$\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A) \quad \textit{for each} \quad A \subset S$$

hold.

A subset A of S will be called $\mathbf{U}$-closed if $\mathbf{U}(A) = A$. The set of all $\mathbf{U}$-closed subsets of S will be denoted by $\mathscr{F}(\mathbf{U})$.

(5) *If $A \subset S$, then $\mathbf{U}(A) = \bigcap_{i \in I} A_i$ where A_i $(i \in I)$ are all $\mathbf{U}$-closed subsets of S such that $A \subset A_i$.*

Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. Then we define

$$\mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathbf{U}(A) \subset \mathbf{V}(A) \quad \textit{for each} \quad A \subset S . \tag{6}$$

We have

$$\mathscr{F}(\mathbf{U} \vee \mathbf{V}) = \mathscr{F}(\mathbf{U}) \cap \mathscr{F}(\mathbf{V}), \tag{7}$$

$$\mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathscr{F}(\mathbf{V}) \subset \mathscr{F}(\mathbf{U}) . \tag{8}$$

We shall denote by $\mathscr{Q}(S)$ the set of all $\mathscr{Q}$-closure operations for a set S, i.e. $\mathscr{Q}(S) \subset \mathscr{C}(S)$ and for every $\mathbf{U} \in \mathscr{Q}(S)$ and for every $A \subset S$,

$$\mathbf{U}(A) = \bigcup_{x \in A} \mathbf{U}(x) \tag{9}$$

holds.

Let $\mathbf{U} \in \mathscr{C}(S)$. We define $\mathbf{U}^* \in \mathscr{Q}(S)$.

$$\textit{If} \quad A \subset S \quad \textit{then} \quad x \in \mathbf{U}^*(A) \quad \textit{if and only if} \quad \mathbf{U}(x) \cap A \neq \emptyset . \tag{10}$$

For $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$ we have

$$\mathbf{U} \leqq \mathbf{V} \Rightarrow \mathbf{U}^* \leqq \mathbf{V}^* , \tag{11}$$

$$\mathbf{U}(x) = \mathbf{U}^{**}(x) \quad \textit{for every} \quad x \in S , \tag{12}$$

$$\mathbf{U}^{**} \leqq \mathbf{U} , \tag{13}$$

$$\mathbf{U}^{***} = \mathbf{U}^* . \tag{14}$$

Put $\mathbf{O}(A) = A$ for each $A \subset S$. Then $\mathbf{O} \in \mathscr{Q}(S)$ and

$$\mathbf{O} \leqq \mathbf{U} \quad \textit{holds for every} \quad \mathbf{U} \in \mathscr{C}(S) . \tag{15}$$

See [10].

Let $\mathbf{U} \in \mathscr{C}(S)$. We shall introduce the equivalence $\overline{\mathbf{U}}$ on S by: for $x, y \in S$, $x\overline{\mathbf{U}}y$ if and only if $\mathbf{U}(x) = \mathbf{U}(y)$. For any element x of S, let $\mathbf{U}_x$ denote the $\overline{\mathbf{U}}$-class of S containing x. If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$ then we have

$$\mathbf{U} \leqq \mathbf{V} \Rightarrow \overline{\mathbf{U}} \subset \overline{\mathbf{V}} , \tag{16}$$

$$x\overline{\mathbf{U}}y \Leftrightarrow x \in \mathbf{U}(y) \quad \textit{and} \quad y \in \mathbf{U}(x) . \tag{17}$$

See [9].

Let S be an arbitrary semigroup. Put $\mathbf{P}(\emptyset) = \emptyset$. If $A \subset S$ ($A \neq \emptyset$), then by $\mathbf{P}(A)$ we denote the subsemigroup generated by all elements of A. Evidently $\mathbf{P} \in \mathscr{C}(S)$ and $\mathscr{F}(\mathbf{P})$ is the set of all subsemigroups of S (including $\emptyset$). See [10]. Let $\mathbf{K} = \mathbf{P}^* \vee \mathbf{P}^{**}$. Then $\mathbf{K} = \mathbf{K}^*$ and $x\overline{\mathbf{K}}y$ if and only if there exist positive integers n, m such that $x^n = y^m$. See [9].

By an *ordered semigroup*, we mean a semigroup S with an order which is compatible with the semigroup operation:

$$a, b, c \in S \quad \textit{and} \quad a \leqq b \quad \textit{imply} \quad ac \leqq bc \quad \textit{and} \quad ca \leqq cb\,.$$

A subset A of S is called *convex* if for every $a, b \in A$ and for every $c \in S$

$$(18) \qquad a \leqq c \leqq b \quad \textit{implies} \quad c \in A\,.$$

We shall denote by $\mathbf{C}(A)$ the convex hull of a subset $A \subset S$. It is clear that $\mathbf{C} \in \mathscr{C}(S)$ and $\mathscr{F}(\mathbf{C})$ is the set of all convex subsets of S. It follows from (12) that $\mathbf{C}^{**} = \mathbf{O}$.

Put $\mathbf{P_C} = \mathbf{P} \vee \mathbf{C}$. It follows from (7) that $\mathscr{F}(\mathbf{P_C})$ is the set of all convex subsemigroups of S (including $\emptyset$).

Lemma 1. *Let $u, x \in S$. Then $u \in \mathbf{P_C}(x)$ if and only if $x^n \leqq u \leqq x^m$ for some positive integers n, m.*

Proof. Let $A = \{v \in S / x^n \leqq v \leqq x^m$ for some positive integers $n, m\}$. Since $\mathbf{P_C}(x)$ is a convex subsemigroup of S containing x, hence by (18), $A \subset \mathbf{P_C}(x)$.

If $v, w \in A$, then $x^{n_1} \leqq v \leqq x^{m_1}$ and $x^{n_2} \leqq w \leqq x^{m_2}$ for some positive integers n_1, m_1, n_2, m_2. This implies that $x^{n_1+n_2} \leqq vw \leqq x^{m_1+m_2}$ and thus we have $vw \in A$. Hence, A is a subsemigroup of S. If $v \leqq z \leqq w$ for $v, w \in A$ and for $z \in S$, then $x^{n_1} \leqq v \leqq z \leqq w \leqq x^{m_2}$ for some positive integers n_1, m_2. This means that $z \in A$. According to (18), A is a convex subsemigroup of S. It follows from (5) that $\mathbf{P_C}(x) \subset$ $\subset A$. Therefore, $A = \mathbf{P_C}(x)$.

Lemma 2. *Let $x, y \in S$. Then $\mathbf{P_C}(x) \cap \mathbf{P_C}(y) \neq \emptyset$ if and only if $x^n \leqq y^r \leqq x^m$ for some positive integers n, r, m.*

Proof. If $x^n \leqq y^r \leqq x^m$ for some positive integers n, r, m, then it follows from Lemma 1 that $y^r \in \mathbf{P_C}(x)$. Evidently $y^r \in \mathbf{P_C}(y)$. Hence we have $\mathbf{P_C}(x) \cap \mathbf{P_C}(y) \neq \emptyset$.

Let $\mathbf{P_C}(x) \cap \mathbf{P_C}(y) \neq \emptyset$. Then there exists an element $u \in \mathbf{P_C}(x) \cap \mathbf{P_C}(y)$. Lemma 1 implies that $x^{n_1} \leqq u \leqq x^{m_1}$ and $y^{n_2} \leqq u \leqq y^{m_2}$ for some positive integers n_1, m_1, n_2, m_2. Then we have $x^n = x^{n_1 n_2} \leqq u^{n_2} \leqq y^{n_2 m_2} = y^r \leqq u^{m_2} \leqq x^{m_1 m_2} = x^m$ where $n = n_1 n_2$, $r = n_2 m_2$ and $m = m_1 m_2$.

Lemma 3. *Let $A \subset S$. Then $A \in \mathscr{F}(\mathbf{P^*_C})$ if and only if for every $x \in S$*

$$(19) \qquad x^n \leqq u \leqq x^m\,, \quad u \in A \Rightarrow x \in A\,.$$

Proof. Let $A \in \mathscr{F}(\mathbf{P^*_C})$. If $x^n \leqq u \leqq x^m$ for some positive integers n, m and for some $u \in A$, then by Lemma 1, $u \in \mathbf{P_C}(x)$. It follows from (10) and (2) that $x \in \mathbf{P^*_C}(u) \subset$ $\subset \mathbf{P^*_C}(A) = A$.

Let (19) hold for every $x \in S$. Evidently $\mathbf{P^*_C} \in \mathscr{Q}(S)$. If $A \neq \emptyset$, then by (9) we have $\mathbf{P^*_C}(A) = \bigcup_{x \in A} \mathbf{P^*_C}(x)$. If $y \in \mathbf{P^*_C}(A)$, then $y \in \mathbf{P^*_C}(x)$ for some $x \in A$. It follows from (10) and (12) that $x \in \mathbf{P}_C(y)$. Then by Lemma 1 and (19), $y \in A$. Thus we have $\mathbf{P}_C^*(A) \subset A$. It follows from (3) that $A = \mathbf{P}_C^*(A) \in \mathscr{F}(\mathbf{P}_C^*)$.

Lemma 4. *Let $A \subset S$. Then $A \in \mathscr{F}(\mathbf{P}_C^{**})$ if and only if for every $x \in S$*

$$u^n \leqq x \leqq u^m\,, \quad u \in A \Rightarrow x \in A\,. \tag{20}$$

Proof. Let $A \in \mathscr{F}(\mathbf{P}_C^{**})$. If $u^n \leqq x \leqq u^m$ for some positive integers n, m and for some $u \in A$, then by Lemma 1, (12) and (2), $x \in \mathbf{P}_C(u) = \mathbf{P}_C^{**}(u) \subset \mathbf{P}_C^{**}(A) = A$.

Let (20) hold for every $x \in S$. Since $\mathbf{P}_C^{**} \in \mathscr{Q}(S)$, hence by (9) and (12) we have $\mathbf{P}_C^{**}(A) = \bigcup_{x \in A} \mathbf{P}_C(x)$. If $y \in \mathbf{P}_C^{**}(A)$, then $y \in \mathbf{P}_C(x)$ for some $x \in A$. According to Lemma 1 and (20), $y \in A$. Therefore, $\mathbf{P}_C^{**}(A) \subset A$. It follows from (3) that $A = \mathbf{P}_C^{**}(A) \in \in \mathscr{F}(\mathbf{P}_C^{**})$.

Definition 1. $\mathbf{K}_C = \mathbf{P}_C^* \vee \mathbf{P}_C^{**}$.

Lemma 5. $\mathbf{K}_C = \mathbf{K}_C^*$.

Proof. Evidently $\mathbf{P}_C^* \leqq \mathbf{K}_C$ and $\mathbf{P}_C^{**} \leqq \mathbf{K}_C$. It follows from (11) and (14) that $\mathbf{P}_C^{**} \leqq \mathbf{K}_C^*$ and $\mathbf{P}_C^* = \mathbf{P}_C^{***} \leqq \mathbf{K}_C^*$. This implies that $\mathbf{K}_C = \mathbf{P}_C^* \vee \mathbf{P}_C^{**} \leqq \mathbf{K}_C^*$. According to (11) and (13), we have $\mathbf{K}_C^* \leqq \mathbf{K}_C^{**} \leqq \mathbf{K}_C$. Hence $\mathbf{K}_C = K_C^*$.

Lemma 6. $\mathbf{K} \leqq \mathbf{K}_C$ and $\overline{\mathbf{K}} \subset \overline{\mathbf{K}}_C$.

Proof. Since $\mathbf{P} \leqq \mathbf{P} \vee \mathbf{C} = \mathbf{P}_C$, hence by (11) we have $\mathbf{P}^* \leqq \mathbf{P}_C^*$ and $\mathbf{P}^{**} \leqq \mathbf{P}_C^{**}$. Therefore $\mathbf{K} = \mathbf{P}^* \vee \mathbf{P}^{**} \leqq \mathbf{P}_C^* \vee \mathbf{P}_C^{**} = \mathbf{K}_C$. According to (16), we have $\overline{\mathbf{K}} \subset \overline{\mathbf{K}}_C$.

Remark 1. Evidently, if $\mathbf{C} = \mathbf{O}$ (e.g. if S is an unordered semigroup) then $\mathbf{K}_C = \mathbf{K}$ and $\overline{\mathbf{K}}_C = \overline{\mathbf{K}}$.

Theorem 1. *Let S be an ordered semigroup and let $x, y \in S$. Then $x\overline{\mathbf{K}}_C y$ if and only if $x^n \leqq y^r \leqq x^m$ for some positive integers n, r, m.*

Proof. If $x^n \leqq y^r \leqq x^m$ for some positive integers n, r, m, then it follows from Lemma 2 that there exists $u \in \mathbf{P}_C(x) \cap \mathbf{P}_C(y)$. (10) and (6) imply that $x \in \mathbf{P}_C^*(u) \subset \subset \mathbf{K}_C(u)$. By (12) and (6), we have $u \in \mathbf{P}_C(x) = \mathbf{P}_C^{**}(x) \subset \mathbf{K}_C(x)$. Then, by (17), we have $x\overline{\mathbf{K}}_C u$. We can similarly prove that $u\overline{\mathbf{K}}_C y$. Hence $x\overline{\mathbf{K}}_C y$.

Let $x\overline{\mathbf{K}}_C y$. Put $A = \{u \in S / x^n \leqq u^r \leqq x^m$ for some positive integers $n, r, m\}$. Evidently $x \in A$. We shall prove that $A \in \mathscr{F}(\mathbf{K}_C) = \mathscr{F}(\mathbf{P}_C^* \vee \mathbf{P}_C^{**}) = \mathscr{F}(\mathbf{P}_C^*) \cap \cap \mathscr{F}(\mathbf{P}_C^{**})$ (see (7)). Let $u, v \in S$. If $u \in A$ and $v^s \leqq u \leqq v^t$ for some positive integers s, t, then $x^n \leqq u^r \leqq x^m$ for some positive integers n, r, m. This implies that $x^{ns} \leqq \leqq u^{rs} \leqq v^{rst} \leqq u^{rt} \leqq x^{mt}$ and thus we have $v \in A$. It follows from Lemma 3 that $A \in \mathscr{F}(\mathbf{P}_C^*)$. If $v \in A$ and $v^s \leqq u \leqq v^t$ for some positive integers s, t, then $x^n \leqq v^r \leqq x^m$

for some positive integers n, r, m. Hence, $x^{ns} \leqq v^{rs} \leqq u^r \leqq v^{rt} \leqq x^{mt}$ so that $u \in A$. Lemma 4 implies that $A \in \mathscr{F}(\mathbf{P}_C^{**})$. Since $x \in A \in \mathscr{F}(\mathbf{K}_C)$, hence by (17) and (2) $y \in \mathbf{K}_C(x) \subset \mathbf{K}_C(A) = A$. Therefore, $x^n \leqq y^r \leqq x^m$ for some positive integers n, r, m.

Definition 2. The equivalence $\overline{\mathbf{K}}_C$ in an ordered semigroup S is called an *Archimedean equivalence.* An equivalence class of S modulo the Archimedean equivalence $\overline{\mathbf{K}}_C$ is called an *Archimedean class.*

Theorem 2. *Every Archimedean class of an ordered semigroup S is convex.*

Proof. Let $x, y \in S$ and $x\overline{\mathbf{K}}_C y$. It follows from Theorem 1 that $x^n \leqq y^r \leqq x^m$ for some positive integers n, r, m. If $x \leqq z \leqq y$ for some $z \in S$, then $x^r \leqq z^r \leqq y^r \leqq x^m$. Theorem 1 implies that $x\overline{\mathbf{K}}_C z$. Thus every Archimedean class of S is convex.

Remark 2. It follows from Theorem 1 and Theorem 2 that the set of all Archimedean classes of an ordered semigroup S is the maximal decomposition into convex unions of subsemigroups of S.

An element x of an ordered semigroup S is called *nonnegative* if $x \leqq x^2$, while y is called *nonpositive* if $y^2 \leqq y$. A subset A of S is called *nonnegatively* (*nonpositively*) *ordered*, if every element of A is nonnegative (nonpositive).

Lemma 7. *If x is a nonnegative element of S, then $x^n \leqq x^m$ for any positive integers n, m ($n \leqq m$).*

Proof is obvious.

We denote by E the set of idempotents of an ordered semigroup S.

Lemma 8. *Let x be a nonnegative periodic element of S. If $x^n = e \in E$ for some positive integer n, then $ex = e = xe$.*

Proof. Evidently $ex = x^{n+1} = xe$. It follows from Lemma 7 that $e = x^n \leqq \leqq x^{n+1} \leqq x^{2n} = e^2 = e$. Therefore, $ex = e = xe$.

Theorem 3. (Cf. [7], Lemma 2.1.) *Let x, y be nonnegative elements of a simple ordered semigroup S. Then $x\overline{\mathbf{K}}_C y$ if and only if there exists a positive integer n such that $x \leqq y \leqq x^n$ or $y \leqq x \leqq y^n$.*

Proof. If $x \leqq y \leqq x^n$ or $y \leqq x \leqq y^n$ for some positive integer n, then it follows from Theorem 1 that $x\overline{\mathbf{K}}_C y$. Suppose that $x\overline{\mathbf{K}}_C y$. According to Theorem 1, we have $y^s \leqq x^n \leqq y^r$ for some positive integers s, n, r. If $x \leqq y$, then, by Lemma 7, we obtain $x \leqq y \leqq y^s \leqq x^n$. If $y \leqq x$, then $y \leqq x \leqq x^n \leqq y^r$.

The following order dual of Theorem 3 holds:

Theorem 4. *Let x, y be nonpositive elements of a simple ordered semigroup S. Then $x\overline{\mathbf{K}}_{\mathbf{C}}y$ if and only if there exists a positive integer n such that $x^n \leqq y \leqq x$ or $y^n \leqq x \leqq y$.*

Theorem 5. *Let x be a nonnegative element of an ordered semigroup S and let y be a nonpositive element of S. Then $x\overline{\mathbf{K}}_{\mathbf{C}}y$ if and only if there exists a positive integer n such that $x^n = y^n = e \in E$. If $x\overline{\mathbf{K}}_{\mathbf{C}}y$, then $x \leqq y$ and $xy = e = yx$.*

Proof. If $x^n = y^n = e \in E$ for some positive integer n, then according to Theorem 1 we have $x\overline{\mathbf{K}}_{\mathbf{C}}y$. Suppose that $x\overline{\mathbf{K}}_{\mathbf{C}}y$. Theorem 1 implies that $x^k \leqq y^r \leqq x^m$ for some positive integers k, r, m. It follows from Lemma 7 and its dual that $x \leqq x^k \leqq y^r \leqq y$ and $y^n \leqq y^r \leqq x^m \leqq x^n$ where $n = \max(r, m)$. Then $x^n \leqq y^n \leqq x^n$ so that $x^n = y^n$. Now we put $e = x^n = y^n$ and so, by Lemma 7 and its dual, $e = x^n \leqq x^{2n} = e^2 = y^{2n} \leqq y^n = e$. Hence $e = e^2 \in E$.

If $x\overline{\mathbf{K}}_{\mathbf{C}}y$, then it follows from Lemma 7 and its dual that $x \leqq x^n = e = y^n \leqq y$. Lemma 8 implies that $xy \leqq ey = e = xe \leqq xy$ and $yx \leqq ye = e = ex \leqq yx$. Therefore, $xy = e = yx$.

Corollary. (Cf. [7], Corollary 2.4.) *Every Archimedean class of an ordered semigroup S contains at most one idempotent.*

Definition 3. If an Archimedean class A of an ordered semigroup S contains one idempotent, then A is called a *periodic Archimedean class*. Otherwise A is called a *nonperiodic Archimedean class*.

Theorem 6. *If x is a periodic element of an ordered semigroup S, then $\mathbf{K}_{\mathbf{C}x} = \mathbf{K}_x$.*

Proof. Obviously, $x^n = e \in E$ for some positive integer n. It follows from Lemma 6 that $\mathbf{K}_x \subset \mathbf{K}_{\mathbf{C}x}$. Let $u \in \mathbf{K}_{\mathbf{C}x}$. Then $x\overline{\mathbf{K}}_{\mathbf{C}}u$ and so, by Theorem 1, $x^r \leqq u^s \leqq x^t$ for some positive integers r, s, t. Since $e = x^{nr} \leqq u^{ns} \leqq x^{nt} = e$, hence $u^{ns} = e$ and thus we have $u \in \mathbf{K}_e = \mathbf{K}_x$. Therefore $\mathbf{K}_x = \mathbf{K}_{\mathbf{C}x}$.

Corollary 1. *Every element of a periodic (nonperiodic) Archimedean class is periodic (nonperiodic).*

Corollary 2. *If S is a periodic ordered semigroup, then $\overline{\mathbf{K}}_{\mathbf{C}} = \overline{\mathbf{K}}$.*

Theorem 7. *If e is an idempotent of an Archimedean class A having only nonnegative and nonpositive elements, then e is a zero element in A.*

Proof follows from Theorem 6 and from Lemma 8 and its dual.

Corollary. *If e is an idempotent of a simple ordered Archimedean class A, then e is a zero element in A.*

A subset A of an ordered semigroup S is called *nonnegatively* (*nonpositively*) *ordered in the strict sense*, if $x \leqq xy$ and $x \leqq yx$ ($xy \leqq x$ and $yx \leqq x$) for every $x, y \in A$.

Theorem 8. *The following conditions on a simple ordered periodic Archimedean class A are equivalent*:

1. *A is nonnegatively ordered in the strict sense,*
2. *A is nonnegatively ordered,*
3. *An idempotent of A is the greatest element in A.*

Proof. $1 \Rightarrow 2$. Evident.

$2 \Rightarrow 3$. If $x \in A$, then $x^n = e \in E$ for some positive integer n. Lemma 7 implies that $x \leqq x^n = e$.

$3 \Rightarrow 1$. Let $x, y \in A$. Suppose that $xy < y$. This implies that $x^{k+1}y \leqq x^k y$ for every positive integer k. Evidently, $x^n = e \in E$ for some positive integer n. It follows from Corollary to Theorem 7 that $e = ey = x^n y \leqq x^{n-1}y \leqq \ldots \leqq xy < y \leqq e$, which is a contradiction. Thus we have $y \leqq xy$. We can prove $y \leqq yx$ in a similar way. Thus A is nonnegatively ordered in the strict sense.

Theorem 9. *The following conditions on a simple ordered periodic Archimedean class A are equivalent*:

1. *A is nonpositively ordered in the strict sense,*
2. *A is nonpositively ordered,*
3. *An idempotent of A is the least element in A.*

Proof is order dual to that of Theorem 8.

Lemma 9. *Let $x \in S$. If $x^n \leqq x^{n+k}$ for some positive integers n and k, then there exists a positive integer m such that $x^m \leqq x^{2m}$.*

Proof. It is clear that there exist positive integers r and q such that $n + r = qk$. Since $x^n \leqq x^{n+k}$, hence $x^{qk} = x^{n+r} \leqq x^{n+r+k} = x^{(q+1)k}$. This implies that

$$x^{qk} \leqq x^{(q+1)k} \leqq x^{(q+2)k} \leqq \ldots \leqq x^{2qk} .$$

Putting $m = qk$, we have $x^m \leqq x^{2m}$.

Definition 4. We say that a nonperiodic Archimedean class A of an ordered semigroup S satisfies *Condition* (P) if it holds:

(P) *for every $x \in A$ and for any positive integers n, m such that $x^n \leqq x^m$, we have $n \leqq m$.*

We say that a nonperiodic Archimedean class A of S satisfies *Condition* (N) if it holds:

(N) *for every* $x \in A$ *and for any positive integers* n, m *such that* $x^n \leqq x^m$, *we have* $n \geqq m$.

Remark 3. Condition (P) can be replaced by

(P′) $x^n \parallel x^{n+k}$ *or* $x^n < x^{n+k}$ *for every* $x \in A$ *and for any positive integers* n, k.

Similarly, Condition (N) is equivalent to

(N′) $x^n \parallel x^{n+k}$ *or* $x^n > x^{n+k}$ *for every* $x \in A$ *and for any positive integers* n, k.

Remark 4. A nonperiodic Archimedean class A satisfies Conditions (P) and (N) if and only if for every $x \in A$ and for any positive integers n, m $(n \neq m)$

$$x^n \parallel x^m .$$

Theorem 10. *Every nonperiodic Archimedean class* A *of an ordered semigroup* S *satisfies at least one of Conditions* (P) *and* (N).

Proof. Suppose that A does not satisfy Conditions (P) and (N). Then there exist elements $x, y \in A$ such that $x^n \geqq x^{n+k}$ and $y^m \leqq y^{m+l}$ for some positive integers n, k, m, l. It follows from Lemma 9 and its dual that $x^r \geqq x^{2r}$ and $y^s \leqq y^{2s}$ for some positive integers r, s. Evidently $x^r, y^s \in A$. By Theorem 5, A has an idempotent and so A is a periodic Archimedean class, which is a contradiction.

Theorem 11. *Let* x *be a nonperiodic element of an ordered semigroup* S. *If a nonperiodic Archimedean class* $\mathbf{K}_{\mathbf{C}x}$ *satisfies Conditions* (P) *and* (N), *then* $\mathbf{K}_{\mathbf{C}x} = \mathbf{K}_x$.

Proof. By Lemma 6, we have $\mathbf{K}_x \subset \mathbf{K}_{\mathbf{C}x}$. Let $u \in \mathbf{K}_{\mathbf{C}x}$. Then $x\overline{\mathbf{K}}_{\mathbf{C}}u$ and Theorem 1 implies that $x^n \leqq u^r \leqq x^m$ for some positive integers n, r, m. According to Remark 4, we have $n = m$ and $x^n = u^r$. Hence $x\overline{\mathbf{K}}u$ and so $u \in \mathbf{K}_x$. Therefore $\mathbf{K}_x = \mathbf{K}_{\mathbf{C}x}$.

A subset A of an ordered semigroup S is called *positively* (*negatively*) *ordered in the strict sense*, if $x < xy$ and $x < yx$ ($xy < x$ and $yx < x$) for every $x, y \in A$.

Theorem 12. (Cf. [7], Lemma 2.5.) *Every simple ordered nonperiodic Archimedean class* A *satisfying Condition* (P) *is positively ordered in the strict sense.*

Proof. It follows from (P′) of Remark 3 that $x < x^2$ for every $x \in A$. Let $x, y \in A$. If $x \leqq y$, then $x < x^2 \leqq xy$. If $y < x$, then by Theorem 3 we have $x \leqq y^n$ for some positive integer n. Next we suppose that $xy \leqq x$. Then $x^2 \leqq xy^n \leqq xy^{n-1} \leqq \dots$ $\dots \leqq xy \leqq x$ and so $x^2 \leqq x < x^2$, which is a contradiction. Thus $x < xy$. Similarly we can prove $x < yx$. Thus A is positively ordered in the strict sense.

Dually, we have the following

Theorem 13. *Every simple ordered nonperiodic Archimedean class A satisfying Condition* (N) *is negatively ordered in the strict sense.*

A non-empty set A of a semigroup S is called *commutative* if $xy = yx$ for every $x, y \in A$.

Theorem 14. *An Archimedean class A of an ordered semigroup S is a convex subsemigroup of S if one of the following conditions is satisfied*:

1. *A is simple ordered,*
2. *A is nonnegatively ordered in the strict sense,*
3. *A is nonpositively ordered in the strict sense,*
4. *A is commutative.*

Proof. It suffices to prove only that A is a subsemigroup of S (see Theorem 2).

1. Let A be a simple ordered Archimedean class of S. If $x, y \in A$, then $x^2, y^2 \in A$. Since $x \leqq y$ or $y \leqq x$, hence $x^2 \leqq xy \leqq y^2$ or $y^2 \leqq xy \leqq x^2$. By Theorem 2, we have $xy \in A$.

2. Let A be a nonnegatively ordered Archimedean class in the strict sense of S. If $x, y \in A$, then it follows from Theorem 1 that $y^n \leqq x^m$ for some positive integers n, m. Since A is nonnegatively ordered in the strict sense, hence $x \leqq xy \leqq xy^2 \leqq \dots$ $\dots \leqq xy^n \leqq x^{m+1}$. By Theorem 1, we have $xy \in A$.

3. Dual to 2.

4. Let A be a commutative Archimedean class of S. If $x, y \in A$, then it follows from Theorem 1 that $x^n \leqq y^r \leqq x^m$ for some positive integers n, r, m. Thus we have $x^{n+r} \leqq x^r y^r = (xy)^r = x^r y^r \leqq x^{m+r}$. By Theorem 1, we have $xy \in A$.

Remark 5. Let every Archimedean class A of an ordered semigroup S satisfy one of the conditions of Theorem 14. Then it follows from Remark 2 and Theorem 14 that the set of all Archimedean classes of S is the maximal decomposition into convex subsemigroups of S. See [8].

Author's Note. When the paper had already been in print, the author's attention was drawn to the paper by Saitô T.: *Note on the Archimedean Property in Ordered Semigroup,* Bul. Tokyo Gakugei Univ. *22* (1970), 8—12, where Archimedean properties of *simple* ordered semigroups are studied.

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