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CONCERNING ALMOST REALCOMPACTIFICATIONS

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INTRODUCTION

It is known that every completely regular Hausdorff space has a realcompactification υX, contained in the Stone-Čech compactification βX, with the property that every continuous function from X to a realcompact space can be extended to υX. In [1], FROLÍK has defined and investigated almost realcompactness. The purpose of this paper is to show that every Hausdorff space has an almost realcompactification ϱX, which is contained in the Katětov H-closed extension $\varkappa X$, which is a projective maximum in the class of almost realcompactifications of X, and which has an extension property similar to (although necessarily weaker than) that of υX.

1. PRELIMINARIES

1.1. Definitions. An *open filter* is a non-empty collection of open sets $\mathcal{U}$ such that

(1) $\emptyset \notin \mathcal{U}$, and
(2) if $U, V \in \mathcal{U}$ and $G = \text{int}(G) \supset U \cap V$, then $G \in \mathcal{U}$.

An open filter $\mathcal{U}$ is said to have *the countable closure intersection property* (abbreviated c.c.i.p.) provided that for each countable subcollection $C \subset \mathcal{U}$, $\bigcap\{\text{cl}\, U \mid U \in C\} \neq \emptyset$. An *open ultrafilter* is an open filter which is maximal in the collection of open filters.

1.2. Definitions. Let X be a Hausdorff space. X is said to be

(1) *H-closed* provided that every open ultrafilter on X converges; or, equivalently, X is closed in every Hausdorff space in which it is embedded.
(2) *almost realcompact* provided that every open ultrafilter on X with c.c.i.p. converges.

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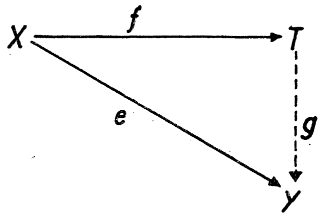
All H-closed, all Lindelöf and all realcompact spaces are almost realcompact. However, (unlike the situation for α-spaces [7]) there are spaces which are not almost realcompact; e.g. the space of all ordinals less than the first uncountable ordinal.

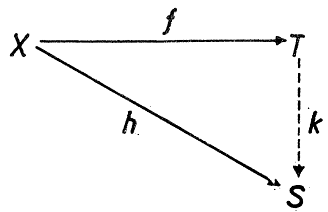
1.3. Definitions. An *almost realcompactification* (resp. *H-closed extension*) of a Hausdorff space X is a pair (Y, e) where Y is an almost realcompact space (resp. H-closed space) and $e : X \to Y$ is a dense embedding.

If $f : X \to T$ is a dense embedding, then (T, f) is called a *ϱ-extension* of X provided that for any almost realcompactification (Y, e) of X, there exists a continuous function $g : T \to Y$ such that $g \,.\, f = e$, i.e.,



commutes. If, in addition, T is almost realcompact, then (T, f) is called a *projective maximum* in the class of almost realcompactifications of X.

Notice that by the definition, a projective maximum (T, f) in the class of almost realcompactifications of X is essentially unique, i.e., if (S, h) is any other projective maximum, then there exists a homeomorphism $k : T \to S$ such that



commutes.

In [8] KATĚTOV has constructed an H-closed extension $(\varkappa X, e)$ for each Hausdorff space X, which in [5] has been shown to be a projective maximum in the class of H-closed extensions of X. We briefly indicate the construction of $\varkappa X$. Let $X^\vee$ be the set of all non-convergent open ultrafilters on X. Let $\varkappa X = X \cup X^\vee$ topologized as follows: a base for the neighborhoods of a point p in $\varkappa X$ is its neighborhood system in X if $p \in X$, and is $\{\{p\} \cup G \mid G = \operatorname{int} G \in p\}$ if $p \in X^\vee$. $e : X \to \varkappa X$ is the inclusion function (which is open).

1.4. Lemma. *If X is a dense subset of Y and $\mathscr{U}'$ is an open ultrafilter on Y, then $\mathscr{U} = \{U \cap X \mid U \in \mathscr{U}'\}$ is an open ultrafilter on X which converges in Y if and only if $\mathscr{U}'$ converges in Y.*

1.5. Lemma. *Suppose that X is dense in Y and $\mathscr{U}$ is an open ultrafilter on X. Then $\mathscr{U}' = \{G \mid G$ is open in Y and $G \cap X \in \mathscr{U}\}$ is an open ultrafilter on Y which converges in Y if and only if $\mathscr{U}$ converges in Y.*

1.6. Corollary. *If X is an open, dense subset of Y, then $\mathscr{U}$ and $\mathscr{U}'$, as described in the above lemmas, are related by: $\mathscr{U} = \{U \in \mathscr{U}' \mid U \subset X\}$.*

2. AN ALMOST REALCOMPACTIFICATION OF X

2.1. Definition. For any Hausdorff space X, X^c will be the set of all non-convergent open ultrafilters on X with c.c.i.p., and ϱX will be $X \cup X^c$, considered as a subspace of $\varkappa X$.

2.2. Lemma. *Suppose that each of T and Y contains X as a dense subset, where Y is almost realcompact. Then every continuous $g : T \to Y$ whose restriction on X is the inclusion map, can be extended to a continuous mapping from ϱT into Y.*

Proof. Let $\mathscr{P} \in \varrho T - T = T$. Then $\mathscr{P}$ is a non-convergent open ultrafilter on T with c.c.i.p. Let $\mathscr{U} = \{P \cap X \mid P \in \mathscr{P}\}$ and $\mathscr{G} = \{G$ open in $Y \mid G \cap X \in \mathscr{U}\}$. By Lemmas 1.4. and 1.5, $\mathscr{G}$ is an open ultrafilter on Y. We wish to show that $\mathscr{G}$ has c.c.i.p. in Y.

Suppose there exists a countable collection $G_n \in \mathscr{G}$ such that $\bigcap_n \mathrm{cl}_Y G_n = \emptyset$. Let $g^{-1}[G_n] = P_n$. Since $P_n \cap X = G_n \cap X \in \mathscr{U}$, by the maximality of $\mathscr{P}$ and Lemma 1.5, we have that $P_n \in \mathscr{P}$. Now, $g[P_n] \subset G_n$ implies that $g[\mathrm{cl}_T P_n] \subset \mathrm{cl}_Y g[P_n] \subset \mathrm{cl}_Y G_n$. Therefore $\bigcap_n g[\mathrm{cl}_T P_n] = \emptyset$, and consequently $\bigcap_n \mathrm{cl}_T P_n = \emptyset$. But this contradicts the fact that $\mathscr{P}$ has c.c.i.p. in T. Since Y is almost realcompact, there exists some point $p \in Y$ such that $\mathscr{G}$ converges to p in Y.

Let us define $f(\mathscr{P}) = p$ for $\mathscr{P} \in \varrho T - T$ and $f(t) = g(t)$ for $t \in T$.

It is clear that f is continuous at each $t \in T$ because T is open in ϱT. Consider $\mathscr{P} \in \varrho T - T$, where $f(\mathscr{P}) = p$ as above. Let W be an open neighborhood of p in Y. Since $W \in \mathscr{G}$, it follows that $G = g^{-1}[W] \in \mathscr{P}$. Thus $G \cup \{\mathscr{P}\}$ is an open neighborhood of $\mathscr{P}$ in ϱT such that $f[G \cup \{\mathscr{P}\}] = g[G] \cup \{p\} \subset W$.

2.3. Theorem. *Let $e : X \to \varrho X$ be the inclusion map. Then $(\varrho X, e)$ is an almost realcompactification of X which is a projective maximum in the class of almost realcompactifications of X.*

Proof. Since X is open an dense in $\varkappa X$, it is clear that it is open and dense in ϱX. Let $\mathscr{U}$ be an open ultrafilter on ϱX with c.c.i.p. in ϱX. We wish to show that $\mathscr{U}$ converges in ϱX. Let $\mathscr{P} = \{U \subset X \mid U \in \mathscr{U}\}$. By Lemma 1.4 and Corollary 1.6, $\mathscr{P}$ is an open ultrafilter on X, whose convergence in X implies the convergence of $\mathscr{U}$ in ϱX. Suppose that $\mathscr{P}$ does not converge in X. We will show that it then belongs to X^c. Let $U_n \in \mathscr{P}$, $n = 1, 2, \ldots$ Clearly, by the construction of ϱX,

$$\mathrm{cl}_{\varrho X} U_n = \mathrm{cl}_X U_n \cup A_n \quad \text{where} \quad A_n = \{\mathscr{V} \in X^c \mid U_n \in \mathscr{V}\}\,.$$

Notice that when $m \neq n$, $\mathrm{cl}_X U_n \cap A_m = \emptyset$. Thus since $\mathscr{U}$ has c.c.i.p. in ϱX, we have

$$\emptyset \neq \bigcap_n (\mathrm{cl}_X U_n \cup A_n) = (\bigcap_n \mathrm{cl}_X U_n) \cup (\bigcap_n A_n)\,.$$

If $\bigcap_n A_n = \emptyset$, then $\bigcap_n \mathrm{cl}_X U_n \neq \emptyset$. If $\hat{\mathscr{P}} \in \bigcap_n A_n$, then each $U_n \in \hat{\mathscr{P}}$. Since $\hat{\mathscr{P}}$ has c.c.i.p. in X, it follows that $\bigcap_n \mathrm{cl}_X U_n \neq \emptyset$. Consequently $\mathscr{P}$ has c.c.i.p. in X, i.e., $\mathscr{P} \in X^c$. Now every open neighborhood of $\mathscr{P}$ in ϱX contains a member of $\mathscr{P}$ (which is contained in $\mathscr{U}$). Thus $\mathscr{U}$ converges to $\mathscr{P}$. To show that $(\varrho X, e)$ is a projective maximum in the class of almost realcompactifications of X, let (Y, j) be an almost realcompactification of X. By Lemma 2.2 there is a continuous extension of j to a map from ϱX to Y.

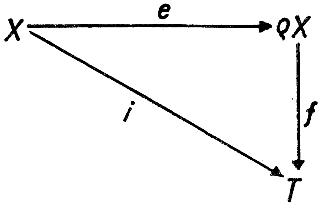
2.4. Theorem. *The following hold*:

(1) $\varrho X = X$ *if and only if* X *is almost realcompact.*

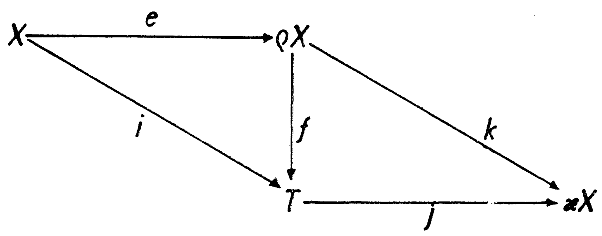
(2) ϱX *is the largest* ϱ*-extension of* X.

(3) ϱX *is the smallest almost realcompact space between* X *and* $\varkappa X$.

Proof (cf. [6, Theorem 3.10]) (1) is immediate from Theorem 2.3 and the construction of ϱX. To see (2) suppose that T is a ϱ-extension of X. By Lemma 2.2 ϱT is also a ϱ-extension of X, so that by the essential uniqueness of ϱX, we have $T \subset \varrho T \approx \varrho X$. For (3), suppose that $X \overset{i}{\hookrightarrow} T \overset{j}{\hookrightarrow} \varkappa T$, where T is almost realcompact. By the projective maximality of $(\varrho X, e)$, there is a continuous function $f : \varrho X \to T$ such that



commutes. Let $k : \varrho X \to \varkappa X$ be the inclusion. Since e is a dense map,



commutes. Thus since k and j are inclusions, f must be an inclusion.

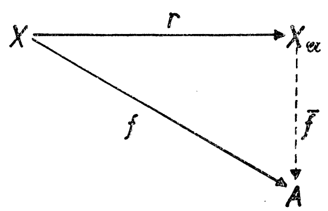
Corollary. *If X is dense in T, then the following are equivalent*:

(1) *T is a ϱ-extension of X.*
(2) *T is homeomorphic to a subset of ϱX.*
(3) $\varrho X \approx \varrho T$.

3. EPI-REFLECTIONS

For categorical notations not specifically defined below the reader should see MITCHELL [9]. In all categories under consideration, we will not distinguish among isomorphic objects.

3.1. Definition. A full subcategory $\mathfrak{A}$ of a category $\mathfrak{C}$ is said to be *epi-reflective in* $\mathfrak{C}$ provided that for each object X in $\mathfrak{C}$, there is an object $X_{\mathfrak{A}}$ in $\mathfrak{A}$ and a $\mathfrak{C}$-epimorphism $r : X \to X_{\mathfrak{A}}$ such that for each object A in $\mathfrak{A}$ and each $\mathfrak{C}$-morphism $f : X \to A$, there exists a morphism $\bar{f} : X_{\mathfrak{A}} \to A$ such that



commutes. (Note that since r is an epimorphism, $\bar{f}$ must be unique.)

There are many examples of epi-reflections in general topology. The Stone-Čech compactification, the Hewitt realcompactification and the Banaschewski zero-dimensional compactification are examples of epi-reflections where all categories in question are full subcategories of the category **Haus** of Hausdorff spaces and continuous functions. For more examples see [2]. In [3] it was shown that although the category of H-closed spaces and all continuous functions between them is not epi-reflective in the category **Haus**, when the class of morphisms is suitably restricted, an epi-reflective situation does occur, thereby giving rise to a situation analogous to the Stone-Čech compactification. In this section it will be shown that with a similar restriction, the almost realcompactification of the previous section yields an epi-reflective situation analogous to the Hewitt realcompactification.

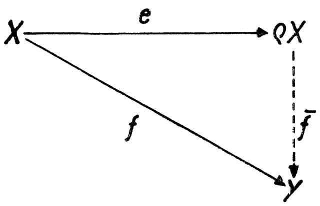
KENNISON [4] has shown that every epi-reflective subcategory of **Haus** must be closed under products and closed subspaces. Thus since closed subspaces of almost realcompact spaces need not be almost realcompact, [1, p. 132] the category of almost realcompact spaces and all continuous functions between them is not epi-reflective in **Haus**.

3.2. Definition. [3] A function $f : X \to Y$ is said to be *demi-open* (resp. *semi-open*) provided that for each $A \subset X$, $\operatorname{int} A \neq \emptyset \Rightarrow \operatorname{int}(\operatorname{cl} f[A]) \neq \emptyset$ (resp. $\operatorname{int}(A) \neq \emptyset \Rightarrow \operatorname{int} f[A] \neq \emptyset$) Clearly every open map is semi-open, every semi-open map is demi-open, and semi-open and demi-open maps are closed under composition.

3.3. Lemma. *If $X \subset Y$ and $f : X \to Z$ is demi-open (resp. semi-open), then an extension of f to* $\operatorname{cl} X$ *must be demi-open (resp. semi-open).*

3.4. Theorem. *The full subcategory of almost realcompact spaces is epi-reflective in the category of Hausdorff spaces and continuous demi-open functions.*

Proof. Let X be any space. Clearly the embedding $e : X \to \varrho X$ is open, so it is demi-open. Now let Y be almost realcompact and $f : X \to Y$ be demi-open and continuous. We wish to find a demi-open continuous function $\bar{f} : \varrho X \to Y$ such that



commutes.

For each $x \in X$, let $\bar{f}(e(x)) = f(x)$. Now if $\mathcal{U} \in \varrho X - e[X]$, then $\mathcal{U}$ is an open ultra-filter on X with c.c.i.p. Since f is demi-open, one can easily show that

$$\hat{\mathcal{U}} = \{W \subset Y : W = \operatorname{int} W, \text{ and for some } U \in \mathcal{U} \operatorname{int}(\operatorname{cl} W) \supset \operatorname{int}(\operatorname{cl} f[U])\}$$

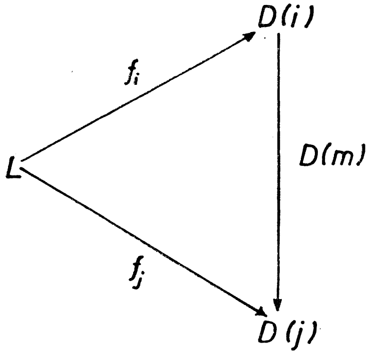
is an open filter on Y. If W is an open set which meets every member of $\hat{\mathcal{U}}$, then $f^{-1}[W] \cap U \neq \emptyset$ for each $U \in \mathcal{U}$. Thus since $\mathcal{U}$ is maximal, $f^{-1}[W] \in \mathcal{U}$. Consequently, since $\operatorname{int}(\operatorname{cl} W) \supset \operatorname{int}(\operatorname{cl}(f(f^{-1}[W])))$, W must be in $\hat{\mathscr{U}}$. Therefore $\hat{\mathscr{U}}$ is an open ultrafilter.

We wish to show that $\hat{\mathscr{U}}$ has c.c.i.p. If $V_n \in \hat{\mathscr{U}}$, $n = 1, 2, \ldots$, then for each n, there is some U_n such that $\operatorname{int}(\operatorname{cl} V_n) \supset \operatorname{int}(\operatorname{cl} f[U_n])$. Since $\mathscr{U}$ has c.c.i.p., there is some $x \in \bigcap_n \operatorname{cl} U_n$. If $f(x) \in W = \operatorname{int} W$, then $f^{-1}[W]$ meets each U_n. Since f is demi-open, W meets each $\operatorname{int}(\operatorname{cl} f[U_n])$. Thus $f(x) \in \bigcap_n \operatorname{cl}(\operatorname{int}(\operatorname{cl} f[U_n])) \subset \bigcap_n \operatorname{cl}(V_n)$.

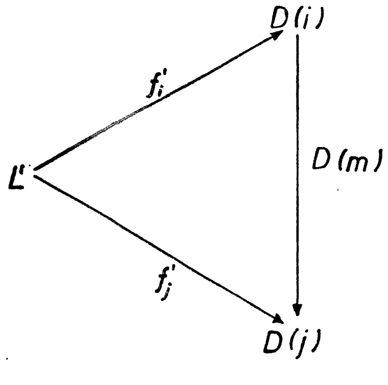
Since $\hat{\mathscr{U}}$ has c.c.i.p. and Y is almost realcompact, $\hat{\mathscr{U}}$ must converge to some point $u \in Y$. For each such $\mathscr{U} \in \varrho X - e[X]$, we let $\bar{f}(\mathscr{U}) = u$. The proof of continuity of $\bar{f}$ is essentially the same as that given in Lemma 2.2. $\bar{f}$ is demi-open by Lemma 3.3.

3.5. Corollary. *The full subcategory of almost realcompact spaces is epi-reflective in the category of Hausdorff spaces and continuous semi-open functions.*

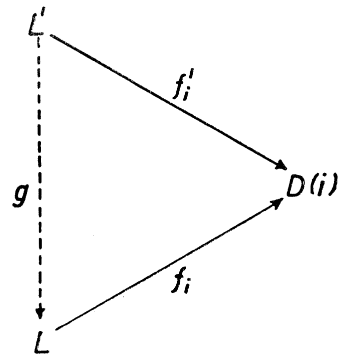
3.6. Definition. A diagram in a category $\mathfrak{A}$ is a functor $D : I \to \mathfrak{A}$, where I is a small category. A *limit* of a diagram $D : I \to \mathfrak{A}$ is an object L together with morphisms $L \xrightarrow{f_i} D(i)$ such that for each morphism $m : i \to j$ in I,



commutes, and whenever $\{L' \xrightarrow{f'_i} D(i)\}$ has the property that for each $m : i \to j$ in I,



commutes, then there is a unique morphism $g : L' \to L$ such that for each i in I

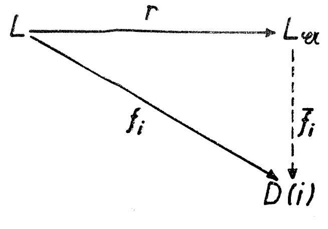


commutes.

It is well-known that epi-reflections are coadjoints of limit preserving inclusions [9, p. 129]. However, this fact cannot be applied in our situation since, for example, the category of Hausdorff spaces and continuous demi-open (or semi-open) functions does not have products. The following strengthened version of the limit preservation property will be useful.

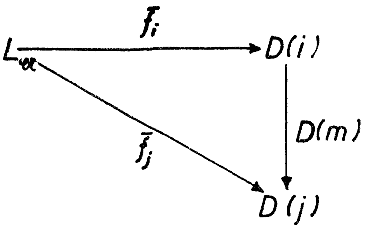
3.7. Theorem. *Let $\mathfrak{A}$ be an epi-reflective subcategory of $\mathfrak{B}$ where $\mathfrak{B}$ is subcategory of $\mathfrak{C}$ having the same objects as $\mathfrak{C}$ and the property that every epimorphism in $\mathfrak{B}$ is an epimorphism in $\mathfrak{C}$. If $\{L \xrightarrow{f_i} D(i)\}$ is a limit in $\mathfrak{C}$ of a diagram $D : I \to \mathfrak{A}$, and if each f_i is in $\mathfrak{B}$, then L must be in $\mathfrak{A}$.*

Proof. Since $\mathfrak{A}$ is epi-reflective in $\mathfrak{B}$ there is an $\mathfrak{A}$-object $L_{\mathfrak{A}}$ and a $\mathfrak{B}$-epimorphism $r : L \to L_{\mathfrak{A}}$. Since each $D(i)$ is in $\mathfrak{A}$ and $f_i : L \to D(i)$ is in $\mathfrak{B}$, there exists a morphism $\bar{f}_i$ such that the diagram

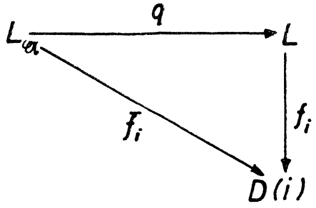


commutes.

Clearly, since r is an epimorphism, for each $m : i \to j$ in I



commutes. Thus there is a $\mathfrak{C}$-morphism $q : L_{\mathfrak{A}} \to L$ such that



commutes for each i. Thus for each i, $f_i \,.\, q \,.\, r = \bar{f}_i \,.\, r = f_i = f_i \,.\, 1_L$. Hence by the uniqueness in the definition of limit, $q \,.\, r = 1_L$. Also $r \,.\, q \,.\, r = r \,.\, 1_L = 1_{L_{\mathfrak{A}}} \,.\, r$, so that since r is an epimorphism in $\mathfrak{C}$, $r \,.\, q = 1_{L_{\mathfrak{A}}}$. Thus r is an isomorphism and consequently L is in $\mathfrak{A}$.

3.8. Corollary. [Frolík]. *Every product of almost realcompact spaces is almost realcompact and every regular-closed subspace of an almost realcompact space is almost realcompact.*

Proof. Let $\mathfrak{B}$ be the category of Hausdorff spaces and continuous demi-open functions. Let $\mathfrak{A}$ be the full subcategory of $\mathfrak{B}$ consisting of all almost realcompact spaces, and let $\mathfrak{C}$ be **Haus**.

Every topological product $\{P \xrightarrow{\pi_i} D(i)\}$ of almost realcompact spaces is a limit of a diagram D from a small discrete category into $\mathfrak{A}$ and each projection π_i is open and thus is demi-open. Hence by the theorem, P must be in $\mathfrak{A}$.

A regular-closed subspace S of an almost realcompact space is an equalizer of demi-open maps, i.e., the limit of a diagram $D : I \to \mathfrak{A}$ where $I = i \underset{n}{\overset{m}{\rightrightarrows}} j$, and $D(m)$ and $D(n)$ are demi open.

Also a regular closed embedding is demi-open. Thus by the theorem, S must be in $\mathfrak{A}$.

3.9. Corollary. *If $\{X_\alpha, \mu_{\beta\alpha}\}$ is an inverse limit spectrum over a directed set, where each X_α is almost realcompact and all spectrum and projection maps are demi-open and continuous, then the inverse limit* $\underleftarrow{\mathrm{Lim}}\, X_\alpha$ *must be almost realcompact.*

Proof. Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be the same categories as in the above corollary. Since inverse limits are particular limits, the result is an immediate consequence of the theorem.

References

[1] *Z. Frolík,* A Generalization of Realcompact Spaces, Czech. Math. J. *13* (*88*) (1963), 127—137.

[2] *H. Herrlich,* Topologische Reflexionen und Coreflexionen, Springer-Verlag, Berlin, Lecture Notes *78* (1968).

[3] *H. Herrlich* and *G. E. Strecker, H*-closed Spaces and Reflective Subcategories, Math. Annalen, *177* (1968), 302—309.

[4] *J. F. Kennison,* Reflective Functors in General Topology and Elsewhere, Trans. Amer. Math. Soc. *118* (1965), 303—315.

[5] *C. T. Liu,* Absolutely Closed Spaces, Trans. Amer. Math. Soc. *130* (1968), 86—104.

[6] *C. T. Liu,* The α-closure αX of a Topological Space X, Proc. Amer. Math. Soc. *23* (1969), 605—607.

[7] *C. T. Liu,* An Equivalent Condition for the Existence of a Measurable Cardinal, Proc. Amer. Math. Soc. *22* (1969), 620—624.

[8] *M. Katětov,* Über *H*-abgeschlossene und bikompakte Räume, Časopis Pěst. Mat. *69* (1940), 36—49.

[9] *B. Mitchell,* Theory of Categories, Academic Press, New York, 1965.

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