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CANTOR-BERNSTEIN THEOREM FOR LATTICE ORDERED GROUPS

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Orthogonally complete lattice ordered groups („l-groups") and K-spaces were studied in the papers [2], [3], [7], [9]. The purpose of this Note is to show that for complete and orthogonally complete l-groups the following proposition analogous to the Cantor-Bernstein theorem is valid: $(*)$ *Let G and H be complete and orthogonally complete l-groups. Let $\bar{G}$ and $\bar{H}$ be the corresponding lattices. Assume that there exists an isomorphism φ of the lattice $\bar{G}$ into $\bar{H}$ and an isomorphism ψ of the lattice $\bar{H}$ into $\bar{G}$ such that $\varphi(\bar{G})$ is a convex sublattice of $\bar{H}$ and $\psi(\bar{H})$ is a convex sublattice of $\bar{G}$. Then the lattice ordered groups G and H are isomorphic.*

In particular, if G and H are l-groups such that $\bar{G} = \bar{H}$ and if G is complete and orthogonally complete, then G and H are isomorphic. The main step in the proof of $(*)$ is the theorem on the representation of positive elements of a singular l-group (Thm. 3.2) that is analogous to the integral representation of elements of a K-space (cf. [8], Chap. III). If the l-groups G and H are not complete or if they are not orthogonally complete, then the assertion of the theorem $(*)$ need not hold.

The standard notations for lattices and lattice ordered groups will be used [1], [5]. Let $G = (G; +, \wedge, \vee)$ be a lattice ordered group. The corresponding lattice $(G; \wedge, \vee)$ will be denoted by $\bar{G}$. The lattice $\bar{G}$ is infinitely distributive. G is said to be complete, if the lattice $\bar{G}$ is conditionally complete. A subset $\{x_i\}_{i\in I}$ of G is disjoint (or orthogonal) if $x_i \geqq 0$ for each $i \in I$ and $x_{i_1} \wedge x_{i_2} = 0$ for any pair of distinct elements $i_1, i_2 \in I$. G is called orthogonally complete if $\bigvee_{i\in I} x_i$ exists in G whenever $\{x_i\}_{i\in I}$ is a disjoint subset of G. Let $a, b \in G$, $a \leqq b$. The interval $[a, b]$ is the set $\{x \in G : a \leqq x \leqq b\}$. Let A be a subset of G such that $a_1, a_2 \in A$, $a_1 \leqq a_2$ implies $[a_1, a_2] \subset A$. Then A is said to be a convex subset of G. Let L_1 be a sublattice of a lattice L. Assume that from $\{x_i\} \subset L_1$, $\bigvee x_i = x \in L$ it follows $x \in L_1$ and that the dual condition also holds. Then L_1 is called a closed sublattice of L. Let $a, e \in G$, $a \geqq 0$, $e > 0$. The element a is singular, if $x \wedge (a - x) = 0$ for each $x \in [0, a]$. The element e is a weak unit, if $e \wedge x > 0$ for each $0 < x \in G$. If e is a weak unit of G, let $B(e)$ be the set of all $e_1 \in [0, e]$ with the property that e_1 has a relative complement in the interval $[0, e]$. Let $\emptyset = X \subset G$. The set $X^\delta = \{y \in G : |y| \wedge |x| = 0$ for each $x \in X\}$ is a polar of G. Any polar X^δ is a closed convex l-subgroup of G and the intersection $X^\delta \cap Y^\delta$ of two polars is a polar [10]. If $X = \{a\}$, $a > 0$, then the element a is a weak unit of the l-group $X^{\delta\delta}$. For any $Y \subset G$ we denote $Y^+ = \{y \in Y : y \geqq 0\}$.

The lattice ordered group G is a K-space provided there can be defined a multiplication λx of elements $x \in G$ with reals λ such that G turns out to be a linear space with the property that $\lambda x > 0$ for each $\lambda > 0$ and $x > 0$.

I. DIRECT PRODUCTS OF l-GROUPS

In this section there are given the basic definitions and described some properties of the direct product of l-groups that we shall need in the sequel. (Cf. also [6].) Let $\{G_i\}_{i \in I}$ be a system of l-groups and let H be the set of all mappings $f : I \to \bigcup G_i$ such that $f(i) \in G_i$ for each $i \in I$. $f(i)$ is the component of f in G_i. The operations $+, \wedge, \vee$ in H are performed componentwise. Then $H = \Pi_{i \in I} G_i$ is the direct product of l-groups G_i. Let G be an l-group and let φ be an isomorphism of G onto H. For each $i \in I$ denote

$$G_i^0 = \{x \in G : \varphi(x)(j) = 0 \text{ for each } j \in I,\ j \neq i\} .$$

G_i^0 is a closed convex l-subgroup of G and G_i^0 is isomorphic to G_i. For each $x \in G$ let x_i be the element of G_i^0 satisfying $\varphi(x)(i) = \varphi(x_i)(i)$. The mapping

$$x \to (\ldots, x_i, \ldots)_{i \in I}$$

is an isomorphism of the l-group G onto $\Pi_{i \in I} G_i^0$. We shall write

$$G = \Pi_{i \in I}^0 G_i^0 .$$

Let $x \in G_i^0$ for some $i \in I$. Then $x_i = x$ and $x_j = 0$ for each $j \in I$, $j \neq i$. If $y \in G_j^0$, $j \neq i$, then $|x| \wedge |y| = 0$.

1.1. *Let $\{X_i\}_{i \in I}$ be a system of convex l-subgroups of an l-group G such that*

(i) $x^i \wedge x^j = 0$ *for any* $0 \leqq x^i \in X_i$ *and any* $0 \leqq x^j \in X_j$ *whenever* $i, j \in I$, $i \neq j$,
(ii) *for each* $0 < x \in G$ *there are elements* $0 \leqq x^i \in X_i$ *such that* $x = \bigvee_{i \in I} x^i$.

Then G is isomorphic to a subgroup of $\Pi_{i \in I} X_i$.

Proof. Since G is infinitely distributive, the elements x^i from (ii) are uniquelly determined. Let $x \in G$, $i \in I$. Denote $x \vee 0 = y$, $-(x \wedge 0) = z$, $x^i = y^i - z^i$. Then it is easy to verify that the mapping $x \to (\ldots, x_i, \ldots)_{i \in I}$ is an isomorphism of G into ΠX_i.

A system $S = \{X_i\}_{i \in I}$ of convex l-subgroups of an l-group G is called orthogonal, if the condition (i) from 1.1 is fulfilled; S is maximal orthogonal, if $S = S'$, whenever

$S' \supset S$ is an orthogonal system of convex l-subgroups of G. An orthogonal system S is maximal orthogonal if and only if for each $0 < g \in G$ there is $i \in I$ and $x_i \in X_i$ such that $0 < g \wedge x_i$.

1.2. *Let $S = \{X_i\}_{i \in I}$ be a maximal orthogonal system of convex l-subgroups of a complete and orthogonally complete l-group G. Assume that each X_i is a closed l-subgroup of G. Then $G = \Pi^0_{i \in I} X_i$.*

Proof. Let $0 < x \in G$, $i \in I$. Denote $x^i = \sup \{y \in X_i : y \leqq x\}$. Since X_i is closed, x^i belongs to X_i. The system $\{x^i\}_{i \in I}$ is disjoint and hence there exists $z = \bigwedge_{i \in I} x^i$ in G and $0 \leqq z \leqq x$. Suppose that $z < x^i$. Then $v = x - z > 0$. Since the system S is maximal orthogonal, there is an element $i \in I$ and $t \in X_i$ such that $0 < v \wedge t = u^i$. Clearly $u^i \in X_i$. We have $x^i < x^i + u^i \leqq x$ and $x^i + u^i \in X_i$, which is a contradiction. Thus $x = \bigvee_{i \in I} x^i$. According to 1.1 the correspondence $\varphi : x \to (\dots, x^i, \dots)$ is an isomorphism of G into $\Pi_{i \in I} X_i$. In order to verify that φ is onto it suffices to show that $\varphi(G^+) = (\Pi_{i \in I} X_i)^+$, since each element of an l-group is a difference of positive elements. For each $i \in I$ let $0 \leqq y^i \in X_i$. Then $\bigvee y^i = x$ does exist in G and $x^i = x^i \wedge x = \bigvee_{j \in I}(x^i \wedge y^j) = x^i \wedge y^i$. Since $y^i \leqq x$, we have $y^i = x^i$, thus $\varphi(x) = (\dots, y_i, \dots)$. This shows that φ is an isomorphism onto. If $x \in X_i$, then $x^i = x$ and $x^j = 0$ for each $j \in I$, $j \neq i$. From this it follows $X_i^0 = X_i$ and therefore we may write $G = \Pi^0_{i \in I} X_i$.

1.3. *Let e be a weak unit of a complete and orthogonally complete l-group G. Assume that $e = \bigvee_{i \in I} e_i$ and $e_{i_1} \wedge e_{i_2} = 0$ for any pair of distinct elements i_1, i_2 of I. Denote $X_i = \{e_i\}^{\delta\delta}$. Then $G = \Pi^0_{i \in I} X_i$.*

Proof. Each X_i is closed convex l-subgroup of G and $e_i \in X_i$. Since $\{e_i\}_{i \in I}$ is a disjoint set in G, the system $S = \{X_i\}_{i \in I}$ is orthogonal. If $0 < g \in G$, then $0 < g \wedge e = \bigvee_{i \in I} g \wedge e_i$, hence $g \wedge e_i > 0$ for some e_i. This shows that S is a maximal orthogonal system of convex l-groups in G. Now it suffices to apply 1.2.

An l-subgroup Y of G is called a direct factor of G if there is a direct decomposition $G = \Pi^0_{j \in J} Y_j$ of G such that $Y = Y_j$ for some $j \in J$.

Each direct factor of G is a closed convex l-subgroup of G. For $g \in G$ the component g_i of g in the direct factor Y_i will be denoted also by $g_i = g(Y_i)$. The following assertions 1.4 and 1.5 are known (cf. [6]):

1.4. *Let Y be a direct factor of G, $0 \leqq g \in G$. Then the component $g(Y)$ of g in Y is the element $g(Y) = \sup \{y \in Y : y \leqq g\}$; therefore $g(Y) \leqq g$. If $g \wedge y = 0$ for each $0 \leqq y \in Y$, then $g(Y) = 0$.*

1.5. *Let Y be a direct factor of G and let $G = \Pi^0_{i \in I} X_i$. Then $Y = \Pi^0_{i \in I}(Y \cap X_i)$, the l-subgroups $Y \cap X_i$ are direct factors of G and for any $g \in G$,*

$$g(Y \cap X_i) = g(Y)(X_i) = g(X_i)(Y).$$

In particular, if $Y \subset X_i$ for some $i \in I$, then $g(Y) = g(X_i)(Y)$.

1.6. *Let A, B be direct factors of G such that $A \cap B = \{0\}$ and let C be the subgroup of G generated by $A \cup B$. Then C is a direct factor of G and $C = A \times B$.*

Proof. Since A, B are direct factors of G there are l-subgroups A', B', of G such that $G = A \times A'$, $G = B \times B'$. According to 1.5 $B = (B \cap A) \times (B \cap A')$ and similarly $A' = (A' \cap B) \times (A' \cap B') = B \times (A' \cap B')$, thus $G = A \times B \times (A' \cap \cap B')$. Denote $C = \{g \in G : g(A' \cap B') = 0\}$. Then clearly $C = A \times B$. Each element $c \in C$ can be written in the form $c = a + b$, with $a \in A$, $b \in B$; hence C is generated by the set $A \cup B$.

1.7. *Let e be weak unit of a complete and orthogonally complete l-group and assume that the element e is singular. Let $e = x_1 + \ldots + x_n$, $0 \leqq x_i$ $(i = 1, \ldots, n)$, $X_i = \{x_i\}^{\delta\delta}$. Then $G = X_1 \times \ldots \times X_n$.*

Proof. From the definition of a singular element it follows that $x_1 \wedge (x_2 + \ldots \ldots + x_n) = 0$, hence $x_1 \wedge x_i = 0$ for $i = 2, \ldots, n$. Since G is commutative, $x_j \wedge x_i = = 0$ for distinct $i, j \in \{1, \ldots, n\}$. Therefore $x_1 + \ldots + x_n = x_1 \vee \ldots \vee x_n$. Our assertion now follows from 1.3.

1.8. *Let e be a weak unit of a complete and orthogonally complete l-group and let e be singular, $0 < a \leqq e$, $a = x + y$, $0 \leqq x$, $0 \leqq y$. Denote $\{a\}^{\delta\delta} = A$, $\{x\}^{\delta\delta} = = X$, $\{y\}^{\delta\delta} = Y$. Then A, X, Y are direct factors of G and $a(X) = x$.*

Proof. Put $a' = e - a$. Then $0 \leqq a'$ and $e = a + a' = x + y + a'$. According to 1.7 A, X and Y are direct factors of G. If $y = 0$, then $Y = \{0\}$ and thus $x(Y) = 0$. If $y > 0$, then y is a weak unit of the l-group Y. Since $x \wedge y = 0$, we have $x \wedge y_i = 0$ for each $0 \leqq y_i \in Y$, thus according to 1.4 $x(Y) = 0$. Therefore $a(X) = (x + y)(X) = = x(X) = x$.

The following lemma is obvious.

1.9. *Let $G = \Pi^0_{i \in I} X_i$. Then G is complete (orthogonally complete) if and only if each X_i is complete (orthogonally complete).*

2. COMPLETE l-GROUPS AND K-SPACES

We need the following result due to Conrad and McAlister:

2.1. ([4], Thm. 4.9, Corollary 2) *Let S be the set of all singular elements of a complete l-group G. Then $G = S^\delta \times S^{\delta\delta}$ and S^δ is a K-space.*

We denote $S^\delta = K(G)$, $S^{\delta\delta} = K'(G)$. Let G, H be complete and orthogonally complete l-groups and let $\bar{G}$, $\bar{H}$ be the corresponding lattices. Assume that

$$\varphi : \bar{G} \to \bar{H}$$

is an isomorphism of the lattice $\bar{G}$ into $\bar{H}$ such that $\varphi(\bar{G})$ is a convex sublattice of $\bar{H}$ and $\varphi(0) = 0$. In this paragraph we shall prove that the l-group $K(G)$ is isomorphic with a convex l-subgroup of $K(H)$. Let S and S' be the set of all singular elements of G and H, respectively.

2.2. *Let $0 \leqq a$ be an element of an l-group G. Then a is singular if and only if $[0, a]$ is a Boolean algebra.*

Proof. If a is singular and $x \in [0, a]$, then the element $a - x$ is the relative complement of x in $[0, a]$, hence $[0, a]$ is a Boolean algebra. Conversely, let $[0, a]$ be a Boolean algebra, $x \in [0, a]$ and let y be a relative complement of x with respect to the interval $[0, a]$. Then $x \wedge y = 0$, hence $y + x = y \vee x = a$, thus $y = a - x$, therefore $x \wedge (a - x) = 0$.

2.3. *Let $x \in G$. Then $x \in S$ if and only if $\varphi(x) \in S'$.*

Proof. According to 2.2 $x \in S$ if and only if $[0, x]$ is a Boolean algebra and this is fulfilled if and only if $[0, \varphi(x)]$ is a Boolean algebra.

2.4. *Let $0 \leqq x \in G$. Then $x \in (S^\delta)^+$ if and only if $\varphi(x) \in (S'^\delta)^+$.*

Proof. Since $x \geqq 0$, we have $\varphi(x) \geqq 0$. Let $s' \in S'$, $\varphi(x) \wedge s' = s_1$. From 2.2 it follows $s_1 \in S'$. Since $0 \leqq s_1 \leqq \varphi(x)$ and $\varphi(\bar{G})$ is a convex sublattice of $\bar{H}$, we have $s_1 \in \varphi(\bar{G})$, thus there is $y \in G$ such that $\varphi(y) = s_1$ and by 2.3 $y \in S$. Clearly $y \leqq x$. If $x \in S^\delta$, then $x \wedge y = 0$, hence $y = 0$. This implies $s_1 = 0$ and therefore $\varphi(x) \in (S'^\delta)^+$. Conversely, assume that $\varphi(x) \in (S'^\delta)^+$ and let $s \in S$. Then by 2.2 $\varphi(s) \in S'$, hence $\varphi(x) \wedge \varphi(s) = 0$ and from this we obtain $x \wedge s = 0$, thus $x \in (S^\delta)^+$.

2.5. *Let $0 \leqq y \in G$. Then $y \in (S^{\delta\delta})^+$ if and only if $\varphi(y) \in (S'^{\delta\delta})^+$.*

Proof. Since $y \geqq 0$, we have $\varphi(y) \geqq 0$. Let $x' \in (S'^\delta)^+$, $\varphi(y) \wedge x' = x'_1$. Then $x'_1 \in (S'^\delta)^+ \cap \varphi(\bar{G})$, thus there is $x_1 \in G$ such that $x'_1 = \varphi(x_1)$. According to 2.4 $x_1 \in S^\delta$ and clearly $0 \leqq x_1 \leqq y$. If $y \in S^{\delta\delta}$, then $x_1 = 0$, hence $x'_1 = 0$ and therefore $\varphi(y) \in (S'^{\delta\delta})^+$. Conversely, let $\varphi(y) \in S'^{\delta\delta}$, $x \in (S^\delta)^+$. Then by 2.4 $\varphi(x) \in (S'^\delta)^+$ and so $\varphi(y) \wedge \varphi(x) = 0$. This implies $y \wedge x = 0$ and thus $y \in (S^{\delta\delta})^+$.

Let H_1 and H_2 be the intersection of all closed convex orthogonally complete l-subgroups of H that contain $\varphi((S^\delta)^+)$ or $\varphi((S^{\delta\delta})^+)$, respectively. According to 2.1 we have

$$H = S'^\delta \times S'^{\delta\delta},$$

and thus S'^δ is a closed convex orthogonally complete l-subgroup of H. By 2.4 $\varphi((S^\delta)^+) \subset S'^\delta$ and therefore H_1 is a closed convex l-subgroup of S'^δ. Since S'^δ is a K-space, H_1 is a K-space as well. Analogously according to 2.5 H_2 is a closed convex l-subgroup of $S'^{\delta\delta}$.

Let $\{x_i\}$ be a maximal disjoint subset of G. Then $x = \bigvee x_i$ exists in G and x is a weak unit in G. Put $x(S^\delta) = e_1$. The element e_1 is a weak unit in S^δ whenever $S^\delta \neq \{0\}$.

2.6. *Let $S^\delta \neq \{0\}$. Then $\varphi(e_1)$ is a weak unit in H_1.*

Proof. Let $0 < y' \in H_1$. If $y' \wedge x' = 0$ for each $x' \in \varphi((S^\delta)^+)$, then $\{y'\}^\delta$ is a closed convex orthogonally complete l-subgroup of H, $\varphi((S^\delta)^+) \subset \{y'\}^\delta$ and thus $H_1 \subset \subset \{y'\}^\delta$. Clearly $y' \notin \{y'\}^\delta$ which is a contradiction. Therefore there is $x' \in \varphi((S^\delta)^+)$ with $y' \wedge x' = x_1' > 0$. Because of $0 < x_1' \leqq x' \in \varphi(\bar{G})$, we have $x_1' \in \varphi(\bar{G})$ and hence there are elements $x, x_1 \in G$ with $\varphi(x) = x'$, $\varphi(x_1) = x_1'$. Then by 2.4 $x \in S^\delta$ and, since S^δ is a convex l-subgroup of G, x_1 belongs to S^δ as well. We obtain $x_1 \wedge e_1 > 0$, thus $y' \wedge \varphi(e_1) \geqq x_1' \wedge \varphi(e_1) > 0$.

2.7. *The l-groups S^δ and H_1 are isomorphic.*

Proof. S^δ and H_1 are orthogonally complete K-spaces with weak units e_1 and $\varphi(e_1)$, respectively. We have defined $B(e_1)$ as the set of all $x \in [0, e_1]$ that have a relative complement in $[0, e_1]$. By 2.6, $\varphi(e_1)$ is a weak unit in H_1 and thus it follows from $\varphi(0) = 0$ that $\varphi(B(e_1)) = B(\varphi(e_1))$, thus the lattices $B(e_1)$ and $B(\varphi(e_1))$ are isomorphic. This implies (cf. [8], 2.21) that the K-spaces S^δ and H_1 are isomorphic.

3. SINGULAR l-GROUPS

An l-group A with the set S of singular elements is said to be singular, if $S^\delta = \{0\}$, or, equivalently, $S^{\delta\delta} = A$. In this section we assume that the l-group $A \neq \{0\}$ is complete, orthogonally complete and singular and we are searching for a representation of positive elements of A by means of elements of an appropriate Boolean algebra.

3.1. *There is a weak unit e of A such that $e \in S$.*

Proof. Let $\{s_i\}_{i \in I}$ be a maximal disjoint subset of S. Since A is orthogonally complete, there exists $e = \bigvee s_i$ in A. From the fact that $\{s_i\}_{i \in I}$ is a maximal disjoint subset of S it follows that e is a weak unit in A. Let $x \in [0, e]$. Then

$$x = \bigvee x_i, \quad x_i = x \wedge s_i .$$

According to 2.2 $[0, s_i]$ is a Boolean algebra, thus there is a relative complement y_i of x_i in the interval $[0, s_i]$. The system $\{y_i\}_{i \in I}$ is disjoint, hence there is $y = \bigvee y_i$ and $y \in [0, e]$. It is easy to verify that y is a relative complement of x with respect to the interval $[0, e]$. By 2.2, e belongs to S.

In this section we shall use several times the lemmas 1.6, 1.7 and 1.8 without mentioning it explicitely. For $a \in A$ we denote $\{a\}^{\delta\delta} = [a]$ and for any $x \in A$ we write $x[a]$ instead of $x([a])$.

In the sequel we suppose that we have chosen a fixed weak unit e of A such that $e \in S$. Let $0 < f \in A$. We construct two sequences

(1) $$e_0, e_1, e_2, \ldots, e_n, \ldots,$$

(2) $$e_1^*, e_2^*, \ldots, e_n^*, \ldots$$

in the following manner.

Put $e_1 = f \wedge e$, $e_0 = e - e_1$. Then we have $e_0 \wedge e_1 = 0$, $e_0 \vee e_1 = e_0 + e_1 = e$,

(3) $$e_0 \wedge f = 0, \quad e_1 \leqq f.$$

Denote

$$(2e_1 - f) \vee 0 = e_1^*.$$

We have $((2e_1 - f) \vee 0) - e_1 = (e_1 - f) \vee (-e_1) \leqq 0$, thus $e_1^* \leqq e_1$. Put $e_1 - e_1^* = e_2$. Then

(4) $$e = e_0 + e_1^* + e_2,$$

therefore according to 1.7

$$G = [e_0] \times [e_1^*] \times [e_2].$$

From (3) it follows $f[e_0] = 0$, whence $f = f_1 + g_2$, $f_1 = f[e_1^*]$, $g_2 = f[e_2]$. Therefore

$$(2e_1 - f) \vee 0 = ((2e_1^* - f_1) \vee 0) + ((2e_2 - g_2) \vee 0).$$

Since $(2e_1 - f) \vee 0 = e_1^* \in [e_1^*]$, we obtain

(5) $$(2e_1^* - f_1) \vee 0 = e_1^*,$$

(6) $$(2e_2 - g_2) \vee 0 = 0.$$

(5) implies $(e_1^* - f_1) \vee (-e_1^*) = 0$, thus $(f_1 - e_1^*) \wedge e_1^* = 0$. Since e_1^* is a weak unit in $[e_1^*]$ and $0 \leqq f_1 - e_1^* \in [e_1^*]$, we get $f_1 - e_1^* = 0$, thus

$$f_1 = f[e_1^*] = e_1^*.$$

From (6) we infer $2e_2 \leqq g_2$ and clearly $g_2 \leqq f$; therefore

$$2e_2 \leqq f.$$

Let $0 < x \leqq e_1^*$. Denote $y = e_1^* - x$. According to (4) $e = e_0 + x + y + e_2$, hence by 1.7

$$G = [e_0] \times [x] \times [y] \times [e_2].$$

Since $[x] \subset [e_1^*]$ we have (cf. 1.5 and 1.8) $f[x] = f[e_1^*][x] = e_1^*[x] = x$, thus

$$(2x - f)[x] = 2x[x] - f[x] = x > 0$$

and therefore $2x \nleqq f$. Let us assume that for some positive integer n we have constructed elements $e_0, e_1, \ldots, e_n, e_{n+1}$ and $e_1^*, \ldots, e_n^*$ with the following properties:

(α) $e_i \geqq 0,\ e_j^* \geqq 0\ (i = 0, \ldots, n + 1;\ j = 1, \ldots, n)$,
(β) $e = e_0 + e_1^* + \ldots + e_n^* + e_{n+1}$,
(γ) $0 < x \leqq e_i^* \Rightarrow (i + 1)\, x \nleqq f \quad (i = 1, \ldots, n)$,
(δ) $(n + 1)\, e_{n+1} \leqq f$,
(ε) $f[e_i^*] = ie_i^* \quad (i = 1, \ldots, n)$.

As we have already proved the conditions (α) – (ε) hold for $n = 1$. Now we distinguish two cases.

(a) Assume that $e_{n+1} = 0$. Then by (β)

$$G = [e_0] \times [e_1^*] \times \ldots \times [e_n^*],$$

hence

$$f = f[e_1^*] + \ldots + f[e_n^*] = e_1^* + 2e_2^* + \ldots + ne_n^*,$$

and since the system $\{ie_i\}_{i=1,\ldots,n}$ is disjoint, we have

$$f = \bigvee_{i=1}^{n} ie_i .$$

In this case we put $e_i^* = e_j = 0$ for $i \geqq n + 1, j \geqq n + 2$.

(b) Suppose that $e_{n+1} > 0$. Denote $f[e_i^*] = f_i\ (i = 1, \ldots, n)$, $f[e_{n+1}] = g_{n+1}$. From (β) it follows

$$G = [e_0] \times [e_1^*] \times \ldots \times [e_n^*] \times [e_{n+1}],$$

hence

$$(n + 2)\, e_{n+1} - f = -f_1 - \ldots - f_n + ((n + 2)\, e_{n-1} - g_{n+1}),$$

therefore

$$((n + 2)\, e_{n+1} - f) \vee 0 = ((n + 2)\, e_{n+1} - g_{n+1}) \vee 0 . \tag{7}$$

Denote $((n + 2)\, e_{n+1} - f) \vee 0 = e_{n+1}^*$. From (7) we get $e_{n+1}^* \in [e_{n+1}]$. Clearly $e_{n+1}^* \geqq 0$.

We have

$$\{((n + 2)\, e_{n+1} - f) \vee 0\} - e_{n+1} = ((n + 1)\, e_{n+1} - f) \vee (-e_{n+1}) \leqq 0$$

because of (α) and (δ), hence $e_{n+1}^* \leqq e_{n+1}$. Denote $e_{n+2} = e_{n+1} - e_{n+1}^*$. Then $e_{n+2} \geqq 0$ and

$$e = e_0 + e_1^* + \ldots + e_n^* + e_{n+1}^* + e_{n+2} .$$

From $e_{n+1} = e_{n+1}^* + e_{n+2}$ we get (since $e_{n+1} \in S$ and e_{n+1} is a weak unit of $[e_{n+1}]$)

$$[e_{n+1}] = [e_{n+1}^*] \times [e_{n+2}] . \tag{8}$$

Put $g_{n+1}[e^*_{n+1}] = f_{n+1}, g_{n+1}[e_{n+2}] = g_{n+2}$. Clearly $f_{n+1} = f[e^*_{n+1}], g_{n+2} = f[e_{n+2}]$. From (7) and (8) it follows

$$e^*_{n+1} = \{((n + 2) e^*_{n+1} - f_{n+1}) \vee 0\} + \{((n + 2) e_{n+2} - g_{n+2}) \vee 0\} ,$$

whence

$$(9) \qquad e^*_{n+1} = ((n + 2) e^*_{n+1} - f_{n+1}) \vee 0 ,$$

$$(10) \qquad 0 = ((n + 2) e_{n+2} - g_{n+2}) \vee 0 .$$

From (9) we get $0 = ((n + 1) e^*_{n+1} - f_{n+1}) \vee (-e^*_{n+1})$, thus

$$0 = (f_{n+1} - (n + 1) e^*_{n+1}) \wedge e^*_{n+1} .$$

Since $f_{n+1} - (n + 1) e^*_{n+1}$ belongs to $[e^*_{n+1}]$ and e^*_{n+1} is a weak unit in $[e^*_{n+1}]$, we get $f_{n+1} - (n + 1) e^*_{n+1} = 0$, therefore

$$f[e^*_{n+1}] = (n + 1) e^*_{n+1} .$$

From (10) we obtain $(n + 2) e_{n+2} \leqq g_{n+2}$ and since $g_{n+2} = f[e_{n-2}] \leqq f$, we have

$$(n + 2) e_{n+2} \leqq f .$$

Let $0 < x \leqq e^*_{n+1}$. Then $f[x] = f[e^*_{n+1}] [x] = (n + 1) e^*_{n+1}[x] = (n + 1) x$, thus

$$((n + 2) x - f) [x] = x > 0 ,$$

therefore $(n + 2) x \nleq f$.

We have proved that the conditions $(\alpha) - (\varepsilon)$ hold for the positive integer $n + 1$. Hence we can construct the sequences (1) and (2) such that the conditions $(\alpha) - (\varepsilon)$ are satisfied for $n = 1, 2, \ldots$

If $e_{k+1} = 0$ for some positive integer k, then according to (a) we have

$$f = \bigvee_{i=1}^{k} i e^*_i .$$

Assume that $e_{k+1} > 0$ for each $k = 1, 2, \ldots$ and consider the system

$$(11) \qquad e_0, e^*_1, e^*_2, \ldots, e^*_n, \ldots$$

Since for each positive integer n the equation (β) holds and $e \in S$, the system (11) is disjoint and therefore there exists the join p of the system (11). Clearly $p \leqq e$, hence $e - p = q \geqq 0$. Assume that $q > 0$. We have $p \wedge q = 0$, hence $e_0 \wedge q = 0$ and $e^*_n \wedge q = 0$ for $n = 1, 2, \ldots$ According to (β)

$$e = e_0 + e^*_1 + \ldots + e^*_n + e_{n+1} = e_0 \vee (\bigvee_{i=1}^{n} e^*_i) \vee e_{n+1} ,$$

$$q = q \wedge e = (q \wedge e_0) \vee (\bigvee_{i=1}^{n} (q \wedge e^*_i)) \vee (q \wedge e_{n+1}) = q \wedge e_{n+1} ,$$

whence $0 \leqq q \leqq e_{n+1}$ for each integer n. According to (δ)

$$(n + 1)\, q \leqq f$$

for each positive integer n. Since G is archimedean, we have a contradiction. Hence $p = e$, and so

$$e = e_0 \vee \left(\bigwedge_{i=1}^{\infty} e_i^*\right).$$

According to 1.3 this implies

$$G = [e_0] \times \prod_{i=1}^{\infty}{}^{0}[e_i^*].$$

Since $f \geqq 0$ and (ε) holds, we have (because of $f[e_0] = 0$)

$$f = \bigvee_{i=1}^{\infty} f[e_i^*] = \bigvee_{i=1}^{\infty} ie_i^*.$$

Let N be the set of all positive integers, $N(f) = \{i \in N : e_i^* \neq 0\}$. Then

(12) $$f = \bigvee ie_i^* \quad (i \in N(f)).$$

By summarizing, we have the following assertion:

3.2. Theorem. *Let G be a complete and orthogonally complete singular l-group, $0 < f \in G$. Let $e \in G$ be a weak unit of G and let the element e be singular. Then there is a subset $N(f) \subset N$ and a disjoint system $\{e_i^*\}$ $(i \in N(f))$ such that $e \geqq e_i^* > 0$ for each $i \in N(f)$ and $f = \bigvee ie_i^*$ $(i \in N(f))$.*

Let us assume that for the given $0 < f \in G$ there exists another subset $N_1 \subset N$ and a disjoint system $\{e_j'\}$ $(j \in N_1)$ such that $e \geqq e_j' > 0$ for each $j \in N_1$ and $f = \bigvee je_j'$ $(j \in N_1)$. Let $j \in N_1$. Then

(13) $$je_j' = je_j' \wedge f = \bigvee(je_j' \wedge ie_i^*) \quad (i \in N(f)),$$

hence there is $i_0 \in N(f)$ such that $je_j' \wedge i_0 e_{i_0}^* > 0$. This implies $e_j' \wedge e_{i_0}^* = x > 0$. Suppose that $j \neq i_0$. If $j < i_0$, then $e_j' = x + y$, $x \wedge y = 0$, thus

$$f[x] = f[e_j'][x] = je_j'[x] = jx, \quad (i_0 x - f)[x] = (i_0 - j)\, x > 0,$$

therefore $i_0 x \nleqq f$. But from $x \leqq e_{i_0}^*$ we obtain $i_0 x \leqq i_0 e_{i_0}^* \leqq f$, which is a contradiction. Thus $j \geqq i_0$. Analogously we can verify that $i_0 \geqq j$ and hence $i_0 = j$. This implies that $N_1 \subset N(f)$ and similarly $N(f) \subset N_1$, thus $N(f) = N_1$. Further we have $e_j' \wedge e_i^* = 0$ whenever i, j are distinct elements of N_1. Hence it follows from (13) $je_j' = je_j' \wedge je_j^*$ and similarly $je_j^* = je_j' \wedge je_j^*$, thus $je_j' = je_j^*$. Therefore $e_j' = e_j^*$ for each $j \in N_1$. We obtain:

3.3. *Under the same assumptions as in* 3.2 *the set* $N(f)$ *and the system* $\{e_i^*\}$ $(i \in N(f))$ *satisfying the assertion of* 3.2 *are uniquelly determined.*

Let $0 < f \in G$, $0 < g \in G$. Let $N(f)$, $\{e_i^* : i \in N(f)\}$ be as in 3.2 and let $N(g)$, $\{e'_j : j \in N(g)\}$ have an analogical meaning with respect to the element g. Put $e^* = \bigvee e_i^* \ (i \in N(f))$, $e' = \bigvee e'_j (j \in N(g))$. Under these denotations we have:

3.4. $f \leqq g$ *if and only if* $e^* \leqq e'$ *and* $e_i^* \wedge e'_j > 0 \Rightarrow i \leqq j$.

Proof. Let $f \leqq g$. Denote $e - e^* = e_0$, $e - e' = e'_0$. Then $e_0(e'_0)$ is the complement of $e^*(e')$ in the Boolean algebra $[0, e]$. Since $g = \bigvee j e'_j \ (j \in N(g))$, we have $g \wedge e'_0 = 0$, thus $f \wedge e'_0 = 0$. Because of $e^* \leqq f$, it is also $e^* \wedge e'_0 = 0$ and hence $e^* \leqq e'$. Let $e_i^* \wedge e'_j = x > 0$ and assume that $i > j$. Then $ix \leqq ie_i^* \leqq f \leqq g$, but according to 3.3 and (γ) from $0 < x \leqq e'_j$ it follows that $ix \nleqq g$. This is a contradiction; therefore $i \leqq j$.

Conversely, let $e^* \leqq e'$ and $i \leqq j$ whenever $e_i^* \wedge e'_j > 0$. Then $e_i^* \leqq e'$ for each $i \in N(f)$, thus

$$e_i^* = e_i^* \wedge e' = \bigvee_{j \in N(g)} (e_i^* \wedge e'_j)$$

and hence

$$e = e_0 \vee \left(\bigvee_{i \in N(f)} \bigvee_{j \in N(g)} (e_i^* \wedge e'_j)\right).$$

Since the system $\{e_0, e_i^* \wedge e'_j\}$ is disjoint, according to 1.3 we have

$$G = [e_0] \times \Pi^0_{(i,j)} (e_i^* \wedge e'_j)\,, \quad (i, j) \in N(f) \times N(g)\,.$$

Further we have $g[e_0] \geqq 0 = f[e_0]$. If $e_i^* \wedge e'_j = 0$, then $g[e_i^* \wedge e'_j] = f[e_i^* \wedge e'_j] = 0$. If $e_i^* \wedge e'_j > 0$, then

$$f[e_i^* \wedge e'_j] = f[e_i^*]\,[e_i^* \wedge e'_j] = ie_i^*[e_i^* \wedge e'_j] = i[e_i^* \wedge e'_j]$$

and similary $g[e_i^* \wedge e'_j] = j[e_i^* \wedge e'_j]$. Since $j \geqq i$, we have $g \geqq f$.

4. ISOMORPHISM OF SINGULAR l-GROUPS

In this section we assume that A and B are complete and orthogonally complete l-groups with weak units e and e', respectively, such that the elements e and e' are singular. Suppose that φ is an isomorphism of the lattice $[0, e]$ onto $[0, e']$. We intend to prove that then the l-groups A and B are isomorphic.

Let $0 < f \in A$. According to 3.2 and 3.3 there is a uniquelly determined disjoint system $\{e_i^*\}$ $(i \in N(f) \subset N)$ such that $0 < e_i^* \leqq e$ for each $i \in N(f)$ and $f = \bigvee ie_i^*$ $(i \in N(f))$. Then $0 < \varphi(e_i^*) \leqq \varphi(e) = e'$ and $\{\varphi(e_i^*)\}$ is a disjoint system in B. Thus there is $f' = \bigvee i\, \varphi(e_i^*)$ $(i \in N(f))$ in B. From 3.2 and 3.3 (applied for the l-group B) it follows that the correspondence

$$\psi : f \to f'\,, \quad \psi(0) = 0$$

is a one-to-one mapping of the set A^+ onto B^+. According to 3.4 for any $f, g \in A^+$ we have

$$f \leqq g \Leftrightarrow f' \leqq g' .$$

Thus we have proved:

4.1. *ψ is an isomorphism of the lattice A^+ onto B^+.*

For any $x \in A$ we put $0\, x = 0$. Let $0 < f \in A$, $0 < g \in A$. Let e_0, e_0' have the same meaning as in § 3 and put $e_0^* = e_0$, $N'(f) = N(f) \cup \{0\}$, $N'(g) = N(g) \cup \{0\}$. Then

$$f = \bigvee ie_i^* \;(i \in N'(f)) , \quad g = \bigvee je_j' \;(j \in N'(g)) ,$$

$$e = \bigvee e_i^* \;(i \in N'(f)) , \quad e = \bigvee e_j' \;(j \in N'(g))$$

and the systems $\{e_i^* : i \in N'(f)\}$, $\{e_j' : j \in N'(g)\}$ are disjoint. Denote $e_i^* \wedge e_j' = h_{ij}$. Then

$$e = \bigvee h_{ij} \;((i, j) \in N'(f) \times N'(g))$$

and the system $\{h_{ij}\}$ is disjoint. Therefore

$$A = \Pi^0_{(i,j)}[h_{ij}] .$$

Denote $f + g = t$ and define $d(i, j)$ as follows:

$$d(i, j) = 0 \quad \text{if either} \quad (i, j) = (0, 0) \quad \text{or} \quad h_{ij} = 0 , \quad \text{and}$$

$$d(i, j) = i + j \quad \text{otherwise} .$$

For $k = 0, 1, 2, \ldots$ put $M_k = \{(i, j) : d(i, j) = k\}$,

$$t_k^* = \bigvee_{(i,j) \in M_k} h_{ij} .$$

If k_1, k_2 are distinct elements of the set $\{0, 1, 2, \ldots\}$, then $M_{k_1} \cap M_{k_2} = \emptyset$, whence the system $\{t_k^*\}$ is disjoint and $0 \leqq t_k^* \leqq e$. Denote

$$t^0 = \bigvee kt_k^* \quad (k = 0, 1, 2, \ldots) . \tag{14}$$

We have

$$t[h_{ij}] = (f + g)\,[h_{ij}] = f[h_{ij}] + g[h_{ij}] = ih_{ij} + jh_{ij} = (i + j)\, h_{ij} ,$$

$$t^0[h_{ij}] = t^0[t_{i+j}^*]\,[h_{ij}] = (i + j)\, t_{i+j}^*[h_{ij}] = (i + j)\, h_{ij}$$

for each $(i, j) \in N(f) \times N'(g)$ and therefore $t^0 = t$. From this and from (14) it follows

$$\psi(f + g) = \psi(f) + \psi(g) ,$$

hence ψ is an isomorphism of the lattice ordered semigroup A^+ onto B^+. Clearly the l-groups A, B are isomorphic if and only if A^+ and B^+ are isomorphic. We obtain:

4.2. *Let A, B be complete and orthogonally complete singular l-groups with weak units e and e', respectively, such that e and e' are singular elements. If the lattices $[0, e]$ and $[0, e']$ are isomorphic, then the l-groups A and B are isomorphic.*

Now let G and H have the same meaning as in § 2. Under the same denotations as in § 2 we have $S^{\delta\delta} = \{0\}$ if and only if $H_2 = \{0\}$. Let us assume that $S^{\delta\delta} \neq \{0\}$. Since $S^{\delta\delta}$ is a singular l-group, according to 3.1 there exists a singular element $0 < e \in S^{\delta\delta}$ such that e is weak unit of $S^{\delta\delta}$. Let such an element e be fixed.

4.3. *$\varphi(e)$ is a weak unit in H_2.*

Proof. Let $0 < y' \in H_2$. Assume that $y' \wedge \varphi(e) = 0$. Let $x' \in \varphi((S^{\delta\delta})^+)$, $x' > 0$, $x' = \varphi(x)$. According to 2.5 $0 < x \in S^{\delta\delta}$. If $y' \wedge x' = x_1' > 0$, then $x_1' \in \varphi(G)$, $x_1' = \varphi(x_1)$, where $0 < x_1 \leqq x$, thus $x_1 \in S^{\delta\delta}$ and therefore $e \wedge x_1 = t > 0$. This implies $y' \wedge \varphi(e) \geqq x_1' \wedge \varphi(e) = \varphi(x_1) \wedge \varphi(e) = \varphi(x_1 \wedge e) > 0$, which is impossible. Therefore $y' \wedge x' = 0$ for each $x' \in \varphi((S^{\delta\delta})^+)$. Denote $X = \{y'\}^\delta$. Then $\varphi((S^{\delta\delta})^+) \subset X$ and X is a closed, convex and orthogonally closed l-subgroup of H. Hence according to the definition of H_2 we have $H_2 \subset X$. Clearly y' does not belong to X and this is a contradiction.

4.4. *The l-group H_2 is singular.*

Proof. Let S_2 be the set of all singular elements of H_2. For any $\emptyset \neq Z \subset H_2$ let $Z^\delta = \{t \in H_2 : |t| \wedge |z| = 0 \text{ for each } z \in Z\}$ (i.e., the operation Z^δ is taken with respect to H_2). We have $\varphi(e) \in S_2$ and hence $\{\varphi(e)\}^{\delta\delta} \subset S_2^{\delta\delta}$. Since $\varphi(e)$ is a weak unit in H_2, $\{\varphi(\{e)\}^\delta = \{0\}$, thus $\{\varphi(e)\}^{\delta\delta} = H_2$. Therefore $S_2^{\delta\delta} = H_2$ and so H_2 is singular.

4.5. *The l-groups $S^{\delta\delta}$ and H_2 are isomorphic.*

Proof. Let e have the same meaning as in 4.3. $S^{\delta\delta}$ and H_2 are complete and orthogonally complete. The element $e(\varphi(e))$ is a weak unit in $S^{\delta\delta}$ (in H_2) and both elements e and $\varphi(e)$ are singular. Moreover, $[0, e]$ is isomorphic to $[0, \varphi(e)]$. Obviously $S^{\delta\delta}$ is singular and by 4.4 H_2 is singular as well. Thus according to 4.2 the l-groups $S^{\delta\delta}$ and H_2 are isomorphic.

4.6. *The l-groups H_1 and H_2 are orthogonal.*

Proof. Let $0 < x \in H_1$, $0 < y \in H_2$ and assume that $x \wedge y = t > 0$. Since H_1 and H_2 are convex in H, we have $t \in H_1 \cap H_2$. Let e_1 be as in § 2 and let e have the same meaning as above. Since $e_1 \in S^\delta$ and $e \in S^{\delta\delta}$ we have $e_1 \wedge e = 0$, thus $\varphi(e_1) \wedge \wedge \varphi(e) = 0$. Since $\varphi(e_1)$ and $\varphi(e)$ are weak units in H_1 and H_2, respectively, we have $0 < t \wedge \varphi(e_1) \in H_2$, $0 < t \wedge \varphi(e_1) \wedge \varphi(e)$, which is impossible.

4.7. *The l-subgroup H_0 of H generated by $H_1 \cup H_2$ is a direct factor of H and the l-groups G, H_0 are isomorphic.*

Proof. The l-subgroups H_1 and H_2 are closed and convex in H. Since H is complete, according to [1], Chap. XIV, Thm. 19 H_1 and H_2 are direct factors of H. Now it follows from 4.6 and 1.6 that $H_0 = H_1 \times H_2$ is a direct factor of H. Then we obtain from 2.1, 2.7 and 4.5 that G and H_0 are isomorphic.

5. PROOF OF THE THEOREM(*)

Let G and H be complete and orthogonally complete l-groups. Assume that there is an isomorphism φ of the lattice $\bar{G}$ into $\bar{H}$ and an isomorphism ψ of the lattice $\bar{H}$ into $\bar{G}$ such that $\varphi(\bar{G})$ is a convex sublattice of $\bar{H}$ and $\psi(\bar{H})$ is a convex sublattice of $\bar{G}$.

For each $g \in G$ put $\varphi_0(g) = \varphi(g) - \varphi(0)$. Then φ_0 is an isomorphism of $\bar{G}$ into $\bar{H}$ such that $\varphi_0(\bar{G})$ is a convex sublattice of $\bar{H}$ and $\varphi_0(0) = 0$. The mapping $\psi_0(h) = \psi(h) - \psi(0)$ of $\bar{H}$ into $\bar{G}$ has similar properties. Hence in proving the theorem (*) we may assume without loss of generality that $\varphi(0) = 0$, $\psi(0) = 0$. Then according to 4.7 there is an isomorphism

$$\varphi_1 : G \to H$$

of the l-group G into H such that $\varphi_1(G)$ is a direct factor of H. Analogously, there is an isomorphism

$$\psi_1 : H \to G$$

of the l-group H into G such that $\psi_1(H)$ is a direct factor of G. Let $\chi(G) = \psi_1(\varphi_1(G)) = A_1$. Then χ is an isomorphism of G onto A_1 and A_1 is a direct factor of $B = \psi_1(H)$. Hence there are l-subgroups C_1, D_1 of G such that

$$B = D_1 \times A_1 \,, \tag{15}$$

$$A_0 = G = C_1 \times D_1 \times A_1 \,. \tag{16}$$

We define by induction C_n, D_n, A_n $(n = 2, 3, \ldots)$ according to the rule $X_n = \chi(X_{n-1})$ for $X = C, D, A$. Then from (16) it follows

$$A_n = C_{n+1} \times D_{n+1} \times A_{n+1} \tag{17}$$

for $n = 1, 2, \ldots$ Put $\bigcap_{n=1}^{\infty} A_n = A^0$. Consider the system $\mathscr{S}$ of l-subgroups

$$A^0, C_i, D_j \quad (i, j = 1, 2, \ldots) \,.$$

Since $A_{i-1} = C_i \times D_i \times A_i$ and $A^0 \subset A_i$, the l-groups C_i, D_i, A^0 are pairwise orthogonal. If $i < j$, then $D_j \subset A_i$ and thus C_i and D_j are orthogonal. Analogously, if $j < i$, then C_i and D_j are orthogonal. Therefore the system $\mathscr{S}$ is orthogonal.

Let $0 < g \in G$ such that $g \wedge c_i = g \wedge d_i = 0$ for each $0 < c_i \in C_i$ and each $0 < d_i \in D_i$ $(i = 1, 2, \ldots)$. Then $g(C_i) = g(D_i) = 0$ and thus according to (17)

$g \in A_i$ for $i = 1, 2, \ldots$, therefore $g \in A^0$. This shows that the system $\mathcal{S}$ is a maximal orthogonal system of convex l-subgroups of G. Since C_i, D_i, A_i are direct factors of G, they are closed and thus A^0 is closed as well. Therefore it follows from 1.2

$$G = \prod_{i=1}^{\infty}{}^{0} C_i \times \prod_{i=1}^{\infty}{}^{0} D_i \times A^0 . \tag{18}$$

From the fact that $\mathcal{S}$ is a maximal orthogonal system and from (15) we obtain that the system

$$A^0, D_1, C_i, D_j \quad (i, j = 2, 3, \ldots)$$

is a maximal orthogonal system in B; therefore

$$B = \prod_{i=2}^{\infty}{}^{0} C_i \times \prod_{i=1}^{\infty}{}^{0} D_i \times A^0 . \tag{19}$$

Obviously C_m is isomorphic to C_n for $n, m = 1, 2, \ldots$ Therefore G is isomorphic to B. Since $B = \psi_1(H)$, the l-groups G and H are isomorphic. The proof of $(*)$ is complete.

As a corollary, we obtain from $(*)$:

$(**)$ *Let G and H be complete and orthogonally complete l-groups. If the corresponding lattices $\bar{G}$ and $\bar{H}$ are isomorphic, then the l-groups G and H are isomorphic.*

6. EXAMPLES

6.1. Let G and H be orthogonally complete l-groups. Assume that there is an isomorphism φ of the l-group G into H and an isomorphism ψ of the l-group H into G such that $\varphi(G)$ is a convex l-subgroup of H and $\psi(H)$ is a convex l-subgroup of G. The l-groups G and H need not be isomorphic.

Example. Let E be the additive l-group of all integers with the natural order. If X, Y are l-groups, their lexicographic product is denoted by $X \circ Y$ (cf. [5]). For $i = 1, 2, \ldots$ let $B_i = E \circ E$ and

$$G = \prod_{i=1}^{\infty} B_i , \quad H = E \times G .$$

Both l-groups G and H are orthogonally complete. Obviously there is an isomorphism φ of the l-group G into H and an isomorphism ψ of the l-group H into G such that $\varphi(G)$ and $\psi(H)$, respectively, is a convex l-subgroup of H or G. The l-groups G and H are not isomorphic.

6.2. Let G and H be complete l-groups. Let φ and ψ be as in 6.1. The l-groups G and H need not be isomorphic.

Example. If $a < b$ are reals we denote by $F(a, b)$ ($B(a, b)$) the set of all real functions (all bounded real functions) defined on $[a, b]$. Let $G = F(0, 1)$, $H = F(0, 1) \times B(2, 3)$. Clearly G is isomorphic with a convex l-subgroup of H. Let G_0 be the set of all $f \in F(0, 1)$ such that f is bounded on $[\frac{2}{3}, 1]$ and $f(t) = 0$ for each $t \in (\frac{1}{3}, \frac{2}{3})$. Then G_0 is a convex l-subgroup of $F(0, 1)$ isomorphic to H. The l-groups G and H are not isomorphic (G is orthogonally complete and H is not).

6.3. If G, H are complete and orthogonally complete and if φ, ψ satisfy the assumptions of $(*)$, $\varphi(0) = 0$, $\psi(0) = 0$, then φ and ψ need not be isomorphisms with respect to the group operation; $\varphi(G)$ and $\psi(H)$ need not be a subgroup of H or G, respectively.

Example. Let $G = H = E$ (= the additive group of all real numbers with the natural order). There exists an isomorphism φ_0 of the lattice $\bar{E}$ onto $(-1, 1)$ such that $\varphi_0(0) = 0$. Put $\varphi = \psi = \varphi_0$. Then $\varphi(\bar{G})$ is not a subgroup of H and $\psi(\bar{H})$ is not a subgroup of G.

6.4. Let G and H be complete l-groups such that the corresponding lattices $\bar{G}$ and $\bar{H}$ are isomorphic. Then the l-groups G and H need not be isomorphic (i.e., the Proposition $(**)$ cannot be generalized for complete l-groups).

An element $0 < e \in G$ is a strong unit if for each $g \in G$ there is a positive integer n satisfying $g \leqq ne$. Let G_0 be the additive l-group of all real functions defined on the interval $(0, \infty)$ the lattice operations being defined by $f \vee g = \max(f, g)$, $f \wedge g = \min(f, g)$. Let G be the set of all bounded functions $f \in G_0$ and let H be the set of all functions $f \in G_0$ with the property

$$|f(x)| \leqq e^{mx}$$

for some positive integer $m = m(f)$ and for each $x \in (0, \infty)$. Let $f_1(x) = 1$ identically on $(0, \infty)$. Then G and H are l-subgroups of G_0 and f_1 is a strong unit in G. On the other hand, H has no strong unit, thus G and H are not isomorphic. Both l-groups G and H are complete.

Denote $g_m(x) = e^{mx}$ ($m = 1, 2, 3, \ldots$) and let $g_0(x) = 0$ for each $x \in (0, \infty)$. For each fixed $x \in (0, \infty)$ let $\varphi_x(y)$ be a real increasing continuous function defined on the set $(-\infty, \infty) = R$ such that

$$\varphi_x(m) = g_m(x), \quad \varphi_x(-m) = -g_m(x) \quad (m = 0, 1, 2, \ldots).$$

Let $f \in G$. We define $\varphi f \in G_0$ by the rule

$$\varphi f(x) = \varphi_x(f(x))$$

for each $x \in (0, \infty)$. If $|f| \leqq nf_1$ for some positive integer n, then $|\varphi f| \leqq g_n$, hence $\varphi f \in H$. Conversely, if $h \in H$, $|h| \leqq g_n$, then there is a uniquelly determined element

$f \in G$ such that $|f| \leqq nf_1$ and $\varphi f = h$. Since φ_x is an automorphism of R, we have

$$f \leqq g \Leftrightarrow \varphi f \leqq \varphi g$$

for any $f, g \in G$. Therefore φ is an isomorphism of the lattice G onto the lattice H.

6.5. Let G and H be orthogonally complete l-groups such that the lattices $\bar{G}$ and $\bar{H}$ are isomorphic. Then the l-groups G and H need not be isomorphic.

Example. Let E be as in 6.1, $H = E \circ (E \times E)$ and let G be the l-group with three generators desribed in [1], p. 216, Example 6. Then $\bar{G}$ and $\bar{H}$ are isomorphic. The l-groups G and H are not isomorphic (H is abelian and $\bar{G}$ is not).

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