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THE GROUP OF AUTOTOPIES OF A DIGRAPH

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In [4] we have defined the concept of isotopy of digraphs. If G_1, G_2 are two digraphs and there exist two one-to-one mappings f_1, f_2 of the vertex set V_1 of G_1 onto the vertex set V_2 of G_2 such that the existence of the edge $\overrightarrow{f_1(u) f_2(v)}$ in G_2 is equivalent with the existence of the edge $\overrightarrow{uv}$ in G_1 for any two vertices u, v of V_1, we say that G_1 and G_2 are isotopic to each other and we call the ordered pair of mappings $\mathfrak{f} = \langle f_1, f_2 \rangle$ an isotopy of G_1 onto G_2. If $f_1 \equiv f_2$, then $\mathfrak{f}$ is called an isomorphism of G_1 onto G_2. If $\mathfrak{f} = \langle f_1, f_2 \rangle$ is an isotopy of G_1 onto G_2 and $\mathfrak{g} = \langle g_1, g_2 \rangle$ is an isotopy of G_2 onto G_3, then the product $\mathfrak{g}\mathfrak{f}$ is the pair of mappings $\langle g_1 f_1, g_2 f_2 \rangle$; it is evidently an isotopy of G_1 onto G_3. The inverse isotopy $\mathfrak{f}^{-1}$ is the pair $\langle f_1^{-1}, f_2^{-1} \rangle$; it is an isotopy of G_2 onto G_1. An isotopy of a digraph G onto itself is called an autotopy of the digraph G. The autotopies of a given digraph G form a group according to the above defined operations. We shall denote it by $\mathfrak{A}_G$. Now by $\mathfrak{B}_G^{(1)}$, $\mathfrak{B}_G^{(2)}$ we denote the set of permutations of the vertex set V of G which occur at the first or at the second place respectively in some autotopy of G. By $\mathfrak{A}_G^{(1)}$, $\mathfrak{A}_G^{(2)}$ we denote the set of autotopies of G which have the the identity mapping e of V at the first or at the second place respectively. The identity autotopy $\langle e, e \rangle$ will be denoted by $\mathfrak{e}$. Finally, the group of all permutations of V will be denoted by $\mathfrak{S}_G$ (it is isomorphic to the symmetric group of order equal to the cardinality of V) and the group of all ordered pairs of these permutations will be denoted by $\mathfrak{S}_G^{(2)}$ (it is isomorphic to the direct product of two groups which are both isomorphic to $\mathfrak{S}_G$). Evidently $\mathfrak{A}_G^{(1)}$ and $\mathfrak{A}_G^{(2)}$ are subgroups of $\mathfrak{A}_G$ and $\mathfrak{A}_G$ is a subgroup of $\mathfrak{S}_G^{(2)}$. The sets $\mathfrak{B}_G^{(1)}$ and $\mathfrak{B}_G^{(2)}$ are subgroups of $\mathfrak{S}_G$.

Theorem 1. *The group $\mathfrak{A}_G^{(1)}$ is a normal subgroup of $\mathfrak{A}_G$ and the factor-group $\mathfrak{A}_G / \mathfrak{A}_G^{(1)}$ is isomorphic to $\mathfrak{B}_G^{(1)}$.*

Proof. Let $\mathfrak{f} \in \mathfrak{A}_G^{(1)}$, $\mathfrak{g} \in \mathfrak{A}_G$. We have $\mathfrak{f} = \langle e, f_2 \rangle$, $\mathfrak{g} = \langle g_1, g_2 \rangle$, where e is the identity mapping of V and f_2, g_1, g_2 are permutations of V. The mapping $\mathfrak{g}\mathfrak{f}\mathfrak{g}^{-1} = \langle g_1 e g_1^{-1}, g_2 f_2 g_2^{-1} \rangle = \langle e, g_2 f_2 g_2^{-1} \rangle \in \mathfrak{A}_G^{(1)}$ and therefore $\mathfrak{A}_G^{(1)}$ is a normal subgroup of $\mathfrak{A}_G$. All autotopies of the class $\mathfrak{g}\mathfrak{A}_G^{(1)}$ have g_1 at the first place. On the other hand, if $\bar{\mathfrak{g}} = \langle g_1, \bar{g}_2 \rangle$ is an autotopy of G, then we form the product $\mathfrak{g}^{-1}\bar{\mathfrak{g}} = \langle e, g_2^{-1}\bar{g}_2 \rangle$. This product is in $\mathfrak{A}_G^{(1)}$, thus $\bar{\mathfrak{g}} = \mathfrak{g}(\mathfrak{g}^{-1}\bar{\mathfrak{g}}) \in \mathfrak{g}\mathfrak{A}_G^{(1)}$. We have proved that the class $\mathfrak{g}\mathfrak{A}_G^{(1)}$ is formed exactly by all autotopies of G which have g_1 at the first place. Thus to any element g_1 of $\mathfrak{B}_G^{(1)}$ we may assign a class $\mathfrak{g}\mathfrak{A}_G^{(1)}$ such that g_1 is at the first place in g; this correspondence is one-to-one. From the definition of multiplication of autotopies it follows that this correspondence is an isomorphism.

Theorem 1'. *The group $\mathfrak{A}_G^{(2)}$ is a normal subgroup of $\mathfrak{A}_G$ and the factor-group $\mathfrak{A}_G/\mathfrak{A}_G^{(2)}$ is isomorphic to $\mathfrak{B}_G^{(2)}$.*

Proof is dual to the proof of Theorem 1.

Theorem 2. *The group $\mathfrak{A}_G^{(1)}$ is isomorphic to a direct product of some groups $\mathfrak{C}_1, \ldots, \mathfrak{C}_k$, each of which is isomorphic to some symmetric group.*

Proof. Let $\mathfrak{f} = \langle e, f_2 \rangle \in B_G^{(1)}$ and let $u \in V$, $v \in V$. The existence of the edge $\overrightarrow{uv}$ is equivalent to the existence of the edge $\overrightarrow{e(u) f_2(v)} = \overrightarrow{uf_2(v)}$. Thus $u \in \Gamma^{-1}v$ if and only if $u \in \Gamma^{-1} f_2(v)$ and therefore $\Gamma^{-1}v = \Gamma^1 f_2(v)$. (The symbols Γ and Γ^{-1} are used following [1].) Let ε^- be such a relation on the set V that $u\varepsilon^- v$ if and only if $\Gamma^{-1}u = \Gamma^{-1}v$. The relation ε^- is evidently an equivalence, therefore there exists a partition $\mathscr{E}^-$ of V into equivalence classes of the relation ε^-. Now let g be a permutation of V such that $g(v)\, \varepsilon^- v$ for any $v \in V$. If $x \in \Gamma^{-1}v$, there exist both the edges $\overrightarrow{xv}$ and $\overrightarrow{xg(v)}$; if $x \notin \Gamma^{-1}v$, there is also $x \notin \Gamma^{-1}\, g(v)$ and none of the edges $\overrightarrow{xv}$ and $\overrightarrow{xg(v)}$ exists. Therefore $\langle e, g \rangle$ is an autotopy of G. We have shown that the group $\mathfrak{A}_G^{(1)}$ is isomorphic with the group of permutations of V which map any element onto an element of the same class of $\mathscr{E}^-$. If $\mathscr{E}^- = \{E_1, \ldots, E_k\}$, then this group is a direct product of the groups $\mathfrak{C}_1, \ldots, \mathfrak{C}_k$, where $\mathfrak{C}_i$ for $i = 1, \ldots, k$ is the group of all permutations of E_i.

Theorem 2'. *The group $\mathfrak{A}_G^{(2)}$ is isomorphic to a direct product of some groups $\mathfrak{C}'_1, \ldots, \mathfrak{C}'_l$, each of which is isomorphic to some symmetric group.*

Proof is dual to the proof of Theorem 2.

Theorem 3. *If $\mathfrak{A}_G$ for some digraph G is Abelian, then to any vertex u of G there exists at most one vertex $v \neq u$ such that $\Gamma^{-1}v = \Gamma^{-1}u$ and at most one vertex $w \neq u$ such that $\Gamma w = \Gamma u$.*

Proof. A group is Abelian only if all of its subgroups are Abelian. Thus if $\mathfrak{A}_G$ is Abelian, also $\mathfrak{A}_G^{(1)}$ and $\mathfrak{A}_G^{(2)}$ are Abelian. As $\mathfrak{A}_G^{(1)}$ is a direct product of groups isomorphic to symmetric groups, all of these groups must be Abelian. A symmetric group is Abelian if and only if it is of order 1 or 2, thus any of the classes of $\mathscr{E}^-$ must consist of one or two vertices; therefore to any vertex u of G there exists at most one vertex $v \neq u$ such that $\Gamma^{-1}v = \Gamma^{-1}u$. Dually we can prove the second part of the theorem.

Now we shall investigate digraphs with the property that $u \neq v$ implies $\Gamma u \neq \Gamma v$, $\Gamma^{-1}u \neq \Gamma^{-1}v$ for any two vertices u, v. In these graphs evidently $\mathfrak{A}_G^{(1)} = \mathfrak{A}_G^{(2)} = \{\mathfrak{e}\}$.

Theorem 4. *Let G be a digraph in which $u \neq v$ implies $\Gamma u \neq \Gamma v$, $\Gamma^{-1}u \neq \Gamma^{-1}v$ for any two vertices u, v. Then $\mathfrak{B}_G^{(1)} \cong \mathfrak{B}_G^{(2)} \cong \mathfrak{A}_G$ and there exists such an isomorphic mapping φ of $\mathfrak{B}_G^{(1)}$ onto $\mathfrak{B}_G^{(2)}$ that any autotopy of G has the form $\langle f, \varphi(f)\rangle$, where $f \in \mathfrak{B}_G^{(1)}$.*

Proof. The assertion that $\mathfrak{B}_G^{(1)} \cong \mathfrak{B}_G^{(2)} \cong \mathfrak{A}_G$ follows immediately from Theorems 1 and 1′. Now let us have two autotopies $\mathfrak{f} = \langle f_1, f_2\rangle$ and $\mathfrak{g} = \langle g_1, g_2\rangle$ such that $f_1 \equiv g_1$. Then $f^{-1}g = \langle f_1^{-1}g_1, f_2^{-1}g_2\rangle = \langle e, f_2^{-1}g_2\rangle \in \mathfrak{A}_G^{(1)}$. As $\mathfrak{A}_G^{(1)} = \{\mathfrak{e}\}$, we must have $\mathfrak{f}^{-1}\mathfrak{g} \equiv \mathfrak{e}$, therefore $\mathfrak{f} = \mathfrak{g}$. We have proved that to any $f_1 \in \mathfrak{B}_G^{(1)}$ at most one (and therefore exactly one) mapping $f_2 \in \mathfrak{B}_G^{(2)}$ exists such that $\langle f_1, f_2\rangle \in \mathfrak{A}_G$. Analogously we can prove that to any $f_2 \in \mathfrak{B}_G^{(2)}$ at most one (and therefore exactly one) mapping $f_1 \in \mathfrak{B}_G^{(1)}$ exists such that $\langle f_1, f_2\rangle \in \mathfrak{A}_G$. We shall define φ as the mapping of $\mathfrak{B}_G^{(1)}$ onto $\mathfrak{B}_G^{(2)}$ such that $\langle f, \varphi(f)\rangle \in \mathfrak{A}_G$ for any $f \in \mathfrak{B}_G^{(1)}$; according to what was proved above there exists exactly one such φ and it is one-to-one. If $\langle f, \varphi(f)\rangle$ and $\langle g, \varphi(g)\rangle$ are two autotopies, then their product $\langle fg, \varphi(f)\,\varphi(g)\rangle$ is also an autotopy; according to the definition, $\varphi(f)\,\varphi(g) = \varphi(fg)$. The inverse mapping to $\langle f, \varphi(f)\rangle$ is $\langle f^{-1}, \varphi(f)^{-1}\rangle$, it is evidently also an autotopy and therefore $\varphi(f)^{-1} = \varphi(f^{-1})$. Therefore φ is an isomorphism of $\mathfrak{B}_G^{(1)}$ onto $\mathfrak{B}_G^{(2)}$.

Corollary 1. *If G is such a digraph that $u \neq v$ implies $\Gamma u \neq \Gamma v$, $\Gamma^{-1}u \neq \Gamma^{-1}v$ for any two vertices u, v, then either all autotopies of G are automorphisms, or there exists no automorphism of G except $\mathfrak{e}$.*

The first case occurs, if φ is an identity mapping of $\mathfrak{B}_G^{(1)}$, else the second case holds.

Now we shall consider the case when φ is induced by an inner automorphism of $\mathfrak{S}_G$. (According to Theorem of Hölder [2] and a contribution to it by Schreier and Ulam [3], all automorphisms of $\mathfrak{S}_G$ are inner if the number of vertices of G is greater than two and different from six.) This means that there exists a permutation λ of V such that $\varphi(f) \equiv \lambda^{-1}f\lambda$ for any $f \in \mathfrak{B}_G^{(1)}$.

Theorem 5. *Let the assumption of Theorem 4 be satisfied and let there exist a mapping $\lambda \in \mathfrak{S}_G$ such that $\varphi(f) \equiv \lambda^{-1}f\lambda$ for any $f \in \mathfrak{B}_G^{(1)}$. Let $\langle e, \lambda\rangle$ be an isotopy of G onto a digraph G' with the same vertex set. Then any autotopy of G is an automorphism of G' and vice versa and G' contains no autotopies except for automorphisms.*

Proof. For any two vertices $u \in V$, $v \in V$ the existence of the edge $\overrightarrow{uv}$ in G is equivalent to the existence of the edge $\overrightarrow{u\,\lambda(u)}$ in G'. Let $f \in \mathfrak{B}_G^{(1)}$. Then $\langle f, \varphi(f)\rangle$ is an autotopy of G and therefore the existence of $\overrightarrow{uv}$ in G is equivalent to the existence of $\overrightarrow{f(u)\,\varphi(f)\,(v)}$ in G. We have $\varphi(f)\,(v) = \lambda^{-1}f\,\lambda(v)$. Thus $\overrightarrow{f(u)\,\varphi(f)\,(v)} = \overrightarrow{f(u)\,\lambda^{-1}f\,\lambda(v)}$. Let us apply the isotopy $\langle e, \lambda\rangle$ to this edge. We obtain that the existence of this edge in G is equivalent to the existence of the edge $\overrightarrow{f(u)\,f\,\lambda(v)}$ in G'. This implies that the existence of $\overrightarrow{u\,\lambda(v)}$ in G' is equivalent to the existence of $\overrightarrow{f(u)\,f\lambda(v)}$ in G' and therefore f is an automorphism of G'. Now let $\mathfrak{g} = \langle g_1, g_2\rangle$ be an autotopy of G'. The existence of the edge $\overrightarrow{xy}$ in G' is equivalent to the existence of the edge $\overrightarrow{g_1(x)\,g_2(y)}$ in G'. But the existence if $\overrightarrow{xy}$ and $\overrightarrow{g_1(x)\,g_2(y)}$ in G' is equivalent to the existence of the edges $\overrightarrow{x\lambda^{-1}(y)}$ and $\overrightarrow{g_1(x)\,\lambda^{-1}\,g_2(y)}$ in G. Thus $\langle g_1, \lambda^{-1}g_2\lambda\rangle$ is an autotopy of G. This implies $\lambda^{-1}g_2\lambda = \lambda^{-1}g_1\lambda$, i.e. $g_1 = g_2$ and $\mathfrak{g}$ is an automorphism of G'.

Now we shall continue the study of the groups $\mathfrak{A}_G^{(1)}$ and $\mathfrak{A}_G^{(2)}$. By $\tilde{\mathfrak{A}}_G^{(1)}$, $\tilde{\mathfrak{A}}_G^{(2)}$ we shall denote the set of mappings which occur at the second or at the first place respectively in autotopies of $\mathfrak{A}_G^{(1)}$ or $\mathfrak{A}_G^{(2)}$ respectively. It is quite evident that $\tilde{\mathfrak{A}}_G^{(1)} \cong \mathfrak{A}_G^{(1)}$, $\tilde{\mathfrak{A}}_G^{(2)} \cong \mathfrak{A}_G^{(2)}$. Further let $\mathfrak{A}_G^{(0)}$ be the product of the groups $\mathfrak{A}_G^{(1)}$ and $\mathfrak{A}_G^{(2)}$ (they are evidently groups). This product is direct, because evidently any element of $\mathfrak{A}_G^{(1)}$ is commutative with any element of $\mathfrak{A}_G^{(2)}$. The group $\mathfrak{A}_G^{(0)}$ is formed by all ordered pairs of mappings in which the first element is in $\mathfrak{A}_G^{(2)}$ and the second element is in $\mathfrak{A}_G^{(1)}$.

Lemma 2. *Let $\mathfrak{f} = \langle f_1, f_2\rangle \in \mathfrak{A}_G$ and $f_1 \in \tilde{\mathfrak{A}}_G^{(2)}$. Then $f_2 \in \tilde{\mathfrak{A}}_G^{(1)}$ and $\mathfrak{f} \in \mathfrak{A}_G^{(0)}$.*

Proof. As $f_1 \in \tilde{\mathfrak{A}}_G^{(2)}$, we have $\bar{\mathfrak{f}} = \langle f_1, e\rangle \in \mathfrak{A}_G^{(2)}$. Let us make the product $\mathfrak{f}\bar{\mathfrak{f}}^{-1} = \langle e, f_2\rangle$. This is an element of $\mathfrak{A}_G^{(1)}$ and thus $f_2 \in \tilde{\mathfrak{A}}_G^{(1)}$.

Lemma 2'. *Let $\mathfrak{f} = \langle f_1, f_2\rangle \in \mathfrak{A}_G$ and $f_2 \in \tilde{\mathfrak{A}}_G^{(1)}$. Then $f_1 \in \tilde{\mathfrak{A}}_G^{(2)}$ and $\mathfrak{f} \in \mathfrak{A}_G^{(0)}$.*

Proof is dual to the proof of Lemma 2.

Now we shall prove a theorem which is a generalization of Theorem 4.

Theorem 6. *Let G be a digraph. Then $\tilde{\mathfrak{A}}_G^{(2)}$ is a normal subgroup of $\mathfrak{B}_G^{(1)}$, $\tilde{\mathfrak{A}}_G^{(1)}$ is a normal subgroup of $\mathfrak{B}_G^{(2)}$ and $\mathfrak{B}_G^{(1)}/\tilde{\mathfrak{A}}_G^{(2)} \cong \mathfrak{B}_G^{(2)}/\tilde{\mathfrak{A}}_G^{(1)} \cong \mathfrak{A}_G/\mathfrak{A}_G^{(0)}$ and there exists an isomorphic mapping φ of $\mathfrak{B}_G^{(1)}/\mathfrak{A}_G^{(2)}$ onto $\mathfrak{B}_G^{(2)}/\mathfrak{A}_G^{(1)}$ such that any autotopy of G has the form $\langle f_1, f_2\rangle$, where $f_1 \in \mathfrak{B}_G^{(1)}$, $f_2 \in \mathfrak{B}_G^{(2)}$, $f_2\tilde{\mathfrak{A}}_G^{(1)} = \varphi(f_1\tilde{\mathfrak{A}}_G^{(2)})$ and all ordered pairs of mappings satisfying this are autotopies of G.*

Proof. Let $f \in \tilde{\mathfrak{A}}_G^{(2)}$, $g_1 \in \mathfrak{B}_G^{(1)}$. This means that there exist autotopies $\mathfrak{f} = \langle f, e\rangle$, $\mathfrak{g} = \langle g_1, g_2\rangle$ in $\mathfrak{A}_G$, where g_2 is an element of $\mathfrak{B}_G^{(2)}$. We have $\mathfrak{g}\mathfrak{f}\mathfrak{g}^{-1} = \langle g_1 f g_1^{-1}, e\rangle \in \mathfrak{A}_G^{(2)}$, thus $g_1 f g_1^{-1} \in \tilde{\mathfrak{A}}_G^{(2)}$ and $\tilde{\mathfrak{A}}_G^{(2)}$ is a normal subgroup of $\mathfrak{B}_G^{(1)}$. Analogously we can prove that $\tilde{\mathfrak{A}}_G^{(1)}$ is a normal subgroup of $\mathfrak{B}_G^{(2)}$. Now let $\mathfrak{f} = \langle f_1, f_2\rangle \in \mathfrak{A}_G^{(0)}$ and $\mathfrak{g} = \langle g_1, g_2\rangle \in \mathfrak{A}_G$. We have $f_1 \in \tilde{\mathfrak{A}}_G^{(2)}$, $g_1 \in \mathfrak{B}_G^{(1)}$, $\tilde{\mathfrak{A}}_G^{(2)}$ is a normal subgroup of $\mathfrak{B}_G^{(1)}$, thus $g_1 f_1 g_1^{-1} \in \tilde{\mathfrak{A}}_G^{(2)}$. Analogously $g_2 f_2 g_2^{-1} \in \tilde{\mathfrak{A}}_G^{(1)}$. Therefore $\mathfrak{g}\mathfrak{f}\mathfrak{g}^{-1} = \langle g_1 f_1 g_1^{-1}, g_2 f_2 g_2^{-1}\rangle \in \mathfrak{A}_G^{(0)}$ and $\mathfrak{A}_G^{(0)}$ is a normal subgroup of $\mathfrak{A}_G$. Now assume that we have two autotopies $\mathfrak{f} = \langle f_1, f_2 \rangle$, $\mathfrak{g} = \langle g_1, g_2 \rangle$ such that f_1, g_1 are elements of the same class of $\mathfrak{B}_G^{(1)}$ according to $\tilde{\mathfrak{A}}_G^{(2)}$, i.e. $f_1 \tilde{\mathfrak{A}}_G^{(2)} = g_1 \tilde{\mathfrak{A}}_G^{(2)}$. Let us take the product $\mathfrak{f}\mathfrak{g}^{-1} = \langle f_1 g_1^{-1}, f_2 g_2^{-1} \rangle$. As f_1, f_2 belong to the same class of $\mathfrak{B}_G^{(1)}$ according to $\tilde{\mathfrak{A}}_G^{(2)}$, we have $f_1 g_1^{-1} \in \tilde{\mathfrak{A}}_G^{(2)}$. According to Lemmas 2 and 2′ we have $f_2 g_2^{-1} \in \tilde{\mathfrak{A}}_G^{(1)}$, thus f_2 and g_2 are elements of the same class of $\mathfrak{B}_G^{(2)}$ according to $\tilde{\mathfrak{A}}_G^{(1)}$. Analogously vice versa. Thus to any class $f_1 \tilde{\mathfrak{A}}_G^{(2)}$ of $\mathfrak{B}_G^{(1)}$ according to $\tilde{\mathfrak{A}}_G^{(2)}$ we can assign a class $\varphi(f_1 \tilde{\mathfrak{A}}_G^{(2)})$ of $\mathfrak{B}_G^{(1)}$ according to $\tilde{\mathfrak{A}}_G^{(1)}$ by a one-to-one manner so that any autotopy of G whose first element is in $f_1 \tilde{\mathfrak{A}}_G^{(2)}$ has the second element in $\varphi(f_1 \tilde{\mathfrak{A}}_G^{(2)})$. Analogously as in the proof of Theorem 4 we can prove that φ is an isomorphism. Now let $g_1 \in f_1 \tilde{\mathfrak{A}}_G^{(2)}$, $g_2 \in f_2 \tilde{\mathfrak{A}}_G^{(1)} = \varphi(f_1 \tilde{\mathfrak{A}}_G^{(2)})$. Therefore there exists an autotopy of G whose first element is in $f_1 \tilde{\mathfrak{A}}_G^{(2)}$ and second element is in $f_2 \tilde{\mathfrak{A}}_G^{(1)}$; without any loss of generality we may assume that this autotopy is $\langle f_1, f_2 \rangle$. Thus $f_1^{-1} g_1 \in \tilde{\mathfrak{A}}_G^{(2)}$, $f_2^{-1} g_2 \in \tilde{\mathfrak{A}}_G^{(1)}$. The pair $\langle f_1^{-1} g_1, f_2^{-1} g_2 \rangle \in \mathfrak{A}_G^{(0)} \subset \mathfrak{A}_G$. Multiplying the autotopies $\langle f_1, f_2 \rangle$ and $\langle f_1^{-1} g_1, f_2^{-1} g_2 \rangle$ we obtain $\langle g_1, g_2 \rangle$. This pair of mappings is a product of two autotopies of G and must be also an autotopy of G. Thus any ordered pair of mappings satisfying the assumption is an autotopy of G. It remains to prove the assertion concerning $\mathfrak{A}_G / \mathfrak{A}_G^{(0)}$. Let $\mathfrak{f} = \langle f_1, f_2 \rangle \in \mathfrak{A}_G$ and construct the class $\mathfrak{f}\mathfrak{A}_G^{(0)}$. We have $f_2 \mathfrak{A}_G^{(1)} = \varphi(f_1 \mathfrak{A}_G^{(2)})$, thus any element of $\mathfrak{f}\mathfrak{A}_G^{(0)}$ has the first element in $f_1 \mathfrak{A}_G^{(2)}$ and the second in $\varphi(f_1 \mathfrak{A}_G^{(2)})$. Evidently $\mathfrak{f}\mathfrak{A}_G^{(0)}$ is formed exactly by all such elements and we can define the isomorphic mapping of $\mathfrak{B}_G^{(1)} / \mathfrak{A}_G^{(2)}$ onto $\mathfrak{A}_G / \mathfrak{A}_G^{(0)}$ such that $\psi(f_1 \mathfrak{A}_G^{(2)})$ is the set of all autotopies whose first element is in $f_1 \mathfrak{A}_G^{(2)}$ and whose second element is in $\varphi(f_1 \mathfrak{A}_G^{(2)})$.

At the end we shall investigate the relation ε^- and the dual relation ε^+ defined so that $u \varepsilon^+ v$ for two vertices u, v of G if and only if $\Gamma u = \Gamma v$. Also the partition $\mathscr{E}^+$ is defined analogously to $\mathscr{E}^-$.

Theorem 7. *Let A be a class of $\mathscr{E}^+$ and B be a class of $\mathscr{E}^-$. Then either from any vertex of A an edge goes into any vertex of B, or there is no edge going from a vertex of A into a vertex of B.*

Proof. Let $u \in A$, $u' \in A$, $v \in B$, $v' \in B$. Let $\overrightarrow{uv}$ be an edge of G. We have $\Gamma u = \Gamma u'$, thus also $v \in \Gamma u'$. This means $u' \in \Gamma^{-1} v$. As $\Gamma^{-1} v = \Gamma^{-1} v'$, we have $u' \in \Gamma^{-1} v'$ and $\overrightarrow{u'v'}$ is an edge of G. We have proved that if from one vertex of A an edge goes into a vertex of B, then from any vertex of A an edge goes into a vertex of B.

Theorem 8. *Let $A \in \mathscr{E}^+$. To any class $f \mathfrak{A}_G^{(2)}$, where $f \in \mathfrak{B}_G^{(1)}$, there exists a class $B \in \mathscr{E}^+$ such that the image of any vertex of A in any of the mappings of $f \mathfrak{A}_G^{(2)}$ is in B.*

Proof. Let $u \in A$. By B denote the class of $\mathscr{E}^+$ which $f(u)$ belongs to. Now let $g \in f \mathfrak{A}_G^{(2)}$; this means that $g = hf$, where $h \in \mathfrak{A}_G^{(2)}$. We have $g(u) = hf(u)$. As $h \in \mathfrak{A}_G^{(2)}$, according to the proof of Theorem 2 h maps B onto itself and thus also $g(u) \in B$. If $v \in A$, there exists a mapping $h' \in \mathfrak{A}_G^{(2)}$ such that $v = h'(u)$. We have $f(v) = fh'(u)$. The mapping fh' is in $f \mathfrak{A}_G^{(2)}$, thus this vertex is also in B.

Theorem 8'. *Let $A \in \mathscr{E}^-$. To any class $f\mathfrak{A}_G^{(1)}$, where $f \in \mathfrak{B}_G^{(2)}$, there exists a class $B \in \mathscr{E}^-$ such that the image of any vertex of A in any of the mappings of $f\mathfrak{A}_G^{(1)}$ is in B.*

Proof is dual to the proof of Theorem 8.

Corollary 2. *Let $G(\mathscr{E}^+, \mathscr{E}^-)$ be the digraph whose vertices are classes of $\mathscr{E}^+$ and $\mathscr{E}^-$ in a digraph G such that from a class A of $\mathscr{E}^+$ an edge goes into a class B of $\mathscr{E}^-$ in $G(\mathscr{E}^+, \mathscr{E}^-)$ if and only if in G from any vertex of A an edge goes into any vertex of B. Let to any vertex of $G(\mathscr{E}^+, \mathscr{E}^-)$ a value equal to the cardinality of the corresponding class be assigned. Then the group of automorphisms of this valuated digraph (i.e. automorphisms which preserve the values of vertices) is isomorphic to $\mathfrak{A}_G/\mathfrak{A}_G^{(0)}$.*

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