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REPRESENTATIONS OF PRESHEAVES OF SEMIUNIFORMISABLE SPACES, AND REPRESENTATION OF A PRESHEAF BY THE PRESHEAF OF ALL CONTINUOUS SECTIONS IN ITS COVERING-SPACE

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INTRODUCTION

Let $\mathcal{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a presheaf of closure spaces over a topological space X, P its covering space and $\mathcal{S}' = \{A_U; \tilde{\varrho}_{UV}; X\}$ its natural representation. That is to say, we cnow that every $a \in S_U$ can be regarded in a natural way as a section $a(x)$ over U in P. Denoting the assignement $a \to a(x)$ by p_U, then to the set S_U there corresponds a set $A_U = p_U(S_U)$ of sections over U. Moreover, let us denote by $\tilde{\varrho}_{UV} : A_U \to A_V$ the map defined as follows: $a(x) \in A_U \to \tilde{\varrho}_{UV}(a(x)) = a(x)/V = [\varrho_{UV}(a)](x)$. Then $\mathcal{S}' = \{A_U; \tilde{\varrho}_{UV}; X\}$ is a presheaf of sets over X. If $\mathcal{S}$ satisfies convenient natural requirements, every p_U is injective and thus $\mathcal{S}'$ is a natural representation of $\mathcal{S}$ with help of the presheaf of certain sets of sections in P. Let us denote by $\mathcal{B}(X)$ the set of all open subsets in X. We say, that a nonempty family $\mathcal{K}$ of subsets of a set L is cofilter base, if the following holds: $K_1, K_2 \in \mathcal{K} \Rightarrow K_1 \cup K_2 \subset K_3$ for some $K_3 \in \mathcal{K}$.

We say, that to the presheaf $\mathcal{S} = (S_U; \varrho_{UV}; X\}$ there is given a cofiltration $\varkappa = \{\mathcal{K}_a^U; U \in \mathcal{B}(X), a \in S_U\}$, if for every $U \in \mathcal{B}(X)$, $a \in S_U$ there is given a cofilter base $\mathcal{K}_a^U$ in U such that for $V \in \mathcal{B}(U)$ we have: ,,To every $K \in \mathcal{K}^V_{\varrho_{UV}(a)}$ there exists $L \in \mathcal{K}_a^U$, such that $K \subset L$".

Let be given a closure t in P. Further let for every $U \in \mathcal{B}(X)$, q_U be some method, which enables us to form a closure $q_U(t)$ in A_U from t. Then $\mathcal{S}' = \{(A_U, q_U(t)); \tilde{\varrho}_{UV}; X\}$ is a presheaf of sets A_U with the closures $q_U(t)$. Now we can try to find a closure t in P, so that all the natural maps $p_U : (S_U, \tau_U) \to (A_U, q_U(t))$ would be homeomorphisms. Every such closure we call normal. Here we restrict ourselves to the case, when the method q_U is the closure of uniform convergence on some cofiltration $\varkappa$. (Briefly u.c. on $\varkappa$).

In the paper two questions are deald with:

1. If there is given a cofiltration $\varkappa$ and if $k_U(t)$ is a closure of uniform convergence on $\varkappa$, we study, when there exists a normal closure t in P.

2. We study, if there exists in P a closure t generating a representation, i.e. for which the following is satisfied:

(a) There exists a cofiltration $\varkappa$ for which t is normal.

(b) If for $U \in \mathscr{B}(X)$ $\Gamma(U, t)$ is the set of all continuous sections over U, then $\Gamma(U, t) = A_U$ for all $U \in \mathscr{B}(X)$.

That is to say, if t generates such a representation, then $\mathscr{S}' = \{(A_U, k_U(t)); \tilde{\varrho}_{UV}; X\}$ is not only a set representation, but even a topological representation of $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$, i.e. $\mathscr{S}$ can be represented even topologically using the presheaf of all continuous sections in its covering space. ($k_U(t)$ is the closure of u.c. on $\varkappa$).

In the first chapter we study the case, when $\mathscr{S} = \{(S_U, \eta_U); \varrho_{UV}; X\}$ is a presheaf of semiuniform spaces and when the stalks $\psi^{-1}(x)$ in its covering space P are isomorphic under the family of maps $\Psi = \{\psi_{xy}; x, y \in X;\ \psi_{xy} : \psi^{-1}(x) \to \psi^{-1}(y)\}$. If t is a closure in P such that every induced closure t_x in $\psi^{-1}(x)$ is generated by a semiuniformity η_x and $\psi_{xy} : (\psi^{-1}(x), \eta_x) \to (\psi^{-1}(y), \eta_y)$ are isomorphisms, then for a given cofiltration $\varkappa$ we can define the semiuniformity $n(t)$ of uniform convergence on $\varkappa$ using the transferring of η_x by ψ_{xy}. We try to find a normal closure t in P, i.e. such a t for which the all natural maps $p_U : (S_U, \eta_U) \to (A_U, n(t))$ are isomorphisms.

In 1.3.20,21 we find two necessary, in 1.3.22 a necessary and sufficient condition for the existence of a normal closure. In §4 we construct a semiuniformity η in P such that if 1.3.22 holds, the closure t_1 generated by η is normal.

In the second chapter we try to find a closure t in the covering space P of a presheaf $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$, generating a representation, i.e. such a t, for which there exists a cofiltration $\varkappa$ such that all the natural maps $p_U : (S_U, \tau_U) \to (A_U, k(t))$ are homeomorphisms, and for every U $A_U = \Gamma(U, t)$ ($k(t)$ is the closure of uniform convergence on $\varkappa$, $\Gamma(U, t)$ is the set of all continuous sections over U in the space (P, t)). Theorem 2.1.7 contains a necessary and sufficient condition for the fulfillment of the inclusion $\Gamma(U, t) \subset A_U$. In §2 we show that every continuous map $f : U \to (S_U, \tau_U)$ generates $\varphi^f \in \Gamma(U, t)$ if $t \geqq t^*$ (t^* is the natural closure from [5], (2.6.2)). From this follows a necessary condition 2.2.4 for the inclusion $\Gamma(U, t) \subset A_U$ where $t \geqq t^*$, which implies that even for very reasonable presheaves the inclusion $\Gamma(U, t^*) \subset A_U$ need not be satisfied. The main results are 2.4.9, 2.5.3, for they are sufficient conditions for the existence of a closure generating a representation.

The paper is a continuation of my papers "Representations of presheaves of closure spaces" [5] and "Modifications of closure collections" [4]. The notations and some results from them are used here, therefore it is suitable to have them at hand. Because we could not avoid a complicated notations, it is necessary before reading to go through the agreements and notations.

Agreements and notations

The set of all open subsets of a topological space X we shall denote by $\mathscr{B}(X)$.

0.1. Definition. A *presheaf* of the sets over the topological space X is a system

$$\mathscr{S} = \{S_U; \varrho_{UV}; X\}, \tag{0.2}$$

where S_U for $U \in \mathscr{B}(X)$ are sets, and ϱ_{UV} for every $U, V \in \mathscr{B}(X)$, $V \subset U$ is a map $\varrho_{UV} : S_U \to S_V$ such that the following holds:

(1) For $V' \subset V \subset U$ there is $\varrho_{UV'} = \varrho_{VV'} \circ \varrho_{UV}$.

(2) $\varrho_{UU} = i_U$ – identity map of S_U onto S_U for all $U \in \mathscr{B}(X)$.

We say, that the presheaf $\mathscr{S}$ in (0.2) is a presheaf of closure spaces (semiuniform spaces) over X, if in every set S_U there is given a closure τ_U (a semiuniformity η_U) and the maps ϱ_{UV} are continuous (uniformly continuous) maps of the closure (semiuniform) spaces $\varrho_{UV} : (S_U, \tau_U) \to (S_V, \tau_V)$ $((S_U, \eta_U) \to (S_V, \eta_V))$. Where it is not important to repeat the all data, we shall speak only about a presheaf. Mostly the system of maps ϱ_{UV} we shall not distinguish, and we shall write simply $\mathscr{S} = \{(S_U, \tau_U); X\}$.

If $\mathscr{S}$ is a presheaf over X from (0.2), we can to every $x_0 \in X$ assign the system

$$\mathscr{S}_{x_0} = \{S_U; \varrho_{UV}; U, V \in \mathscr{B}(X);\ x_0 \in U, V\}. \tag{0.3}$$

Because ϱ_{UV} satisfies the conditions (1), (2) from (0.1), we can form the set $\mathscr{F}_{x_0} = \varinjlim \mathscr{S}_{x_0}$ – the inductive limit of $\mathscr{S}_{x_0}$.

0.4. Definition. The set $\mathscr{F}_{x_0}$ will be called *fiber over the point* x_0. For every $U \in \mathscr{B}(X)$ containing x_0, there exists a natural map of the set S_U into $\mathscr{F}_{x_0}$.

0.5. Notation. Let $x_0 \in U \in \mathscr{B}(X)$. The natural map of S_U into $\mathscr{F}_{x_0}$ will be denoted by ξ_{Ux_0}. If $a \in S_U$, then the element $\xi_{Ux_0}(a) \in \mathscr{F}_{x_0}$ will be called germ of the element a over the point x_0.

Let us set $P = \bigcup_{x \in X} \mathscr{F}_x$. Further let ψ be a map of the set P onto X, constructed in this way: If $\alpha \in P$, then there exists the unique $x_0 \in X$ such that $\alpha \in \mathscr{F}_{x_0}$. Let us set $\psi(\alpha) = x_0$.

0.6. Notation. The set P we shall call covering space of the presheaf $\mathscr{S}$. The map ψ we shall call projection.

0.7. Remark. Clearly there is $\mathscr{F}_{x_0} = \psi^{-1}(x_0)$. Thus for the fiber over x_0 we shall more frequently use the symbol $\psi^{-1}(x_0)$. The capital P will in the next denote only the covering space of $\mathscr{S}$.

0.8. Definition. Let $U \in \mathscr{B}(X)$. Every map $r : U \to P$ for which $\psi \circ r = i_U$ is the identity map of U onto U we shall call *section over* U.

If $U \in \mathscr{B}(X)$, $a \in S_U$, then we can to every $x \in U$ assign an element $h_a(x) \in P$ as follows: $h_a(x) = \xi_{Ux}(a)$ (see (0.5)).

According to our way of introduction of ξ_{Ux} and ψ, there is $(\psi \circ h_a)(x) = \psi(\xi_{Ux}(a)) = x$. Thus the map h_a is a section over U.

If we assign in this way to every $a \in S_U$ the section h_a over U, we get a certain set of sections over U.

0.9. Notation. The map which to every $a \in S_U$ assigns the section h_a over U we shall denote by p_U and we shall call it natural map. The set of sections $\{p_U(a); a \in S_U\}$ we shall denote by A_U. Instead of h_a we shall write briefly $a(x)$, where x denotes the variable, taking values from U. Thus $a(x)$ is the section over U such that

$$p_U(a) = a(x)\,. \tag{0.10}$$

The element $\xi_{Ux_0}(a)$ we shall denote with accordance with our agreements by $a(x_0)$ and we shall say that the section $a(x)$ goes through the point $\alpha = a(x_0)$. Thus there is

$$\xi_{Ux_0}(a) = a(x_0)\,. \tag{0.11}$$

0.12. Remark. If $U, V \in \mathscr{B}(X)$, $V \subset U$ and $a(x) \in A_U$ is a section over U from (0.9), we can to $a(x)$ assign a section from A_V as follows: $a(x) \to a(x)/V$, i.e. the restriction of $a(x)$ to V. Let us denote this map by $\tilde{\varrho}_{UV}$. Then $\tilde{\varrho}_{UV}(a(x)) \in A_V$ and we have the commutative diagram

$$\begin{array}{ccc} S_U & \xrightarrow{p_U} & A_U \\ \downarrow{\scriptstyle \varrho_{UV}} & & \downarrow{\scriptstyle \tilde{\varrho}_{UV}} \\ S_V & \xrightarrow{p_V} & A_V \end{array} \tag{0.13}$$

0.14. Remark. For $x_0 \in U$ there exists a natural map $\tilde{\xi}_{Ux_0} : A_U \to \psi^{-1}(x_0)$. Namely, if there is $a(x) \in A_U$, then we set $\tilde{\xi}_{Ux_0}(a(x)) = a(x_0)$. According to (0.10,11) we have $\tilde{\xi}_{Ux_0}(p_U(a)) = \xi_{Ux_0}(a) = a(x_0)$. Thus the following diagram is commutative:

$$\begin{array}{ccccc} S_U & & \xrightarrow{p_U} & & A_U \\ & {\scriptstyle \xi_{Ux_0}} \searrow & & \swarrow {\scriptstyle \tilde{\xi}_{Ux_0}} & \\ & & \mathscr{F}_{x_0} & & \end{array} \tag{0.15}$$

0.16. Remark. The natural map p_U from (0.10) need not be injective. This can be reached by adding this assumption:

0.17. Assumption. *Let $U \in \mathscr{B}(X)$, $a, b \in S_U$ and let $\mathscr{V}$ be an open covering of U. If $\varrho_{UV}(a) = \varrho_{UV}(b)$ for all $V \in \mathscr{V}$, then $a = b$.*

This assumption implies the injectivity of p_U. In the next we suppose stadily, that this assumption is satisfied.

0.18. Definition. Let $\alpha \in P$, $x_0 = \psi(\alpha)$. Then there exists $U \in \mathscr{B}(X)$ such that for some $a \in S_U$ there is $\xi_{Ux_0}(a) = a(x_0) = \alpha$.
Every such a we shall call *generating element for* α. Similarly the section $a(x) \in A_U$ will be called *generating section for* α.

0.19. Notation. Let $a \in S_U$, let $A \subset U$ be an arbitrary subset. Let us denote

$$\xi_{UA}(a) = \bigcup_{y \in A} \xi_{Uy}(a)$$

and further, more generally, if $M \subset S_U$ is an arbitrary subset,

$$\xi_{UA}(M) = \bigcup_{a \in M} \xi_{UA}(a) .$$

Thus $\xi_{UA}(a)$ and $\xi_{UA}(M)$ are subsets of P. By (0.5) for $x_0 \in A$ there is

$$\xi_{UA}(M) \cap \psi^{-1}(x_0) = \xi_{Ux_0}(M) .$$

In the same way (with respect to (0.9,14) we proceed if $M \subset A_U$. Thus, for example, if $A \subset U$, $M \subset \psi^{-1}(A)$:

$$\xi_{UA}^{-1}(M) = \{a;\ a \in S_U,\ a(y) \in M,\ y \in A\} ,$$
$$\bar{\xi}_{UA}^{-1}(M) = \{a(x);\ a(x) \in A_U,\ a(y) \in M,\ y \in A\} .$$

0.20. Notation. The set $\xi_{UA}(a)$ from (0.19) we shall denote by $\operatorname{gr}_A a$ and we shall call it graph of the section $a(x)$ (resp. of the element a) over A. By (0.9,11) there is $\xi_{UA}(a) = \bigcup_{y \in A} a(y)$, and thus the word graph has the objective meaning.

0.21. Notation. Let (X, t) be a closure space, M its subset.

A. If $M \subset X$, then every filter-base of t-neighborhoods of M we shall denote by $\Delta(M; t)$.

B. By the symbol $\operatorname{ind}_M t$ we shall denote the closure in M, induced by restriction of t to M. If $x \in M$ and $\Delta(X; t)$ is a filter-base of t-neighborhoods of x, then the filter of $\operatorname{ind}_M t$-neighborhoods of x we denote briefly $M \cap \Delta(X; t)$. The relation „the closure u is finer than v" we denote briefly by $u \leqq v$.

C. If $\mathscr{F}$ is such a filter in X, that for every $F \in \mathscr{F}$ there is $M \subset F$, we say, that $\mathscr{F}$ is a filter round M. If $\mathscr{F}$ and $\mathscr{G}$ are two filter-bases and $\mathscr{F}$ majorizes $\mathscr{G}$, we write briefly $\mathscr{F} \leqq \mathscr{G}$. If $\mathscr{F} \leqq \mathscr{G}$, $\mathscr{G} \leqq \mathscr{F}$, we write $\mathscr{F} \sim \mathscr{G}$.

D. For $U \in \mathscr{B}(X)$, $M \subset U$ let us set

$$B(M; U) = \{V; V \in \mathscr{B}(U), M \subset V\}.$$

0.22. Notation. In the set X let us have a nonempty family Ω of closures. The coarsest (finest) closure in X, finer (coarser) then every closure from Ω we shall denote by $\varinjlim \Phi$ (resp. $\varprojlim \Omega$).

0.23. Remark. Let $\{(X_\alpha; \tau_\alpha); \alpha \in A\}$ be a nonempty family of closure spaces, let X be a set, and for every $\alpha \in A$ let φ_α be a map $\varphi_\alpha : (X_\alpha, \tau_\alpha) \to X$ (resp. $\varphi_\alpha : X \to \to (X_\alpha, \tau_\alpha)$). Then if $\tau = \varinjlim \tau_\alpha$ (resp. $\tau = \varprojlim \tau_\alpha$) is the inductive (resp. projective) limit of the closures τ_α, then (0.22) is in keeping with this notation.

0.24. Remark. The map $f : (Q, u) \to (X, \varprojlim \tau_\alpha)$ is continuous iff for every α the map $\varphi_\alpha \circ f : (Q, u) \to (X_\alpha, \tau_\alpha)$ is continuous. A similar remark is true for $\varinjlim \tau_\alpha$.

0.25. Notation. If $\varphi : M \to N$ is a map, let $\bar{\varphi} : M \times M \to N \times N$ be the map defined as follows: $(x, y) \in M \times M \to \bar{\varphi}(x, y) = (\varphi(x), \varphi(y))$.

0.26. Agreement. When speaking about a compact space in a topological space X, we suppose, that X is Hausdorff space.

0.27. Agreement. Let $U \in \mathscr{B}(X)$. By the symbol $\Pi_U(\Pi_U^0)$ we denote the set of all open coverings (of all finite open coverings) of U.

0.28. Notation. Let X, Y be two sets, let $f : X \to Y$ be a map and let $\mathscr{F}$ be a filter-base in X. Then the filter base $\{f(F); F \in \mathscr{F}\}$ in Y we denote by $f(\mathscr{F})$.

0.29. Notation. For a semiuniform space (X, η) let us denote by $\mathscr{D}(X; \eta)$ the filter base of η-neighborhoods of the diagonal in $X \times X$.

0.30. Notation. For a set X let us denote by d the discrete topology in X, and by h the coarsest topology in X, where the only open sets are $\emptyset$ and X.

0.31. Definition. We say that the presheaf $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ is *projective*, if the following condition holds: ,,If $U = \bigcup_\alpha V_\alpha$, $U, V_\alpha \in \mathscr{B}(X)$, and if there exist the elements $a_\alpha \in S_{V_\alpha}$ such that for $V_\alpha \cap V_\beta$ there is $\varrho_{V_\alpha, V_\alpha \cap V_\beta}(a_\alpha) = \varrho_{V_\beta, V_\alpha \cap V_\beta}(a_\beta)$, then there exists $a \in S_U$ such that $\varrho_{UV_\alpha}(a) = a_\alpha$ for all α.

Because we assume, that (0.17) holds, there exists the unique such $a \in S_U$.

0.32. Definition. We say that $\mathscr{S}$ is a presheaf with the unique continuation, if the following conditions are satisfied:

1. X is locally connected,

2. if $U \in \mathscr{B}(X)$ is connected, $a, b \in S_U$, and $\xi_{Ux_0}(a) = \xi_{Ux_0}(b)$ for some $x_0 \in U$, then $a = b$.

0.33. Remark. Let P be the covering space of the presheaf $\mathscr{S} = \{S_U, X\}$. If $U \in \mathscr{B}(X)$, then by (0.9) every $a(x) \in A_U$ is a section over U. Let us set $\Omega' = \{t;\, t$ is a closure in P such that for every $U \in \mathscr{B}(X)$ every $a(x) \in A_U$ is a continuous map of U into the closure space $(P, t)\}$.

0.34. Definition, notation. The closure $\varinjlim \Omega'$ we call *sheaf topology* and we denote it by t_s. If t is a closure in P, and $U \in \mathscr{B}(X)$, then the set of all continuous sections over U we denote by $\Gamma(U, t)$.

0.35. Remark. If the presheaf is projective, then $\Gamma(U, t_s) = A_U$ (see [4]). t_s is the finest of all closures t in P, for which there is $A_U \subset \Gamma(U, t)$ for all $U \in \mathscr{B}(X)$.

0.36. Notation. Let (X_α, u_α) be closure spaces. The space (X, u) will be called topological sum of the spaces (X_α, u_α) if $X = \bigcup_\alpha X_\alpha$ and $u = \varinjlim u_\alpha$.

0.37. Notation. A nonempty family $\mathscr{K}$ of subsets of the set L we shall call cofilter base (resp. cofilter) if the following holds: $K_1, K_2 \in \mathscr{K} \Rightarrow K_1 \cup K_2 \subset K_3$ for some $K_3 \in \mathscr{K}$, resp. $K_1, K_2 \in \mathscr{K} \Rightarrow K_1 \cup K_2 \in \mathscr{K}$.

Chapter 1.

UNIFORM CONVERGENCE ON COFILTRATION. THE SEMIUNIFORM CASE

Let $\mathscr{S} = \{(S_U, \eta_U);\, \varrho_{UV};\, X\}$ be a presheaf of semiuniform spaces from (0.1) ($\varrho_{UV} : (S_U; \eta_U) \to (S_V; \eta_V)$ uniformly continuous), P its covering space, t a closure in P. We introduce the semiuniformity of uniform convergence on the cofiltration $\varkappa = \{\mathscr{K}^U_a;\, U \in \mathscr{B}(X),\, a \in S_U\}$ and we try to find a normal closure in P.

1. Basic assumptions

For every $U \in \mathscr{B}(X)$ let be given a cofilter $\mathscr{K}^U$ of sets in U such that for $U, V \in \mathscr{B}(X)$, $V \subset U$ there is

$$\mathscr{K}^V = \{K;\, K \in \mathscr{K}^U,\, K \subset V\}\,. \tag{1.1.1}$$

In the whole chapter we assume

1.1.2. Assumption. The presheaf $\mathcal{S}$ have the following properties: For $U' \in \mathcal{B}(X)$, $x, y \in U'$ there exists an injective collection of maps ${}^{U'}\Phi_{xy}$ from the presheaf S_x into S_y (see (0.3)) such that

(a) If for $U \in B(x; U')$ and $V \in B(y; U')$ the map ${}^{U'}\varphi_{xy}^{UV} \in {}^{U'}\Phi_{xy}$, ${}^{U'}\varphi_{xy}^{UV}: S_U \to S_V$ is defined, then to every $U_1 \in B(x; U)$ there exists $V_1 \in B(y; V)$ for which ${}^{U'}\varphi_{xy}^{U_1V_1}$ is defined.

(b) ${}^{U'}\varphi_{xy}^{UV} : (S_U, \eta_U) \to (S_V, \eta_V)$ is an isomorphism.

(c) ${}^{U'}\varphi_{yz}^{VV_1} \circ {}^{U'}\varphi_{xy}^{UV} = {}^{U'}\varphi_{xz}^{UV_1}$, if both maps on the left are defined.

(d) ${}^{U'}\varphi_{xx}^{UU} = i_U$ – identity – if the map on the left is defined.

(e) The diagram

$$\begin{array}{ccc} S_U & \xrightarrow{\;{}^{U'}\varphi_{xy}^{UV}\;} & S_V \\ \Big\downarrow{\scriptstyle \varrho_{UU_1}} & & \Big\downarrow{\scriptstyle \varrho_{VV_1}} \\ S_{U_1} & \xrightarrow{\;{}^{U'}\varphi_{xy}^{U_1V_1}\;} & S_{V_1} \end{array}$$

is commutative, if all the maps here are defined. (Notation from (0.25)).

(f) For $U', U'' \in \mathcal{B}(X)$, $x, y \in U' \cap U''$ there is ${}^{U'}\varphi_{xy}^{UV} = {}^{U''}\varphi_{xy}^{UV}$, if both maps here are defined.

1.1.3. Remark. A. Let G be a topological group with the group operation $+$, $X \in \mathcal{B}(G)$, and $\mathcal{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ a presheaf over X, where for $U \in \mathcal{B}(X)$ S_U is the set of all continuous real functions on U and η_U is, for example, the uniformity of uniform convergence. Let $U' \in \mathcal{B}(X)$, $x_0, y_0 \in U'$. Then there exists $U \in B(x_0, U')$ and $V \in B(y_0; U')$ such that

1. $V = U + (y_0 - x_0)$.

2. If $f(x) \in S_U$, then $\varphi(y) = f(y - y_0 + x_0) \in S_V$.

3. If we assign in this way to every $f \in S_U$ a function $\varphi \in S_V$, we get an injective map ${}^{U'}\varphi_{x_0y_0}^{UV} : S_U \to S_V$, which is surjective.

4. The size of the neighborhoods U and V of the points x_0 and y_0, for which ${}^{U'}\varphi_{x_0y_0}^{UV}$ exists, depends on the size of U' and the location of x_0 and y_0 in U'.

5. It can be easily seen, that for the functor ${}^{U'}\Phi_{x_0y_0} = \{{}^{U'}\varphi_{x_0y_0}^{UV}\}$ (1.1.2) holds.

B. If there exists ${}^{U'}\varphi_{xy}^{UV}$, we may assume the existence of ${}^{U'}\varphi_{yx}^{VU}$, since we may put ${}^{U'}\varphi_{yx}^{VU} = ({}^{U'}\varphi_{xy}^{UV})^{-1}$.

1.1.4. Corollary. *If $U' \in \mathcal{B}(X)$, $x, y \in U'$, then there exists a natural map ${}^{U'}\psi_{xy}$ of the stalk $\psi^{-1}(x)$ onto $\psi^{-1}(y)$, generated by the set of maps ${}^{U'}\Phi_{xy}$. For $U', U'' \in$*

$\in \mathscr{B}(X)$, $x, y \in U' \cap U''$ *there is* ${}^{U'}\psi_{xy} = {}^{U''}\psi_{xy}$, *therefore we write briefly* ψ_{xy} *Here* ψ_{xy} *is injective and for* $x, y, z \in X$ *we have*

(a) $\psi_{yz}\psi_{xy} = \psi_{xz}$.

(b) $\psi_{xx} = \textit{identity}$.

Proof. The existence of ${}^{U'}\psi_{xy}$ follows from (1.1.2e). The property (a) resp. (b) follows from (1.1.2c) resp. (1.1.2d). The equality ${}^{U'}\psi_{xy} = {}^{U''}\psi_{xy}$ follows from (1.1.2f). We are going to prove the injectivity of ψ_{xy}. Let $\alpha, \beta \in \psi^{-1}(x)$ and $\psi_{xy}(\alpha) = \psi_{xy}(\beta)$. Let $\alpha \in S_U$ resp. $b \in S_U$ be a generating element for α resp. β. We may assume, that ${}^X\varphi_{xy}^{UV}$ is defined for some $V \in B(y, X)$. Then for some $V_1 \in B(y, V)$ there is $\varrho_{VV_1}\,{}^X\varphi_{xy}^{UV}(a) = \varrho_{VV_1}\,{}^X\varphi_{xy}^{UV}(b)$. By (1.1.2a), (1.1.3B), ${}^X\varphi_{yx}^{V_1U_1}$ is defined for some $U_1 \in \in B(X; U)$. Hence

$$(1.1.5) \qquad {}^X\varphi_{yx}^{V_1U_1}\varrho_{VV_1}\,{}^X\varphi_{xy}^{UV}(a) = {}^X\varphi_{yx}^{V_1U_1}\varrho_{VV_1}\,{}^X\varphi_{xy}^{UV}(b)\,.$$

By (1.1.2e) we get

$${}^X\varphi_{yx}^{V_1U_1}\varrho_{VV_1}\,{}^X\varphi_{xy}^{UV}(a) = {}^X\varphi_{yx}^{V_1U_1}\,{}^X\varphi_{xy}^{U_1V_1}\varrho_{UU_1}(a) = {}^X\varphi_{xx}^{U_1U_1}\varrho_{UU_1}(a) = \varrho_{UU_1}(a)\,.$$

(1.1.5) implies $\varrho_{UU_1}(a) = \varrho_{UU_1}(b)$, hence $\alpha = \beta$.

1.1.6. Remark. Let us set

$$(1.1.7) \qquad \Phi = \{{}^{U'}\Phi_{xy};\, U' \in \mathscr{B}(X),\, x, y \in U'\};\ \Psi = \{\psi_{xy};\, x, y \in X\}\,.$$

For the presheaf $\mathscr{S}$ there could exist several systems Φ and they could generally give us various systems Ψ of stalk-isomorphisms. Thus we have to take a fixed Φ, form the system Ψ corresponding to it, and in the next to study only the system Ψ. For another system Ψ (for instance if we took another Φ) we should get similar results, depending on our choice of Ψ. Further let Φ be fixed and let Ψ be the system from (1.1.7), corresponding to it. As well we could simply suppose, that we have a system Ψ of stalk-isomorphisms from (1.1.7), which satisfies (a), (b) from (1.1.4).

1.1.8. Notation. Let t be a closure in P. Let us denote

$$(1.1.9) \qquad z(t) = \{\eta_x;\, x \in X\}\,,$$

a family of semiuniformities, such that

(a) η_x is a semiuniformity in $\psi^{-1}(x)$ for all $x \in X$.

(b) η_x generates the closure $\operatorname{ind}_{\psi^{-1}(x)} t$.

(c) $\psi_{xy} : (\psi^{-1}(x); \eta_x) \to (\psi^{-1}(y); \eta_y)$ is an isomorphismus.

For the closure t let $Z(t)$ be the set of all systems $z(t)$ from (1.1.8). (For some closures t in P there is $Z(t) = \emptyset$.)

2. Introduction of notions

Let t be a closure in P, $z(t) \in Z(t)$, $x_0 \in X$, $K \subset X$ an arbitrary set and let $M \in \in \mathscr{D}(x_0; \eta_{x_0})$ (see (0.29)). Let us set (notation from (0.25)):

$$(1.2.1) \qquad F(K; M) = \bigcup_{y \in K} \bar{\psi}_{x_0 y}(M),$$

$$(1.2.2) \qquad \mathscr{F}_K(x_0) = \{F(K; M);\ M \in \mathscr{D}(x_0; \eta_{x_0})\}.$$

It is clear, that $\mathscr{F}_K(x_0)$ is a filter base in $\psi^{-1}(K) \times \psi^{-1}(K)$ round the diagonal. If $x_1 \in X$, then $\mathscr{F}_K(x_0) \sim \mathscr{F}_K(x_1)$, for $(\psi^{-1}(x_0); \eta_{x_0})$ and $(\psi^{-1}(x_1); \eta_{x_1})$ are isomorphic under the map $\psi_{x_0 y_0}$ and (1.1.4a) holds. Thus we may write simply $\mathscr{F}_K$.

For $U \in \mathscr{B}(X)$, $K \in \mathscr{K}^U$ (see (1.1.1)) and $F \in \mathscr{F}_K$ let

$$(1.2.3) \qquad P(U, K, F) = (\bar{\xi}_{UK} \times \bar{\xi}_{UK})^{-1}(F) = \\ = \{(a(x), b(y));\ a(x), b(x) \in A_U,\ (a(y), b(y)) \in F \text{ for all } y \in K\}.$$

$$(1.2.4) \qquad \mathscr{P}(U) = \{P(U, K, F);\ K \in \mathscr{K}^U,\ F \in \mathscr{F}_K\}.$$

1.2.5. Definition. Clearly $\mathscr{P}(U)$ is a filter base in $A_U \times A_U$ round diagonal. The semiuniformity in the sets A_U generated by $\mathscr{P}(U)$ we call *semiuniformity of uniform convergence on the cofiltration* $\varkappa$ from (1.1.1) and denote it by $n_z(t)$, by which we express its dependence on t and $z(t) \in Z(t)$. The natural map $p_U : (S_U, \eta_U) \to (A_U, n_z(t))$ we denote by $p_U^{n_z(t)}$. The closure t in P we call *normal*, if $Z(t) \neq \emptyset$ and for some $z(t) \in \in Z(t)$ the all $p_U^{n_z(t)}$ are isomorphisms.

3. Normal closure

1.3.1. Notation. Let $U \in \mathscr{B}(X)$, $K \in \mathscr{K}^U$. Every map $\mu : B(K; U) \to \prod_{\beta(K,U)} \mathscr{D}(S_V; \eta_V)$ such that for $V \in B(K; U)$ there is $\mu(V) = N^V \in \mathscr{D}(S_V; \eta_V)$ we shall call choice. The chain from U to K is the family

$$(1.3.2) \qquad \mathscr{R}(U, K, N^V) = \{N^V;\ V \in B(K; U),\ N^V = \mu(V)\}.$$

For $x_0 \in X$ let

$$(1.3.3) \qquad \mathscr{S}(\mathscr{R}(U, K, N^V)) = \bigcup_{K \subset V \subset U} \bigcup_{y \in K} \bar{\psi}_{y x_0} \bar{\xi}_{V y}(N^V),$$

$$(1.3.4) \qquad \mathscr{B}(K; U) = \{\mathscr{S}(\mathscr{R}(U, K, N^V));\ \mathscr{R}(U, K, N^V) \text{ is a chain from } U \text{ to } K\}.$$

1.3.5. Proposition. Let $U, U' \in \mathscr{B}(X)$, $K \in \mathscr{K}^U \cap \mathscr{K}^{U'}$. *Then* $\mathscr{B}(K, U')$, $\mathscr{B}(K; U)$ *are equivalent filter bases in* $\psi(x_0) \times \psi(x_0)$ *round the diagonal.*

Proof. If $S = \mathscr{S}(\mathscr{R}(U, K, N^V)) \in \mathscr{B}(K; U)$, let us set for $V \in B(K; U')$, $\tilde{N}^V = \bar{\varrho}^{-1}_{VV \cap U}(N^{V \cap U})$. Then $\tilde{S} = \mathscr{S}(\mathscr{R}(U', K, \tilde{N}^V)) \in \mathscr{B}(K; U')$ and $\tilde{S} \subset S$.

1.3.6. Corollary. Instead of $\mathscr{B}(K; U)$ we can write $\mathscr{B}(K)$, and use allways the more convenient base $\mathscr{B}(K; U)$. The semiuniformity, generated in $\psi^{-1}(x_0)$ by the bases $\mathscr{B}(K)$ we denote by η_K. Further for $\mathscr{K}^X = \bigcup_{U \in \mathscr{B}(X)} \mathscr{K}^U$ let

$$\eta^* = \varinjlim_{K \in \mathscr{K}^X} \eta_K . \tag{1.3.7}$$

Of course, it does not mater, in which stalk $\psi^{-1}(x_0)$ we form the semiuniformities η_K and η^*, for all the stalks $\psi^{-1}(x)$, $\psi^{-1}(y)$ are isomorphic under the map ψ_{xy}, which we use for transferring of semiuniformities, and (1.1.4) holds.

For the sake of brevity we introduce for the uniform continuity the abbreviation u.c.

1.3.8. Proposition. *Let t be a closure in P, $z(t) \in Z(t)$. All the maps $p_U^{n_z(t)}$ are u.c. iff in $\psi^{-1}(x_0)$ we have $\eta^* \leqq \eta_{x_0}$.*

Proof. Let all the $p_U^{n_z(t)}$ be u.c., $K \in \mathscr{K}^x$ (see (1.3.6)). We shall prove, that $\eta_K \leqq \eta_{x_0}$. Let $U \in \mathscr{B}(X)$, $V \in B(K; U)$, $M \in \mathscr{D}(\psi^{-1}(x_0); \eta_{x_0})$.

Then according to (1.2.1) $F = F(K; M) \in \mathscr{F}_K$ and by (1.2.3–5) $P = P(V; K, F) \in \mathscr{D}(A_V; n_z(t))$. Because $p_U^{n_z(t)}$ is u.c., we have $\bar{p}_V(N^V) \subset P$ for some $N^V \in \mathscr{D}(S_V; \eta_V)$. Hence by (1.2.1,3) there is $\bar{p}_V(N^V) \subset (\xi_{VK} \times \xi_{VK})^{-1} (\bigcup_{y \in K} \bar{\psi}_{x_0 y}(M))$, i.e. $N^V \subset \xi^{-1}_{VK}(\bigcup_{y \in K} \bar{\psi}_{x_0 y}(M))$. Thus for all $y \in K$ we have $\xi_{Vy}(N^V) \subset \bar{\psi}_{x_0 y}(M)$, $\bar{\psi}_{y x_0} \xi_{Vy}(N^V) \subset M$ and hence

$$\bigcup_{y \in K} \bar{\psi}_{y x_0} \xi_{Vy}(N^V) \subset M . \tag{1.3.9}$$

To every $V \in \mathscr{B}(U)$ we find in this way N^V for which (1.3.9) holds. Then

$$S = \mathscr{S}(\mathscr{R}(U, K, N^V)) \subset M \tag{1.3.10}$$

and $S \in \mathscr{D}(\psi^{-1}(x_0); \eta_K)$. By (1.3.10) we have $\eta_K \leqq \eta_{x_0}$. Because it holds for every $K \in \mathscr{K}^X$, we get $\eta^* \leqq \eta_{x_0}$. Conversely, let $\eta^* \leqq \eta_{x_0}$. Thus for every $K \in \mathscr{K}^X$ we have $\eta_K \leqq \eta_{x_0}$. Let $U \in \mathscr{B}(X)$ and $P = P(U, K, F) \in \mathscr{D}(A_U; u_z(t))$ for $F = F(K; M) \in \mathscr{F}_K$, where $M \in \mathscr{D}(\psi^{-1}(x_0); \eta_{x_0})$. By (1.3.6) we can find $S = \mathscr{S}(\mathscr{R}(U, K, N^V)) \in \mathscr{B}(K; U)$ such that $S \subset M$. By (1.3.3) there is $\bigcup_{y \in K} \bar{\psi}_{y x_0} \xi_{Uy}(N^U) \subset M$ and therefore

$$\xi_{Uy}(N^U) \subset \bar{\psi}_{x_0 y}(M) \tag{1.3.11}$$

for all $y \in K$. By (1.2.1,2) we have $F = F(K; M) = \bigcup_{y \in K} \bar{\psi}_{x_0 y}(M) \in \mathscr{F}_K$ and by (1.3.11) there is $\bar{p}_U(N^U) \subset P(U; K, F)$, which finishes the proof.

If $K \subset X$ is an arbitrary set and $D \in \mathscr{D}(\psi^{-1}(x_0); \eta^*)$, let

$$G(K; M) = \bigcup_{y \in K} \bar{\psi}_{x_0 y}(M)\,, \tag{1.3.12}$$

$$\mathscr{G}_K = \{G(K; M);\quad M \in \mathscr{D}(\psi^{-1}(x_0); \eta^*)\}\,, \tag{1.3.13}$$

which is a filter base in $\psi^{-1}(K) \times \psi^{-1}(K)$ round the diagonal. Here for $K \subset L$, $\mathscr{G}_K \sim \mathscr{G}_L \cap \psi^{-1}(K)$.

For $U \in \mathscr{B}(X)$, $K \in \mathscr{K}^U$, $G \in \mathscr{G}_K$ let

$$E(U; K, G) = (\bar{\xi}_{UK} \times \bar{\xi}_{UK})^{-1}(G) \tag{1.3.14}$$

$$\mathscr{E}(U) = \{E(U; K, G);\ K \in \mathscr{K}^U,\ G \in \mathscr{G}_K\}\,. \tag{1.3.15}$$

1.3.16. Notation. It is clear, that $\mathscr{E}(U)$ is a filter base in $A_U \times A_U$ round the diagonal. The semiuniformity generated there by it we shall denote by n. The natural map $p_U : (S_U, \eta_U) \to (A_U, n)$ we denote by p_U^n.

1.3.17. Proposition. A. *All the maps p_U^n are u.c.*

B. *For the closure t in P, and $z(t) \in Z(t)$ the all $p_U^{n_z(t)}$ are u.c. iff $n \leqq n_z(t)$ in every A_U.*

Proof. A. If $U \in \mathscr{B}(X)$, $E = E(U; K, G) = (\bar{\xi}_{UK} \times \bar{\xi}_{UK})^{-1}(G) \in \mathscr{D}(A_U, n)$ (where $G = G(K; M)$ for some $M \in \mathscr{D}(\psi^{-1}(x_0); \eta^*)$), then by (1.3.7,5) $S \subset M$ for some $S = \mathscr{S}(\mathscr{R}(U, K, N^V)) \in \mathscr{B}(K; U)$. By (1.3.3) we get $\bigcup_{y \in K} \bar{\psi}_{y x_0} \bar{\xi}_{Uy}(N^U) \subset M$. Thus for all $y \in K$ there is $\bar{\xi}_{Uy}(N^U) \subset \bar{\psi}_{x_0 y}(M)$, hence $\bar{p}_U(N^U) \subset E$, which we were to prove.

B. From u.c. of all $p_U^{n_z(t)}$ (by (1.3.8)) follows $\eta^* \leqq \eta_{x_0}$ in $\psi^{-1}(x_0)$. Therefore for every $K \in \mathscr{K}^X$ there is $\mathscr{G}_K \leqq \mathscr{F}_K$ (compare (1.3.12,13) and (1.2.1,2)). Thus by (1.3.14,15) and (1.2.3,4) there is $n \leqq n_z(t)$ in all A_U. Conversely, if $n \leqq n_z(t)$ in all A_U, A. implies u.c. of all $p_U^{n_z(t)}$. From (1.3.17) we get immediately

1.3.18. Proposition. *For $z(t) \in Z(t)$ let the all $(p_U^{n_z(t)})^{-1}$ be u.c. Then $n_z(t) \leqq n$ in all A_U. If the all $p_U^{n_z(t)}$ are isomorphisms, $n_z(t) = n$ in all A_U.*

1.3.19. Definition. The closure t in P we call *polonormal*, if there exists $z(t) \in Z(t)$ such that $n_z(t) = n$ in all A_U. (1.3.18) implies the following two theorems:

1.3.20. Theorem. *The necessary condition for the existence of a normal closure in P is the u.c. of all $(p_U^n)^{-1}$.*

1.3.21. Theorem. *The necessary condition for the existence of a normal closure is the existence of a polonormal closure.*

1.3.22. Theorem. *If there exists a polonormal closure t_1 in P then there exists a normal closure in P iff all the $(p^n_U)^{-1}$ are u.c. If this condition holds, then t_1 is normal.*

Proof. If t is normal, then by (1.3.18) the all $(p^n_U)^{-1}$ are u.c. Conversely, from u.c. of all $(p^n_U)^{-1}$ follows, that t_1 is normal.

4. Polonormal closure

1.4.1. Notation. Let $\mathscr{G}_X$ be the base from (1.3.13) for $K = X$, which is a filter base in $P \times P$ round the diagonal. Thus it generates a semiuniformity v there. For $x \in X$ let $v_x = \operatorname{ind}_{\psi^{-1}(x)} v$. From (1.3.12–15) follows immediately.

1.4.2. Proposition. (a) *For $x, y \in X$, $\psi_{xy} = (\psi^{-1}(x); v_x) \to (\psi^{-1}(y); v_y)$ is an isomorphismus.*

(b) *If t_0 is the closure generated in P by v, then $Z(t_0) \neq \emptyset$, because by* (a) $z_0 = z_0(t_0) = \{v_x; x \in X\} \in Z(t_0)$.

(c) $n_{z_0}(t_0) = n$ *in all* A_U.

Thus t_0 is polynormal.

(1.3.22) implies

1.4.3. Theorem. *The necessary and sufficient condition for the existence of a normal closure t in P is the u.c. of all $(p^n_U)^{-1}$. If this condition holds, then t_0 is normal.*

1.4.4. Remark. Let $x, y \in X$, $x \neq y$. From (1.3.12) we get easily, that the stalks $\psi^{-1}(x)$, $\psi^{-1}(y)$ are mutually separated under the closure t_0. Every stalk $\psi^{-1}(x)$ is a clopen (closed and open) set in (P, t_0). Therefore there is not $A_U \subset \Gamma(U, t_0)$ (see (0.34)).

1.4.5. Notation. Let t_s be the sheaf topology from (0.34). Let us denote by t_1 the topological sum of the closures t_0 and t_s (see (0.36)). Then $\operatorname{ind}_{\psi^{-1}(x)} t_s = \operatorname{ind}_{\psi^{-1}(x)} t_0$ for all $x \in X$ and thus by (1.4.2) and (0.35) we get.

1.4.6. Theorem. (a) $Z(t_1) \neq \emptyset$, *because* $z_1 = z_1(t_1) = \{v_x; x \in X\} \in Z(t_1)$.

(b) $n_{z_1}(t_1) = n$ *in all A_U. thus t_1 is polonormal.*

(c) $A_U \subset \Gamma(U, t_1)$ *for all* $U \in \mathscr{B}(X)$.

Thus the theorem (1.4.3) *holds for the closure t_1.*

Now we shall prove the continuity of all $(p^n_U)^{-1}$ in one special case.

1.4.7. Agreement. For $x_0 \in X$ let η_{x_0} be a semiuniformity in $\psi^{-1}(x_0)$. Let $T \in \mathscr{D}(\psi^{-1}(x_0), \eta_{x_0})$, $U \in \mathscr{B}(X)$, $K \subset U$ an arbitrary set. Then the set $F = F(K, T)$

resp. $P(U; K, F)$ of the form (1.2.1) resp. (1.2.3) we denote by $\text{transf}_K T$ resp. ind $\text{transf}_K T$. The set of the form (1.2.2) resp. (1.2.4) we denote by $\text{transf}_K \mathscr{D}(x_0; \eta_{x_0})$ resp. ind $\text{transf}_K \mathscr{D}(x_0, \eta_{x_0})$. The semiuniformity, which is generated in A_U by these bases we denote by ind η_{x_0}.

1.4.8. Proposition. *Let $\mathscr{S} = \{(S_U, \eta_U); \varrho_{UV}; X\}$ be a presheaf of some sets of continuous functions on a topological group X with the group operation +. For $\mathscr{S}$ let be given a cofiltration $\varkappa = \{\mathscr{K}^U; U \in \mathscr{B}(X)\}$ such that every $K \in \mathscr{K}^U$ is relatively compact in U and (1.1.1) holds. Let η_U be the semiuniformity of uniform convergence on $\mathscr{K}^U$. For $U \in \mathscr{B}(X)$, $x, y \in U$ let the functor ${}^U\Phi_{xy}$ from (1.1.2) be as in (1.1.3). Let $\Psi = \{\psi_{xy}, x, y \in X\}$ be the family corresponding by (1.1.4,6) to it. If some $\mathscr{K}^U$ contains a single point $x_0 = K_0 \in \mathscr{K}_0$, then the all $(p_U^n)^{-1}$ are u.c. (n is the closure from (1.3.16) formed for $\varkappa$).*

Proof. If $x_0 = K_0 \in \mathscr{K}^U$, then for η_{K_0} from (1.3.6) there is $\eta_{K_0} = \eta_{x_0} = \varinjlim_{V \in B(X_0;U)} \eta_V$. Because $\mathscr{S}$ satisfies (1.1.2), it can be easily seen, that if $\tilde{\eta}_y = \varinjlim_{V \in B(y;U)} \eta_V$, then $\psi_{xy} : (\psi^{-1}(x); \tilde{\eta}_x) \to (\psi^{-1}(y); \tilde{\eta}_y)$ is an isomorphismus.

Let $z \in U \in \mathscr{B}(X)$, $\text{col}_U \varepsilon_V = \{\varepsilon_V; V \in B(z; U), \varepsilon_V > 0\}$, $N^z \text{col}_U \varepsilon_V = \{(\text{germ}_z g_1^V, \text{germ}_z g_2^V); V \in B(z; U), (g_1^V, g_2^V) \in S_V \times S_V, |g_1^V(z) - g_2^V(z)| < \varepsilon_V\}$. Then for every $U \in \mathscr{B}(X)$ $\mathscr{D}(\psi^{-1}(z); \tilde{\eta}_z) \sim \{N^z \text{col}_U \varepsilon_V; \text{col}_U \varepsilon_V\} = C_U$. Let $U' \in \mathscr{B}(X)$, $K \in \mathscr{K}^{U'}$, $x, y \in U'$. The definition of ${}^{U'}\Phi_{xy}$ implies easily, that if ${}^{U'}\varphi_{xy}^{UV}$ is defined, then $\psi_{xy}(N^x \text{col}_U \varepsilon_{\tilde{V}}) = N^y \text{col}_V \varepsilon_{\tilde{V}}$. There exists $V_0 \in B(x_0; U')$ such that if $x \in K$, then $V_x = V_0 + x - x_0 \in B(x; U')$, and ${}^{U'}\varphi_{x_0x}^{V_0V_x}$ is defined. Let $N_0 = N^{x_0} \text{col}_{V_0} \varepsilon_{\tilde{V}} \in \mathscr{D}(\psi^{-1}(x_0); \eta_{x_0})$, $\varepsilon_{V_0} = \varepsilon_0$. Let

$$L(K; \varepsilon_0) = \{(f, g); (f, g) \in S_{U'} \times S_{U'}, |f(x) - g(x)| < \varepsilon_0, x \in K\} . \tag{1.4.9}$$

Then $L(K; \varepsilon_0) \in \mathscr{D}(S_{U'}; \eta_{U'})$ and $\bar{p}_U(L(K; \varepsilon_0)) \subset \text{ind transf}_K N_0$, which proves the u.c. of all maps $p_U : (S_U, \eta_U) \to (t_U, \text{ind } \eta_{x_0})$. Because $\eta_{x_0} \leqq \eta^*$ and η^* is the finest of the all semiuniformities λ, for which the all $p_U : (S_U, \eta_U) \to (A_U, \text{ind } \lambda)$ are u.c., we get $\eta_{x_0} = \eta^*$. Now let $L(K; \varepsilon) \in \mathscr{D}(S_{U'}; \eta_{U'})$ be of the form (1.4.9). Let us take $N_0 = N^{x_0} \text{col}_U \varepsilon_{\tilde{V}} \in \mathscr{D}(\psi^{-1}(x_0), \eta^*)$ such that $\varepsilon_{\tilde{V}} \leqq \varepsilon$ for all $\tilde{V} \in B(x_0; V_0)$. Then $(\bar{p}_{U'})^{-1}$ (ind $\text{transf}_K N_0) \subset L(K; \varepsilon)$, which proves the u.c. of $(p_{U'}^n)^{-1}$.

Chapter 2.

SECTIONS IN THE COVERING SPACE

Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a presheaf of closure spaces, P its covering space, t a closure in P, $\Gamma(U, t)$ the set of all continuous sections over $U \in \mathscr{B}(X)$, and q some method, which enables us in every A_U to construct from t a closure $q(t)$. We study, when there exist q and t such that all the natural maps $p_U : (S_U, \tau_U) \to (A_U, q(t))$ are homeomorphisms, and such that $A_U = \Gamma(U, t)$ for every $U \in \mathscr{B}(X)$. In the whole chapter let t^* be the natural closure in P from [5], (2.6.2).

1. Relation between $\Gamma(U, t)$ and the set of solvable sections

2.1.1. Definition. Let $U \in \mathscr{B}(X)$. By *collection* we shall call every set

$$[\varphi] = \{(a_x, V_x);\ x \in U\} \tag{2.1.2}$$

where for every $x \in U$ there is $V_x \in B(x, U)$, and $a_x \in S_{V_x}$. A collection $[\varphi]$ we call *solvable*, if there exists $a \in S_U$ such that $a(x) = a_x(x)$ for all $x \in U$ (see (0.9)). The element a we call *solution of the collection* $[\varphi]$. We say, that a collection $[\varphi'] = \{(b_x, U_x);\ x \in X\}$ refines a collection $[\varphi]$ from (2.1.2), if there is

(a) $U_x \subset V_x$ for all $x \in U$,

(b) $\varrho_{V_x U_x}(a_x) = b_x$.

A collection $[\varphi]$ from (2.1.2) we call *smooth*, if there is

$$\varrho_{V_x, V_x \cap V_y}(a_x) = \varrho_{V_y, V_x \cap V_y}(a_y) \quad \text{for all} \quad x, y \in U\,. \tag{2.1.3}$$

2.1.4. Proposition. *Let $\mathscr{S}$ be a projective presheaf (see (0.31)). A collection $[\varphi]$ is solvable iff there exists a smooth collection $[\varphi']$, refining $[\varphi]$.*

Proof. Let $[\varphi]$ from (2.1.2) be solvable, a its colution. Thus $a(x) = a_x(x)$ for all $x \in U$. But it means, that to every $x \in U$ there exists $U_x \in B(x; V_x)$ such that $b_x = \varrho_{V_x U_x}(a_x) = \varrho_{U U_x}(a)$. Then the collection

$$[\varphi'] = \{(b_x, U_x);\ x \in X\} \tag{2.1.5}$$

refines $[\varphi]$. Here for $x, y \in U$ we have $\varrho_{U_x, U_x \cap U_y}(b_x) = \varrho_{U, U_x \cap U_y}(a) = \varrho_{U_y, U_x \cap U_y}(b_y)$, thus $[\varphi']$ is smooth. For the proof of this implication we did not need the projectivity of $\mathscr{S}$.

Conversely, if $[\varphi']$ from (2.1.5) is smooth, then there exists (as a result of the projectivity of $\mathscr{S}$) an element $a \in S_U$ such that $\varrho_{U, U_x}(a) = b_x$ for all $x \in U$. Because $[\varphi']$ refines $[\varphi]$, $[\varphi]$ is solvable by a. The proof is finished.

Every section φ over U (see (0.8)) determines (not uniquely) some collection. That is because for every $x \in U$ the germ $\varphi(x)$ has a representative (i.e. a generating element) $a_x \in S_{V_x}$ for some $V_x \in B(x, U)$. Then $\{(a_x, V_x);\ x \in U\}$ is a collection determined by φ.

2.1.6. Notation. Every collection determined by φ we call collection of the section φ and denote it by $[\varphi]$. It is clear, that every collection $[\varphi']$, refining $[\varphi]$ is again a collection of φ.

2.1.7. Proposition. *Let t be a closure in P, $U \in \mathscr{B}(X)$. Then $\Gamma(U, t) \subset A_U$ iff every $\varphi \in \Gamma(U, t)$ has a solvable collection.*

Proof. Let $\varphi \in \Gamma(U, t)$. If its collection $[\varphi]$ is solvable by a, then $a(x) = \varphi(x)$ for all $x \in U$, hence $p_U(a) = \varphi \in A_U$. The converse implication is trivial.

2.1.8. Remark. According to (2.1.4) we can see, that the solvability of a collection depends only on the inner structure of the sets A_U and not on the way of defining the closures τ_U in S_U, or on the closure t in P. Thus the set of all solvable collections is given in advance.

2. Relation between $\Gamma(U, t)$ and $S(U, S_U)$

2.2.1. Notation. If $U \in \mathscr{B}(X)$, let u be the topology in U induced from X. Let $S(U, S_U)$ denote the set of all continuous maps $f : (U, u) \to (S_U, \tau_U)$. If $f : U \to S_U$ is any (not necessary continuous) map, we denote by $V_f : U \to S_U \times U$ the map constructed as follows: $x \in U \Rightarrow V_f(x) = (f(x), x)$. Let δ_U be the natural map from [5], (2.6.1), i.e. the map $\delta_U : S_U \times U \to P$ defined as follows: $a \in S_U$, $x_0 \in U \Rightarrow \delta_U(a, x_0) = a(x_0)$. Then we denote

$$\varphi^f = \delta_U \circ V_f : U \to P . \tag{2.2.2}$$

Every map $f : U \to S_U$ is determined by the set

$$f \equiv \{a_x; x \in U , \ a_x = f(x)\} , \tag{2.2.2A}$$

where for every $x \in U$ there is $a_x \in S_U$. Thus every such f is uniquely determined by the collection $\{(a_x, V_x), x \in U\}$ of the form (2.1.2), where $V_x = U$ for all $x \in U$. This collection we shall call collection corresponding to f, but we shall write it in the form (2.2.2A), regarding, that it is the collection $\{(a_x; U), x \in U\}$ from (2.1.1). We say, that $f : U \to S_U$ is solvable, if it has a solvable collection.

2.2.3. Proposition. *Let the closure t in P be coarser than the natural closure t^*, $U \in \mathscr{B}(X)$. The necessary condition for the inclusion $\Gamma(U, t) \subset A_U$ is the solvability of all $f \in S(U, S_U)$.*

Proof. Let $f : (U, u) \to (S_U, \tau_U)$ be continuous. Then the map $V_f : (U, u) \to (S_U \times U, \tau_U \times u)$ from (2.2.1) is also continuous. Because $t^* \leqq t$, the natural map $\delta_U : (S_U \times U, \tau_U \times u) \to (P, t)$ is continuous (see [5], (2.6.2)). Hence the map $\varphi^f = \delta_U \circ V_f : U \to P$ from (2.2.2) is also continuous. Thus $\varphi^f \in \Gamma(U, t)$. We can see, that the collection (2.2.2A) corresponding to f is as well the collection $[\varphi^f]$ of the section φ^f (see (2.1.6)). Now the assertion follows from (2.1.7).

2.2.4. Remark. Let $\mathscr{S}$ be with the unique continuation (see (0.32)), $U \in \mathscr{B}(X)$ connected, $f \in S(U, S_U)$, $f \equiv \{(a_x, U); x \in U\}$. It can be easily seen, that f is solvable iff $a_x = a_y$ for all $x, y \in U$. Thus if there exists a nonconstant $f \in S(U, \tau_U)$, there is not $\Gamma(U, t) \subset A_U$ for any $t \geqq t^*$. It implies, that there is not $\Gamma(U, t) \subset A_U$ in both examples (2.2.8).

2.2.5. Remark. From (2.1.4) we can see, that the solvability of the collection (2.2.2A) corresponding to f, depends only on the inner structure of S_U. It does not depend on the particular closure in S_U. Thus the set of all solvable collections of type (2.2.2A) is given in advance. If τ_U is such a closure, that some $f \in S(U, S_U)$ is not solvable, then there is not (by (2.2.3)) $\Gamma(U, t) \subset A_U$ for any closure t, $t \geqq t^*$.

2.2.6. Remark. If every $f \in S(U, S_U)$ is solvable, it does not mean yet, that $\Gamma(U, t) \subset \subset A_U$, because every $\varphi \in \Gamma(U, t)$ need not be represented by the map $f \in S(U, S_U)$ such that $\varphi = \varphi^f$ (see (2.2.2)). If $\varphi \in \Gamma(U, t)$, then φ is represented only by its collection $[\varphi]$ of the form (2.2.1). So us to decide whether $\varphi \in A_U$, we have (2.1.7). The proposition (2.2.3) often shows, when there is not $\Gamma(U, t) \subset A_U$, as we shall see in examples. The proposition (2.1.7) says much more, but it is qualified by the continuity of the section $\varphi \in \Gamma(U, t)$ whereas (2.2.3) is qualified by the continuity of $f : U \to S_U$. And it is more difficult to verify the solvability of the collection $[\varphi]$, than the solvability of f from the continuity of $f : U \to S_U$, not regarding the difficulties when deciding, if φ is continuous or not.

2.2.7. Remark. Let $U \in \mathscr{B}(X)$ and let us assume: "If $V \in \mathscr{B}(U)$, $b \in S_V$, then there exists $a \in S_U$ such that $\varrho_{UV}(a) = b$". Then every section φ over U has a collection $[\varphi] = \{(a_x, U); x \in X\}$. The assignement $x \to a_x$ is a map $f : U \to S_U$. But if φ is continuous, the map f, corresponding to it in this way, need not be continuous.

2.2.8. Examples. (1) Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; E_1\}$ be the presheaf of all constant functions over E_1, τ_U the closure of pointwise convergence. We can easily find, that (P, t^*) and E_2 are homeomorphic under the identity map. To every $a \in S_U$ there corresponds a constant section $a(x) = a \in \Gamma(U, t^*)$. The section $\tilde{\varphi}(x) = x \in \Gamma(U, t^*)$ corresponds to the non-solvable map $f \in S(U, S_U)$, where $f(x) = x \in S_U$ for $x \in U$. Thus $\tilde{\varphi}(x) = \varphi^f(x)$ and $\varphi^f \notin A_U$. Hence $A_U \subsetneqq \Gamma(U, t^*)$. Every continuous function on U, which is not constant on any component of U represents a non-solvable continuous map f of U into (S_U, τ_U) and the corresponding φ^f does not lie in A_U. It can be seen, that $\Gamma(U, t^*)$ is (under the map $f \to \varphi^f$) isomorphic to the set of all continuous functions on U.

(2) Let $\mathscr{S} = \{(S_U, \tau_U), \varrho_{UV}; E_1\}$ be the presheaf of all polynomials, τ_U the topology of locally uniform convergence. For $U \in \mathscr{B}(X)$ let us define $f : U \to S_U$ as follows:

$$(2.2.9) \qquad f(x) = 2xt - x^2 ,$$

thus to every $x \in U$ we assign the polynomial $2xt - x^2$ of the variable t. First, if $x_n, x_0 \in U$, $x_n \to x_0$ then $f(x_n) = 2x_nt - x_n^2 \to 2x_0t - x_0^2$ locally uniformly. Thus

$f \in S(U, S_U)$ and therefore $\varphi^f \in \Gamma(U, t^*)$. The map f is not solvable, because $\varphi^f(x) = \text{germ}_x (2xt - x^2)$ is the germ of the polynomial $2xt - x^2$ of the variable t at the point x. If for some $a \in S_U$ there were $a = \varphi^f$, then a would coincide with $2xt - x^2$ on a neighborhood of every $x \in U$, i.e. particulary $a(x) = 2x^2 - x^2 = x^2$. Thus a is necessary the polynomial x^2. But $2xt - x^2$ does not coincide with t^2 in any neighborhood. Thus $\varphi^f \in \Gamma(U, t^*) - A_U$. Similarly, if we set $f(x) = a_0(x) t^n + \ldots + a_n(x)$, we get a map $f : U \to S_U$. If the all $a_j(x)$ are continuous functions, f is continuous and $\varphi^f \in \Gamma(U, t^*)$. If there were $\varphi^f = p_U(a) \in A_U$, then there would be $a = a_0(x) x^n + \ldots + a_n(x)$. If such element does not lie in S_U, i.e. if any a_j is not constant on any component of U, then $\varphi^f \notin A_U$.

The presheafs in the both previous examples have the unique continuation (see (0.32)) and their closure collections are projective (see [4], (1.1.4)) in spite of $A_U \not\subseteq \Gamma(U, t^*)$.

3. The case when the closures $q(t)$ are jointly continuous

Now we are going to try to find a closure t in P and a method q such that

(2.3.1) $$p_U^t : (S_U, \tau_U) \to (A_U, q(t)) \text{ are homeomorphisms },$$

(2.3.2) $j_U^t : (A_U \times U, q(t) \times u) \to (P, t)$ are continuous, i.e. the all $q(t)$ are jointly continuous,

(2.3.3) $$A_U = \Gamma(U, t) \quad \text{for all} \quad U \in \mathscr{B}(X).$$

A. If (2.3.1,2) holds, then the all natural maps (see (2.2.1)) $\delta_U : (S_U \times U, \tau_U \times u) \to (P, t)$ are continuous. By the definition of t^* in [5], (2.6.2) we have $t^* \leqq t$.

B. Let X be locally compact. If (2.3.2) holds, then $q(t) \leqq l(t)$ in every A_U, where $l(t)$ is the closure of uniform convergence on compact sets (see [5] (2.1.2)). (Kelley, General topology – [3] ch. 7).

C. We have clearly the inclusion $A_U \subset \Gamma(U, t^*)$ for all U. Because by A. there must be $t^* \leqq t$, we can see, that in those cases, when there is not $A_U = \Gamma(U, t^*)$, for all $U \in \mathscr{B}(X)$, it is not possible to find a closure t and a method q satisfying (2.3.1–3). For example, (2.1.7) implies.

2.3.4. Proposition. *If for some $U \in \mathscr{B}(X)$ there exists a nonsolvable $\varphi \in \Gamma(U, t^*)$, then (2.3.1–3) is not true for any couple t, q. It happens particulary, if for some $U \in \mathscr{B}(X)$ there exists a non-solvable $f \in S(U, S_U)$. If for every $U \in \mathscr{B}(X)$ every $\varphi \in \Gamma(U, t^*)$ is solvable, then (2.3.2,3) holds for the method $q = l$ and the closure t^* (l is the closure of uniform convergence on compact sets).*

Now the only problem is, whether (2.3.1) is true for $q(t^*) = l(t^*)$. In the convenient cases it can be decided with help of tools from chap. 2. in [5].

4. Sufficient condition for the representation

The requirements (2.3.1–3) lied to the inequality $t^* \leqq t$. But the examples show, that (2.3.3) does not hold even for some very convenient presheaves. Thus we shall omit the requirement (2.3.2).

2.4.1. Definition. We say, that the closure t in P generates a *representation*, if for some method q and the closure t (2.3.1,3) is satisfied.

2.4.2. Notation. Every stalk $\psi^{-1}(x)$ we shall provide with the coarse topology $t_x = h$ (see (0.30)). Let t_h (resp. t_s) be the closure in P, which is the topological sum of the closures t_x, (resp. the sheaf topology from (0.34)). Let t_σ be the topological sum of t_h and t_s.

2.4.3. Proposition. *Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; x\}$ be a projective presheaf, t a closure in its covering space, $t_s \leqq t \leqq t_\sigma$. Then $\Gamma(U, t) = A_U$ for all $U \in \mathscr{B}(X)$.*

Proof. For all U there is $A_U \subset \Gamma(U, t_s) \subset \Gamma(U, t) \subset \Gamma(U, t_\sigma)$ (see (0.34)). We shall prove, that $\Gamma(U, t_\sigma) \subset A_U$ for every U. If $\alpha \in P$, $x_0 = \psi(\alpha)$, then by the definition of t_σ and (0.34) we get: If for some $V \in B(x_0; X)$ the element $a \in S_V$ is a generating one for α (see (0.18)), then

$$\Delta(\alpha; t_\sigma) = \{\mathrm{gr}_{V'}\, a \cup \psi^{-1}(x_0);\ V' \in B(x_0; V)\} \quad \text{– see (0.20)}. \tag{2.4.4}$$

Let $U \in \mathscr{B}(X)$, $\varphi \in \Gamma(U, t_\sigma)$, $[\varphi] = \{(a_x, V_x); x \in U\}$ a collection of the section φ. If $x \in U$, then a_x is a generating element for $\varphi(x)$, thus

$$a_x(x) = \varphi(x). \tag{2.4.5}$$

According to (2.4.4) $O = \mathrm{gr}_{V_x} a_x \cup \psi^{-1}(x) \in \Delta(\varphi(x), t_\sigma)$. Thus there exists $\tilde{V}_x \in B(x; V_x)$ such that for $y \in \tilde{V}_x$ there is $\varphi(y) \in O$. The form of the set O implies

$$\varphi(y) = a_x(y) \quad \text{for} \quad y \in \tilde{V}_x - x. \tag{2.4.6}$$

Hence by (2.4.5), (2.4.6) holds for all $y \in \tilde{V}_x$. Moreover let us set $b_x = \varrho_{V_x \tilde{V}_x}(a_z)$. We have

$$\varphi(y) = b_x(y) \quad \text{for} \quad y \in \tilde{V}_x. \tag{2.4.7}$$

To every $x \in U$ let us construct in this way $\tilde{V}_x$ and $b_x \in S_{\tilde{V}_x}$, such that (2.4.7) holds. The collection $[\varphi'] = \{(b_x, \tilde{V}_x); x \in U\}$ is a collection of φ. Let $x, y \in U$ such that $\tilde{V}_x \cap \tilde{V}_y \neq \emptyset$. For $z \in \tilde{V}_x \cap \tilde{V}_y$ (2.4.7) implies $b_x(z) = \varphi(z) = b_y(z)$. By (0.17) we have $\varrho_{\tilde{V}_x \tilde{V}_x \cap \tilde{V}_y}(b_x) = \varrho_{\tilde{V}_y \tilde{V}_x \cap \tilde{V}_y}(b_y)$, and thus $[\varphi']$ is smooth. By (2.1.4) $[\varphi']$ is solvable. (2.1.7) implies $\varphi \in A_U$, which finishes the proof.

In [5], (3.2.4,5) we have for every $x \in X$ denoted by u_x the closure in $\psi^{-1}(x)$ defined as follows: $u_x = \mathrm{ind}_{\psi^{-1}(x)}\, t^*$. We have denoted there by $\bar{t}$ the topological sum of the closures u_x and by $\hat{t}$ the topological sum of $\bar{t}$ and t_s (see (0.36)). Then clearly

$$t_s \leqq \hat{t} \leqq t_\sigma. \tag{2.4.8}$$

2.4.9. Theorem. *Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a projective presheaf, $\mu = \{\tau_U\}$ its closure collection, μ^0 the pointwise modification of μ (see [5], (3.3.4)). If $\mu = \mu^0$, then the closure $\hat{t}$ generates a representation.*

Proof. The equality $\mu = \mu^0$ and (3.2.7) in [5] imply, that $\hat{t}$ is normal, i.e. the all $p_U^{\hat{t}} : (S_U, \tau_U) \to (A_U, b(\hat{t}))$ are homeomorphisms, where $b(\hat{t})$ is the closure of pointwise convergence (see [5], (3.1.2)). From (2.4.8,3) follows, that $\hat{t}$ generates a representation.

5. The sufficient condition for representation in the semiuniform case

2.5.1. Notation. Let $\mathscr{S} = \{(S_U, \eta_U); \varrho_{UV}; X\}$ be a presheaf of semiuniform spaces. As in chap. 1 let for every $U \in \mathscr{B}(X)$ be given a cofilter $\mathscr{K}^U$ in U, such that for $V \subset U$ there is

$$K^V = \{K; K \in \mathscr{K}^U, K \subset V\} . \tag{2.5.2}$$

For the closure t and $z(t) \in Z(t)$ (see (1.1.8)) let $u_z(t)$ be the semiuniformity of uniform convergence on the cofilters $\mathscr{K}^U$ (see (1.2.5)), and let n resp. v be the semiuniformity (1.3.16) resp. (1.4.1). Further let t_0 be the closure in P, generated by v (see (1.1.2b)), and as in (1.4.5) let t_1 be the topological sum of t_0 and the sheaf topology t_s.

2.5.3. Theorem. *Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a presheaf of closure spaces. Let every τ_U be generated by a semiuniformity η_U. Let there exists a cofiltration from (2.5.1) such that if we form $u_z(t)$, n, v, t_0, t_1 for it and the presheaf $\widetilde{\mathscr{S}} = \{(S_U, \eta_U); \varrho_{UV}; X\}$, then all the maps $(p_U^n)^{-1} : (A_U, n) \to (S_U, \eta_U)$ are uniformly continuous. Then the t_1 forms a representation.*

Proof. By (1.4.6,4) all the $p_U^{t_1} : (S_U, \eta_U) \to (A_U, n_{z_1}(t_1))$ are isomorphisms for some $z_1 = z_1(t_1) \in Z(t_1)$. Thus if $q(t_1)$ is a closure in A_U generated by $u_{z_1}(t_1)$, the all $p_U^{t_1} : (S_U, \tau_U) \to (A_U, q(t_1))$ are homeomorphisms. Let t_σ be the closure from (2.4.2). By (1.4.4,5) we get $t_s \leqq t_1 \leqq t_\sigma$. By (2.4.3) we have $\Gamma(U, t_1) = A_U$, which finishes the proof.

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