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ON THE ABSOLUTE CESÀRO SUMMABILITY OF THE ULTRASPHERICAL SERIES

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1. The ultraspherical series associated with a function $f(\theta, \phi)$, defined for the range $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq 2\pi$ on a sphere S, is

$$(1.1) \qquad f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n + \lambda) \iint_S \frac{P_n^{(\lambda)}(\cos \gamma) f(\theta', \phi')\, d\sigma'}{[\sin^2 \theta' \sin^2 (\phi - \phi')]^{1/2-\lambda}} = \frac{1}{2\pi} \sum_{n=0}^{\infty} A_n ,$$

say, where

$$\lambda > 0 ,$$

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') ,$$

$$d\sigma' = \sin \theta'\, d\theta'\, d\phi'$$

and the ultraspherical polynomial $P_n^{(\lambda)}(\cos \gamma)$ is defined by the following expansion

$$(1 - 2z \cos \gamma + z^2)^{-\lambda} = \sum_{n=0}^{\infty} z^n P_n^{(\lambda)}(\cos \gamma) , \quad \lambda > 0 .$$

The Laplace series is a particular case of the series (1.1) for the value $\frac{1}{2}$ of the parameter λ and in view of the relation

$$\lim_{\lambda \to 0} \frac{1}{\lambda} P_n^{(\lambda)}(\cos \theta) = \frac{2}{n} \cos n\theta , \quad n \geqq 1 ,$$

the ultraspherical series (1.1) reduces to the trigonometric series in the limit as $\lambda \to 0$.

Also, on account of the relation [5]

$$\frac{\Gamma(\frac{1}{2} + \lambda)}{\Gamma(\frac{1}{2})\, \Gamma(\lambda)} \int_0^{\pi} (\sin \omega)^{2\lambda - 1} P_n^{(\lambda)}(\cos \gamma)\, d\omega =$$

$$= \frac{\Gamma(n + 1)\, \Gamma(2\lambda)}{\Gamma(n + 2\lambda)} P_n^{(\lambda)}(\cos \theta)\, P_n^{(\lambda)}(\cos \theta') ,$$

where

$$\omega = \phi - \phi',$$

the series (1.1) reduces to the ultraspherical series of the function

$$f(\theta, \phi) \equiv f(\cos\theta) = f(x)$$

and the end points of the linear interval $[-1, +1]$.

In a recent paper GUPTA [2] has discussed the absolute Abel summability of the series (1.1) at a point on the surface of the sphere. His first theorem is an independent result, while the second theorem includes both the theorems of BHATT [1] on the summability $|A|$ of Laplace series as a particular case for $\lambda = \frac{1}{2}$. In a subsequent paper [3] Gupta and the author have studied the absolute Cesàro summability of the ultraspherical series (1.1) at a point on the surface of the sphere. The object of this paper is to establish some new results, in a different line, for the absolute Cesàro summability of the ultraspherical series (1.1).

It is assumed throughout that the function

$$f(\theta', \phi')\,[\sin^2\theta' \sin^2(\phi - \phi')]^{\lambda - 1/2} \tag{1.2}$$

is absolutely integrable (L) on the whole surface of the sphere S.

A generalised mean value of $f(\theta, \phi)$ on the sphere has been defined by KOGBETLIANTZ [4] as follows

$$f(\gamma) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2} + \lambda)}{\Gamma(\lambda)\,2\pi(\sin\gamma)^{2\lambda}} \int_{C_\gamma} \frac{f(\theta', \phi')\,\mathrm{d}S'}{[\sin^2\theta' \sin^2(\phi - \phi')]^{1/2 - \lambda}}, \tag{1.3}$$

where the integral is taken along the small circle whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is γ.

We write

$$\phi(\gamma) = \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2} + \lambda)} f(\gamma) \sin^{2\lambda - 1}\gamma,$$

$$\Phi_p(\varkappa) = \frac{1}{\Gamma(p)} \int_0^\varkappa (\varkappa - t)^{p-1}\,\phi(t)\,\mathrm{d}t,$$

$$\Phi_0(\varkappa) = \phi(\varkappa),$$

$$\phi_p(\varkappa) = \Gamma(p + 1)\,\varkappa^{-p}\,\Phi_p(\varkappa), \quad p \geqq 0;$$

and

$$\Phi_p(\varkappa) = \frac{\mathrm{d}}{\mathrm{d}\varkappa}\,\Phi_{p+1}(\varkappa), \quad -1 < p < 0.$$

The following theorems will be proved.

Theorem 1. *If $\phi(\gamma)$ is of bounded variation in the interval $[\eta, \pi]$, where*

$$\eta = \frac{\mu}{n^{\Delta}}, \quad \Delta = \frac{1}{\alpha + 3\lambda + 1},$$

μ being a large constant and

$$\Phi_1(t) = O[t^{2/\Delta - 1}] \quad \text{as} \quad t \to 0, \tag{1.4}$$

$0 < \lambda < \alpha < 1$, then the series (1.1) is absolutely summable $(c, \alpha + \lambda)$ at the point (θ, ϕ) of the sphere.

Theorem 2. *If $\phi(\gamma)$ is of bounded variation in the range $[\eta, \pi]$, where*

$$\eta = \frac{\mu}{n^{\Delta}}, \quad \Delta = \frac{\alpha}{\alpha + 3\lambda + 1},$$

μ being a large constant and

$$\Phi_\alpha(t) = O[t^{(2\alpha + \lambda + 1)/2\Delta}] \quad \text{as} \quad t \to 0,$$

$0 < \lambda < 1$; $\alpha > \lambda$, then the series (1.1) is summable $|c, \alpha + \lambda|$ at the point (θ, ϕ) on the sphere.

Theorem 3. *If $\phi(\gamma)$ is of bounded variation in the interval $[\eta, \pi]$ where*

$$\eta = \frac{\mu}{n^{\Delta}}, \quad \Delta = \frac{\lambda}{\alpha + 3\lambda + 1},$$

μ being a large constant and

$$\Phi_\alpha(t) = O[t^{(\alpha + 2\lambda + 1)/2\Delta - 1}]\, \gamma(t) \quad \text{as} \quad t \to 0,$$

then the series (1.1) is summable $|c, \alpha + \lambda|$ for $0 < \lambda < \alpha < 1$, at the point (θ, ϕ) on the sphere, provided $\gamma(t)$ is any one of the sequences:

$$\{1/(\log(1/t))^{1+\varepsilon}\}, \ldots, \{1/(\log(1/t)) \ldots (\log \ldots \log_{p-1}(1/t))(\log \ldots \log_p(1/t))^{1+\varepsilon}\},$$
$$\varepsilon > 0.$$

Even the particular cases of these theorems for Laplace series are belived to be not known before. The theorems are independent of each other under different ranges of α and λ.

2. For the proofs of the theorems we need the following lemmas:

Lemma 1. *If $S_n^k(\gamma)$ denotes the n-th Cesàro mean of order k of the series*

$$\sum (n + \lambda)\, P_n^{(\lambda)}(\cos \gamma),$$

then, for $p \geqq 0$ and $\lambda > 0$, we have

$$\frac{\mathrm{d}^p\{s_n^k(\gamma)\}}{\mathrm{d}\gamma^p} = \begin{cases} O(n^{2\lambda+p+1}), & 0 \leqq \gamma \leqq \pi, \quad k > 0; \\ O\left(\dfrac{n^{\lambda+p-k/2-1/2}}{\gamma^{\lambda+k/2+1/2}}\right), & 0 < \gamma \leqq a < \pi \end{cases} \tag{2.1}$$

The proof follows on the lines of Obrechkoff [6].

Lemma 2 [6]. *For $0 < a \leqq \gamma \leqq \pi$, we have uniformly*

$$S_n^k(\gamma) = O(n^{2\lambda-k}) + O(n^{-1}). \tag{2.2}$$

3. Proof of theorem 1. We have

$$nA_n = n(n+\lambda)\iint_S \frac{f(\theta',\phi')\,P_n^{(\lambda)}(\cos\gamma)\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi'}{[\sin^2\theta'\sin^2(\phi-\phi')]^{1/2-\lambda}} =$$

$$= n(n+\lambda)\int_0^\pi \phi(\gamma)\,P_n^{(\lambda)}(\cos\gamma)\sin\gamma\,\mathrm{d}\gamma =$$

$$= n\int_0^\pi \phi(\gamma)\left[\left\{\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}^{(\lambda)}(x)\right\}_{x=\cos\gamma} - \left\{x\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\lambda)}(x)\right\}_{x=\cos\gamma} - \lambda P_n^{(\lambda)}(\cos\gamma)\right]\sin\gamma\,\mathrm{d}\gamma =$$

$$= \int_0^\pi \phi(\gamma)\left[\left\{(n+\lambda+1)\frac{\mathrm{d}}{\mathrm{d}\gamma}P_{n+1}^{(\lambda)}(\cos\gamma)\right\} - \left\{(\cos\gamma)(n+\lambda)\frac{\mathrm{d}}{\mathrm{d}\gamma}P_n^{(\lambda)}(\cos\gamma)\right\} -\right.$$

$$- (\lambda+1)\frac{\mathrm{d}}{\mathrm{d}\gamma}P_{n+1}^{(\lambda)}(\cos\gamma) + \lambda\cos\gamma\frac{\mathrm{d}}{\mathrm{d}\gamma}P_n^{(\lambda)}(\cos\gamma) -$$

$$\left. - \lambda(n+\lambda)\,P_n^{(\lambda)}(\cos\gamma)\sin\gamma + \lambda^2 P_n^{(\lambda)}(\cos\gamma)\sin\gamma\right]\mathrm{d}\gamma\,.$$

Thus,

$$t_n^{\alpha+\lambda}(\gamma) = \left[\int_0^\pi \frac{\mathrm{d}\{s_{n+1}^{\alpha+\lambda}(\gamma)\}}{\mathrm{d}\gamma}\phi(\gamma)\,\mathrm{d}\gamma - \int_0^\pi \frac{d\{s_n^{\alpha+\lambda}(\gamma)}{\mathrm{d}\gamma}\cos\gamma\,\phi(\gamma)\,\mathrm{d}\gamma -\right. \tag{3.1}$$

$$- (\lambda+1)\int_0^\pi \sigma_{n+1}^{\alpha+\lambda}(\gamma)\,\phi(\gamma)\,\mathrm{d}\gamma + \lambda\int_0^\pi \sigma_n^{\alpha+\lambda}(\gamma)\,\phi(\gamma)\cos\gamma\,\mathrm{d}\gamma -$$

$$\left. - \lambda\int_0^\pi s_n^{\alpha+\lambda}(\gamma)\,\phi(\gamma)\sin\gamma\,\mathrm{d}\gamma + \lambda^2\int_0^\pi t_n^{\alpha+\lambda}(\gamma)\,\phi(\gamma)\sin\gamma\,\mathrm{d}\gamma\right]\frac{1}{A_n^{\alpha+\lambda}} =$$

$$= [I_1 + I_2 + I_3 + I_4 + I_5 + I^6]\,.\frac{1}{A_n^{\alpha+\lambda}}\,,$$

say, where $t_n^{\alpha+\lambda}(\gamma)$, $s_n^{\alpha+\lambda}(\gamma)$, $\sigma_n^{\alpha+\lambda}(\gamma)$ and $l_n^{\alpha+\lambda}(\gamma)$ denote the n-th Cesàro mean of order $\alpha + \lambda$ of the sequences

$$\{nA_n\}, \{(n + \lambda) P_n^{(\lambda)}(\cos \gamma)\}, \left\{\frac{d}{d\gamma} P_n^{(\lambda)}(\cos \gamma)\right\} \quad \text{and} \quad \{P_n^{(\lambda)}(\cos \gamma)\}$$

respectively.

Hence, in order to prove the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1} |t_n^{\alpha+\lambda}(\gamma)| < \infty . \tag{3.2}$$

Henceforward we shall denote

$$\frac{d^p\{s_n^{\alpha+\lambda}(\gamma)\}}{d\gamma^p} \quad \text{by} \quad s_n^{(p)}(\gamma) .$$

We consider I_1 first.

$$I_1 = \int_0^\pi s_{n+1}^{(1)}(\gamma) \phi(\gamma) \, d\gamma = \left[\int_0^\eta + \int_\eta^\delta + \int_\delta^\pi\right] = I_{1,1} + I_{1,2} + I_{1,3} , \tag{3.3}$$

say. Now,

$$I_{1,1} = \int_0^\eta s_{n+1}^{(1)}(\gamma) \phi(\gamma) \, d\gamma = \tag{3.4}$$

$$= [s_{n+1}^{(1)}(\gamma) \Phi_1(\gamma)]_0^\eta - \int_0^\eta s_{n+1}^{(2)}(\gamma) \Phi_1(\gamma) \, d\gamma = I_{1,1,1} - I_{1,1,2} ,$$

say.

$$I_{1,1,1} = [\Phi_1(\eta) s_{n+1}^{(1)}(\eta)] = O\left[\eta^{2/\Delta - 1} \cdot \frac{n^{(\lambda+1-\alpha)/2}}{\eta^{-(\alpha+3\lambda+1)/2}}\right] = O(n^{(\lambda-\alpha)/2}) \tag{3.5}$$

$$I_{1,1,2} = \int_0^\eta \Phi_1(\gamma) s_{n+1}^{(2)}(\gamma) \, d\gamma = \left[\int_0^{1/n} + \int_{1/n}^\eta\right] \Phi_1(\gamma) s_{n+1}^{(2)}(\gamma) \, d\gamma = L_1 + L_2 \tag{3.6}$$

say.

Now, we have

$$L_1 = O\left[\int_0^{1/n} |\Phi_1(\gamma)| \, . \, n^{2\lambda+2} \, d\gamma\right] = O\left[\int_0^{1/n} \gamma^{2/\Delta - 1} n^{2\lambda+2} \, d\gamma\right] = O[n^{-2(\alpha+\lambda)}] . \tag{3.7}$$

Also,

$$L_2 = O\left[n \int_{1/n}^\eta n^{(1+\lambda-\alpha)/2} \gamma^{3/2\Delta - 1} \, d\gamma\right] = O[n^{(\lambda-\alpha)/2}] . \tag{3.8}$$

On integration by parts, we have

$$(3.9)\qquad I_{1,2} = [\phi(\gamma)\, s_{n+1}^{\alpha+\lambda}(\gamma)]_\eta^\delta - \int_\eta^\delta \frac{\mathrm{d}}{\mathrm{d}\gamma}\, \phi(\gamma)\, s_{n+1}^{\alpha+\lambda}(\gamma)\, \mathrm{d}\gamma =$$

$$= O(n^{(\lambda-\alpha)/2)}) + O(n^{(\lambda-\alpha)/2)}) \int_\eta^\delta \left|\frac{\mathrm{d}}{\mathrm{d}\gamma}\, \phi(\gamma)\right| \mathrm{d}\gamma = O(n^{(\lambda-\alpha)/2}).$$

Finally, using lemma 2, we get

$$(3.10)\qquad I_{1,3} = \int_\delta^\pi s_{n+1}^{(1)}(\gamma)\, \phi(\gamma)\, \mathrm{d}\gamma =$$

$$= [\phi(\gamma)\, s_{n+1}^{\alpha+\lambda}(\gamma)]_\delta^\pi - \int_\delta^\pi \frac{\mathrm{d}}{\mathrm{d}\gamma}\, \phi(\gamma)\, s_{n+1}^{\alpha+\lambda}(\gamma)\, \mathrm{d}\gamma =$$

$$= O(n^{\lambda-\alpha}) + O\left(\frac{1}{n}\right) + O(n^{\lambda-\alpha}) \int_\delta^\pi \left|\frac{\mathrm{d}}{\mathrm{d}\gamma}\, \phi(\gamma)\right| \mathrm{d}\gamma = O(n^{\lambda-\alpha}).$$

Obviously the order estimates for I_2, I_3, I_4, I_5 and I_6 are included in the orders evaluated for I_1. Thus, combing the relations (3.1), (3.2), (3.3), (3.4) (3.5), (3.7), (3.8), (3.9), and (3.10) we see that the theorem is proved.

4. Proof of theorem 2. Proceeding as in the last section in order to prove the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1}|I_1| < \infty$$

under the hypothesis of the theorem.

We have

$$(4.1)\qquad I_1 = \int_0^\pi s_{n+1}^{(1)}(\gamma)\, \phi(\gamma)\, \mathrm{d}\gamma = \left[\int_0^\eta + \int_\eta^\delta + \int_\delta^\pi\right] = I_{1,1} + I_{1,2} + I_{1,3},$$

say. Let m be the integer such that $\alpha \leqq m < \alpha + 1$. Hence

$$(4.2)\qquad I_{1,1} = \int_0^\eta s_{n+1}^{(1)}(\gamma)\, \phi(\gamma)\, \mathrm{d}\gamma =$$

$$= [\sum_{p=1}^{m} (-1)^{p-1}\, \Phi_p(\gamma)\, s_{n+1}^{(p)}(\gamma)]_0^\eta + (-1)^m \int_0^\eta \Phi_m(\gamma)\, s_{n+1}^{(m+1)}(\gamma)\, \mathrm{d}\gamma =$$

$$= i_1 + (-1)^m\, i_2,$$

say. Now, using lemma 1, we have

$$[\Phi_m(\gamma)\, s_{n+1}^{(m)}(\gamma)]_0^\eta = O[\gamma^{(\alpha+\lambda+1)/2\Delta} n^{m+(\lambda-\alpha-1)/2}] = $$
$$= O(n^{m+(\lambda-\alpha-1)/2})\, \eta^{(\alpha+\lambda+1)/2\Delta} .$$

Thus

$$i_1 = O(n^{m-\alpha-1}) . \tag{4.3}$$

When α is not an integer, we have

$$i_2 = \int_0^\eta \Phi_m(\gamma)\, s_{n+1}^{(m+1)}(\gamma)\, d\gamma =$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_0^\eta s_{n+1}^{(m+1)}(\gamma)\, d\gamma \int_0^\gamma (\gamma - u)^{m-\alpha-1}\, \Phi_\alpha(u)\, du =$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_0^\eta \Phi_\alpha(u)\, du \int_u^\eta (\gamma - u)^{m-\alpha-1} s_{n+1}^{(m+1)}(\gamma)\, d\gamma =$$
$$= \int_0^\eta \Phi_\alpha(u)\, F(\eta, u)\, du ,$$

by Fubini's theorem, since $\Phi_\alpha(u) = O(u^\alpha)$, $m > \alpha$.
The orders of $F(\eta, u)$ can be easily seen to be

$$F(\eta, u) = \begin{cases} O(n^{2\lambda+m+1})\, u^{m-\alpha} + O(u^{m-\alpha-1} n^{2\lambda+m}) , & \text{for } 2u < \eta ; \\ O(n^{(\alpha+\lambda+1)/2} u^{-\alpha/2\Delta}) , & u + \dfrac{1}{n} < \eta . \end{cases}$$

Thus, we get

$$i_2 = \left[\int_0^{1/n} + \int_{1/n}^\eta\right] \Phi_\alpha(u)\, F(\eta, u)\, du = i_{2,1} + i_{2,2} , \tag{4.4}$$

say.

$$i_{2,1} = O\left[\int_0^{1/n} u^{(2\alpha+\lambda+1)/2\Delta} u^{m-\alpha} n^{2\lambda+m+1}\, du\right] + \tag{4.5}$$
$$+ O\left[\int_0^{1/n} u^{(2\alpha+\lambda+1)/2\Delta} u^{m-\alpha-1} n^{2\lambda+m}\, du\right] =$$
$$= O[n^{2\lambda+\alpha}]\, n^{-(2\alpha+\lambda+1)/2\Delta} = O[n^{-\alpha-1}] ,$$

$$i_{2,2} = \int_{1/n}^\eta \Phi_\alpha(u)\, F(\eta, u)\, du = O\left[n^{(\alpha+\lambda+1)/2} \int_{1/n}^\eta u^{(2\alpha+\lambda+1)/2\Delta}\right.$$
$$\left. .\, u^{-(\alpha+3\lambda+1)/2}\, du\right] = O\left[n^{(\alpha+\lambda+1)/2} \int_{1/n}^\eta u^{(\alpha+\lambda+1)/2\Delta}\, du\right] =$$
$$= O[n^{(\alpha+\lambda+1)/2} \eta^{(\alpha+\lambda+1)/2\Delta+1}] = O(n^{-\Delta}) .$$

$I_{1,2}$ and $I_{1,3}$ can be disposed off in exactly the same way as in the last section and we have

$$(4.6) \qquad I_{1,2} = O(n^{(\lambda-\alpha)/2}),$$

and

$$(4.7) \qquad I_{1,3} = O(n^{\lambda-\alpha}).$$

When $\alpha = m$, an integer, we have

$$(4.8) \qquad i_2 = \int_0^\eta \Phi_\alpha(\gamma)\, s_{n+1}^{(\alpha+1)}(\gamma)\, d\gamma = \int_0^{1/n} + \int_{1/n}^\eta = i_{2,1} + i_{2,2},$$

say. Now

$$(4.9) \qquad i_{2,1} = O\left[\int_0^{1/n} \gamma^{(2\alpha+\lambda+1)/2\Delta} n^{2\lambda+\alpha+1}\right] = O[n^{-\alpha-1}].$$

Also

$$(4.10) \qquad i_{2,2,} = \int_{1/n}^\eta \Phi_\alpha(\gamma)\, s_{n+1}^{(\alpha+1)}(\gamma)\, d\gamma = O\left[\int_{1/n}^\eta \gamma^{(\alpha+\lambda+1)/2\Delta} n^{(\alpha+\lambda+1)/2}\, d\gamma\right] =$$
$$= O[n^{(\alpha+\lambda+1)/2} \eta^{(\alpha+\lambda+1)/2\Delta+1}] = O(n^{-\Delta}).$$

This complete the proof of the theorem.

5. Proof of theorem 3. For the proof of the theorem it is sufficient to show that the series.

$$\sum_{n=1}^{\infty} n^{-1}|I_1|$$

is convergent.

We now consider I_1.

$$(5.1) \qquad I_1 = \int_0^\pi \phi(\gamma)\, s_{n+1}^{(1)}(\gamma)\, d\gamma = \int_0^\eta + \int_\eta^\delta + \int_\delta^\pi = J_1 + J_2 + J_3,$$

say. On integration by parts, we have

$$(5.2) \qquad J_3 = [\phi(\gamma)\, s_{n+1}^{\alpha+\lambda}(\gamma)] - \int_\delta^\pi \frac{d}{d\gamma}\phi(\gamma)\, s_{n+1}^{\alpha+\lambda}(\gamma)\, d\gamma = O(n^{\lambda-\alpha}).$$

Now

$$(5.3) \quad J_1 = \int_0^\eta \phi(\gamma)\, s_{n+1}^{(1)}(\gamma)\, d\gamma = \left[\Phi_1(\gamma)\frac{d}{d\gamma} s_{n+1}^{\alpha+\lambda}(\gamma)\right]_0^\eta + \int_0^\eta \Phi_1(\gamma)\, s_{n+1}^{(2)}(\gamma)\, d\gamma =$$
$$= J_{1,1} + J_{1,2},$$

say. From Lemma 1, we observe that

$$\left[\Phi_1(\gamma)\frac{d}{d\gamma}s_{n+1}^{\alpha+\lambda}(\gamma)\right] = O(n^{\lambda-\alpha}).$$

Hence

$$J_{1,1} = O(n^{\lambda-\alpha}). \tag{5.4}$$

Also, following Obrechkoff [6], we have

$$J_{1,2} = \int_0^\eta \Phi_1(\gamma)\, s_{n+1}^{(2)}(\gamma)\, d\gamma = \frac{1}{\Gamma(1-\alpha)}\int_0^\eta \Phi_\alpha(u)\left[\int_u^\eta (\gamma-u)^{-\alpha}\, s_{n+1}^{(2)}(\gamma)\, d\gamma\right] du = \tag{5.5}$$

$$= \left[\int_0^{1/n} + \int_{1/n}^\eta\right]\Phi_\alpha(u)\, F(\eta, u)\, du = J_{1,2,1} + J_{1,2,2},$$

say.

$$J_{1,2,1} = O\left[n^{2\lambda+2}n^{\alpha-1}n^{-(2\lambda+\alpha+1)/2\Delta}\,\gamma\left(\frac{1}{n}\right)\right] = \tag{5.6}$$

$$= O\left[n^{2\lambda+\alpha+1}n^{-(2\lambda+\alpha+1)/2\Delta}\,\gamma\left(\frac{1}{n}\right)\right] =$$

$$= O[n^{-(\alpha+1)/2\Delta}],$$

$$J_{1,2,2} = \int_{1/n}^\eta \Phi_\alpha(u)\, F(\eta, u)\, du = O\left[\int_{1/n}^\eta n^{(\alpha+\lambda+1)/2}u^{-(\alpha+3\lambda+1)/2}\,. \right. \tag{5.7}$$

$$\left. .\, u^{(2\lambda+\alpha+1)/2\Delta-1}\,\gamma(u)\, du\right] =$$

$$= O\left[n^{(\alpha+\lambda+1)/2}\int_{1/n}^\eta u^{(\alpha+\lambda+1)/2\Delta-1}\,\gamma(u)\, du = O[\gamma(\eta)]\right..$$

Also

$$J_2 = O(n^{(\lambda-\alpha)/2}). \tag{5.8}$$

In view of the relation (5,1), (5,2), ..., (5,8) the theorem is completely established.

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