Czechoslovak Mathematical Journal

Ján Jakubík

Weak product decompositions of discrete lattices

Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 3, 399–412

Persistent URL: http://dml.cz/dmlcz/101040

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

WEAK PRODUCT DECOMPOSITIONS OF DISCRETE LATTICES

JÁN JAKUBÍK, Košice

(Received March 24, 1970)

INTRODUCTION

Let $(A_i)_{i\in I}$ be a family of algebras. A weak product of $(A_i)_{i\in I}$ is a subalgebra B of the complete direct product $A = \Pi_{i\in I} A_i$ satisfying the following conditions: (1) two elements in B differ only in a finite number of components; and (2) if an element $a \in A$ differs only in a finite number of components from an element $b \in B$, then $a \in B$ (cf. GRÄTZER [2]). In [6] there were investigated weak product decompositions of universal algebras with pairwise permutable congruence relations. The aim of this Note is to prove that any discrete lattice is a weak product of directly indecomposable factors. This result is then applied for studying isomorphisms of unoriented graphs of modular lattices; there is obtained a generalization of a theorem of [4].

In §1 there is defined the concept of a full subdirect product of lattices and it is proved that any two full subdirect decompositions of a lattice L have a common refinement. In §2 it is shown that any full subdirect decomposition of a discrete lattice L is a weak product decomposition of L and there is constructed the (uniquelly determined) weak product decomposition of a discrete lattice L in which all factors are directly indecomposable. The isomorphisms of graphs of discrete modular lattices are studied in §3.

The notions of the weak product and of the full subdirect product can be defined for relational systems as well; in a forthcomming paper weak products of partially ordered sets will be investigated.

1. FULL SUBDIRECT DECOMPOSITIONS

The symbols $\wedge$, $\vee$ or $\cap$, $\cup$ denote lattice operations and set-theoretical operations, respectively. $A \setminus B$ is the set of all elements of A that do not belong to B. If L is a lattice, $a, b \in L$, $a \leqq b$, then the interval $[a, b]$ is the set of all $x \in L$ with the property $a \leqq x \leqq b$. The interval $[a, b]$ is prime, if card $[a, b] = 2$. A lattice L is said to be discrete, if all bounded chains in L are finite.

Let $\{S_i : i \in I\}$ be a system of lattices. The complete direct product $S = \Pi S_i(i \in I)$ is the set of all mappings $f : I \to \bigcup S_i$ such that $f(i) \in S_i$ for each $i \in I$ with the partial order defined component-wise (i.e., $f \leqq g$ if $f(i) \leqq g(i)$ for each $i \in I$). When $I = \{1, \ldots, n\}$, then S is denoted also by $S = S_1 \times \ldots \times S_n$. $f(i)$ is the i-th component of the element f.

Assume that L is a lattice and that there is an isomorphism φ of L into S. Let x_0 be a fixed element of L, $i \in I$. Denote

$$A_i(x_0) = \{x \in L : \varphi(x)(j) = \varphi(x_0)(j) \text{ for each } j \in I,\ j \neq i\}\,,$$
$$A_i^*(x_0) = \{x \in L : \varphi(x)(i) = \varphi(x_0)(i)\}\,.$$

Clearly $A_i(x_0)$ is a convex sublattice of L (the convexity means that for any $x \in L$ and $a_1, a_2 \in A_i(x_0)$, $a_1 \leqq x \leqq a_2$ implies $x \in A_i(x)$). Analogously, $A_i^*(x_0)$ is a convex sublattice of L and

$$(1) \qquad \operatorname{card}\,[A_i(x_0) \cap A_i^*(x)] \leqq 1$$

for any $x \in L$. The isomorphism φ is said to determine a full subdirect decomposition of L, whenever the following conditions (a) and (b) are satisfied:

(a) for any $i \in I$ and any $a^i \in A_i$ there is $x \in L$ such that $\varphi(x)(i) = a^i$;

(b) for any $i \in I$ and any $x, y \in L$ there exists $z \in L$ such that

$$\varphi(z)(i) = \varphi(x)(i)\,,$$
$$\varphi(z)(j) = \varphi(y)(j) \quad \text{for any} \quad j \in I\,, \quad j \neq i\,.$$

In the whole §1 we assume that φ satisfies (a) and (b). Obviously the element z satisfying (b) belongs to the set $A_i^*(x) \cap A_i(y)$ and conversely, if z belongs to this set, then z fulfils (b). Hence, according to (1), the condition (b) is equivalent to

(b′) $\operatorname{card}\,[A_i(x_0) \cap A_i^*(x)] = 1$ for any $i \in I$, $x_0 \in L$ and any $x \in L$.

The elements of the one-element sets $A_i(x_0) \cap A_i^*(x)$ and $A_i(x) \cap A_i^*(x_0)$ will be denoted by $x(A_i(x_0))$ and $x(A_i^*(x_0))$, respectively. It is easy to verify that the following assertion holds true:

1.1. *Let $i \in I$. If $x \in A_i(x_0)$ $(x \in A_j(x_0), j \neq i)$, then $x(A_i(x_0)) = x$ $(x(A_i(x_0)) = x_0)$. If $x, y \in L$, $x \leqq y$, then $x(A_i(x_0)) \leqq y(A_i(x_0))$. The mapping $\psi : L \to \Pi A_i(x_0) = S'$ defined by $\psi(x)(i) = x(A_i(x_0))$ is an isomorphism of L into S'.*

For the sake of brevity denote $A_i(x_0) = A_i^0$.

Suppose that there is given another isomorphism φ' of L into ΠB_k $(k \in K)$ determining a full subdirect decomposition of L. By analogical denotations let us put $B_i(x_0) = B_i^0$.

1.2. *Let $x \in L$, $x \geqq x_0$. Then $x = \bigvee x(A_i^0)\,(i \in I)$.*

Proof. Let $i \in I$, $x(A_i^0) = y$. According to 1.1 $y(A_i^0) = y = x(A_i^0)$ and for any $j \in I, j \neq i$ we have $y(A_j^0) = x_0 = x_0(A_j^0) \leqq x(A_j^0)$, hence $y \leqq x$. Let $z \in L$, $z \geqq x(A_i^0)$ for each $i \in I$. Then $z(A_i^0) \geqq (x(A_i^0))\,(A_i^0) = x(A_i^0)$ for each $i \in I$, thus $z \geqq x$.

1.3. *Let $i \in I$, $k \in K$, $x \in L$, $x \geqq x_0$. Then $x(A_i^0)\,(B_k^0) = x(A_i^0) \wedge x(B_k^0)$.*

Proof. Denote $x(A_i^0) = u$, $x(B_k^0) = v$, $x(A_i^0)\,(B_k^0) = t$. Obviously $x_0 \leqq u \wedge v = w$, whence $w \in [x_0, u] \cap [x_0, v] \subset A_i^0 \cap B_k^0$. Consider the components of elements t, w with respect to B_k^0 and B_l^0 ($l \in K$, $l \neq k$); we get

$$w(B_k^0) \leqq u(B_k^0) = t = t(B_k^0)\,,$$
$$w(B_l^0) = x_0 = t(B_l^0)\,;$$

thus $w = t$.

As a corollary, we obtain:

1.4. *Let $i \in I$, $k \in K$, $x \in L$, $x \geqq x_0$. Then $x(A_i^0)\,(B_k^0) = x(B_k^0)\,(A_i^0)$.*

In a dual way we can prove the assertions of the lemma 1.3 for the case $x \leqq x_0$.

1.5. *Let $i \in I$, $k \in K$, $x \in L$. Then $x(A_i^0)\,(B_k^0) \in A_i^0$.*

Proof. Put $u = x_0 \wedge x$, $v = x_0 \vee x$. According to 1.3 we have $v(A_i^0)\,(B_k^0) \in [x_0, v(A_i^0)] \subset A_i^0$. Analogously, the dual of 1.3 gives $u(A_i^0)\,(B_k^0) \in A_i^0$. Since $x(A_i^0)\,(B_k^0) \in [u(A_i^0)\,(B_k^0), v(A_i^0)\,(B_k^0)]$, from the convexity of A_i^0 it follows $x(A_i^0)\,.\,(B_k^0) \in A_i^0$.

For each A_i there correspond two congruence relations $R(A_i) = R_i$ and $R'(A_i) = R'_i$ defined as follows:

If $x, y \in L$ and $i \in I$, then we set $x \equiv y(R_i)$ $(x \equiv y(R'_i))$ if $x \in A_i(y)$ $(x \in A_i^*(y))$. For $k \in K$ let R_k, R'_k have an analogical meaning. R_i and R'_i are permutable and $R_i \wedge R'_i$ is the least congruence relation on L. Let $x_1 \in L$ and denote $A_i^1 = A_i(x_1)$, $B_k^1 = B_k(x_1)$. Then for any $z \in L$

$$z \equiv z(A_i^0) \equiv z(A_i^1)\,(R'_i)\,, \quad z(B_k^0) \equiv z(B_k^1)\,(R'_k)\,. \tag{2}$$

1.6. *Let $i \in I$, $k \in K$, $x \in L$. Then $x(A_i^0)\,(B_k^0) = x(B_k^0)\,(A_i^0)$.*

Proof. Put $x_1 = x \wedge x_0$ and denote

$$u = x(A_i^0)\,(B_k^0)\,, \quad v = x(B_k^0)\,(A_i^0)\,,$$
$$u_1 = x(A_i^1)\,(B_k^1)\,, \quad v_1 = x(B_k^1)\,(A_i^1)\,.$$

According to 1.5 $u, v \in A_i^0 \cap B_k^0$, hence $u \equiv v(R_i \wedge R_k)$. By (2)

$$u \equiv x(A_i^0)(R_k'), \quad x(A_i^0) \equiv x(A_i^1)(R_i'), \quad x(A_i^1) \equiv u_1(R_k'),$$

thus $u \equiv u_1(R_i' \vee R_k')$ and analogously $v \equiv v_1(R_i' \vee R_k')$. From 1.4 we get $u_1 = v_1$, therefore $u \equiv v(R_i' \vee R_k')$. This implies $u \wedge v \equiv u \vee v(R_i' \vee R_k')$, hence there are elements $u \wedge v = t_0 \leqq t_1 \leqq \ldots \leqq t_n = u \vee v$ such that for each $m = 1, \ldots, n$ either $t_{m-1} \equiv t_m(R_i')$ or $t_{m-1} \equiv t_m(R_k')$. In the same time $t_{m-1} \equiv t_m(R_i \wedge R_k)$, whence $t_{m-1} = t_m$; therefore $u = v$.

1.6.1. Remark. The assertion of the lemma 1.6 could be deduced also from [3], Theorem 1, where a more general situation (concerning connected partially ordered sets) is dealt with; in the case of lattices, the present proof seems to be simpler.

For any $i \in I$, $k \in K$ denote $A_i^0 \cap B_k^0 = C_{ik}^0$ and let χ be a mapping of L into ΠC_{ik}^0 $(i \in I, k \in K)$, such that $\chi(x)(i, j) = x(A_i^0)(B_j^0)$ for any $x \in L$.

1.7. *χ is an isomorphism of the lattice L into ΠC_{ik}^0 $(i \in I, k \in K)$.*

Proof. Since the mappings $x \to x(A_i^0)$, $x \to x(B_k^0)$ are homomorphisms, χ is a homomorphism as well. We have to verify that χ is one-to-one. Let $x, y \in L$, $\chi(x) = \chi(y)$, $i \in I$. Then $x(A_i^0)(B_k^0) = y(A_i^0)(B_k^0)$ for each $k \in K$, thus $x(A_i^0) = y(A_i^0)$. Since this holds for each $i \in I$, we get $x = y$.

1.8. *Let $x, y \in L$, $i \in I$, $k \in K$. There exists $z \in L$ such that*

$$z(C_{ik}^0) = x(C_{ik}^0),$$
$$z(C_{jl}^0) = y(C_{jl}^0) \quad \textit{for each} \quad (j, l) \in I \times K, \quad (j, l) \neq (i, k).$$

Proof. Let us consider at first the elements $x(A_i^0)(B_k^0)$ and $y(A_i^0)$. Since the mapping φ' satisfies (b), there exists $t \in L$ such that

$$t(B_k^0) = [x(A_i^0)(B_k^0)](B_k^0) = x(A_i^0)(B_k^0),$$
$$t(B_l^0) = y(A_i^0)(B_l^0) \quad \text{for each} \quad l \in K, \quad l \neq k.$$

Further consider the pair t, y. Since φ satisfies (b), there is $z \in L$ such that

$$z(A_i^0) = t(A_i^0),$$
$$z(A_j^0) = y(A_j^0) \quad \text{for each} \quad j \in I, \quad j \neq i.$$

The element z satisfies

$$z(A_i^0)(B_k^0) = t(A_i^0)(B_k^0) = t(B_k^0)(A_i^0) = x(A_i^0)(B_k^0)(A_i^0) = x(A_i^0)(B_k^0).$$

For any $l \in K$, $l \neq k$

$$z(A_i^0)(B_l^0) = t(A_i^0)(B_l^0) = t(B_l^0)(A_i^0) = y(A_i^0)(B_l^0),$$

and for any $j \in I$, $j \neq i$ and any $s \in K$

$$z(A_j^0)(B_s^0) = y(A_j^0)(B_s^0).$$

The proof is complete.

For any $x \in C_{ik}^0$ we have $x(C_{ik}^0) = x(A_i^0)(B_k^0) = x$. From this and from 1.7 and 1.8 it follows:

1.9. *The isomorphism χ determines a full subdirect decomposition of the lattice L.*

The full subdirect decomposition of L with factors C_{ik}^0 determined by χ is a refinement of both full subdirect decompositions with factors $A_i^0(i \in I)$ and $B_k^0(k \in K)$, respectively, in the following sense:

1.10. *Let $i \in I$. Then A_i^0 is a full subdirect product of lattices C_{ik}^0 (the isomorphism of A_i^0 into $\Pi C_{ik}^0(k \in K)$ being determined by the partial mapping $\chi_{A_i^0}$).*

Proof. It suffices to verify that the mapping χ possesses the property (b). Let $x, y \in A_i^0$ and let z have the same meaning as in 1.8. We have to show that z belongs to A_i^0. Denote $u = x \wedge y \wedge x_0$, $v = x \vee y \vee x_0$. Then all elements that were used in the proof of 1.8 belong to the interval $[u, v]$ and $[u, v] \subset A_i^0$; therefore $z \in A_i^0$.

We shall need also the following simple lemma.

1.12. *Let $x, y, z \in L$, $x \leqq y \leqq z$, $y \in A_i(x)$, $y \in A_i^*(z)$. Then there exists $v \in L$ such that $v \in A_i^*(x)$, $v \in A_i(z)$ and v is a relative complement of the element y in the interval $[x, z]$.*

Proof. Let R_i and R_i' have the same meaning as above. Then $x \equiv y(R_i)$, $y \equiv z(R_i')$. Since R_i and R_i' are permutable, there is $v_1 \in L$ with the property $x \equiv v_1(R_i')$, $v_1 \equiv z(R_i)$. Denote $v = (v_1 \wedge z) \vee x$. We have $x \equiv v(R_i')$, $v \equiv z(R_i)$ and $x \leqq v \leqq z$. It remains to prove that v is a relative complement of y in $[x, z]$. From $x \equiv v(R_i')$ we get $x \equiv y \wedge v(R_i')$ and analogously from $x \equiv y(R_i)$ we infer that $x \equiv y \wedge v(R_i)$, therefore $x \equiv y \wedge v(R_i \wedge R_i')$. Since $R_i \wedge R_i'$ is the least congruence relation on L, we have $x = y \wedge v$. By a dual argument, $z = y \vee v$.

1.13. Remark. The concept of a full subdirect product can be applied in the obvious way for universal algebras. Let L be a subalgebra of the complete direct product ΠA_i of universal algebras A_i $(i \in I)$ such that the conditions (a) and (b) are fulfilled; then L is said to be a full subdirect product of algebras A_i. In the case of lattice ordered groups the concept of the weak product (full subdirect product) coincides with the restricted direct product (completely subdirect product [5]) of l-groups A_i.

2. WEAK PRODUCTS

For the definition of a weak product of universal algebras cf. [2] and the Introduction. We restrict ourselves to the case of lattices; then the definition of this concept is as follows:

Let $S = \Pi A_i \, (i \in I)$ be the complete direct product of lattices A_i and let S_1 be a sublattice of S satisfying the following conditions:

(i) if $x, y \in S_1$, then the set $\{i \in I : x(i) \neq y(i)\}$ is finite;

(ii) if $x \in S_1$, $z \in S$ and if the set $\{i \in I : x(i) \neq z(i)\}$ is finite, then $z \in S_1$.

Under these assumptions S_1 is said to be a weak product of lattices A_i. It is easy to verify that any weak product of lattices A_i is a full subdirect product of these lattices. If the set I is finite, then the concepts of the weak product, full subdirect product and complete direct product coincide.

2.1. *Let L be a discrete lattice that is a full subdirect product of lattices $A_i \, (i \in I)$. Then L is a weak product of these lattices.*

Proof. Let $x, x_0 \in L$, $I_1 = \{i \in I : x(i) \neq x_0(i)\}$. Further denote $y = x \wedge x_0$, $I_2 = \{i \in I : x(i) \neq y(i)\}$. Assume that the set I_2 is infinite; then there exist distinct elements $i_1, i_2, i_3, \ldots \in I_2$. According to (b) for each i_k there is $x_k \in L$ such that

$$x_k(i_k) = x(i_k), \quad x_k(i) = y(i) \quad \text{for each} \quad i \in I, \quad i \neq i_k .$$

Put $z_n = x_1 \vee x_2 \vee \ldots \vee x_n \; (n = 1, 2, \ldots)$. Then $y < z_1 < z_2 < \ldots < x$ and this is impossible, since L is discrete. Thus I_2 is finite. Analogously, the set $I_3 = \{i \in I : x_0(i) \neq y(i)\}$ is finite and therefore the set I_1 is finite as well and so the condition (i) is fulfilled. Let $x \in L$, $z \in \Pi A_i \; (i \in I)$ and suppose that the set $\{i \in I : x(i) \neq z(i)\}$ contains only one element i_1. According to (a) there is $y \in L$ such that $y(i_1) = z(i_1)$. Further it follows from (b) that there is $t \in L$ satisfying $t(i_1) = y(i_1)$, $t(i) = x(i)$ for each $i \in I$, $i \neq i_1$. Clearly $t = z$, thus $z \in L$. From this we get by induction that (ii) holds.

Remark. Simple examples show that the assertion of the lemma 2.1 need not hold for non-discrete lattices.

If a lattice L is a full subdirect product of lattices A_i and $x_0 \in L$, then we shall write $L = (fs) \, \Pi A_i(x_0)$.

Let L be a lattice. For any $x_0 \in L$ let $F(x_0)$ be the system of all sublattices A of L such that there exists a full subdirect decomposition

$$(3) \qquad L = (fs) \, \Pi A_i(x_0) \, (i \in I)$$

with the property $A = A_{i_1}(x_0)$ for some $i_1 \in I$.

2.2. *Let* $L = (fs)\,\Pi A_i(x_0)\,(i \in I)$, $L = (fs)\,\Pi B_k(x_0)\,(k \in K)$, $x \in L$, $i \in I$, $k \in K$, $A_i(x_0) = B_k(x_0)$. *Then* $x(A_i(x_0)) = x(B_k(x_0))$ *(i.e., the component of* x *in any* $A \in F(x_0)$ *is uniquely determined by* A*).*

Proof. Denote $x(A_i(x_0)) = y$, $x(B_k(x_0)) = z$. According to 1.6 $y(B_k(x_0)) = z(A_i(x_0))$. Since $y \in A_i(x_0) = B_k(x_0)$ and analogously $z \in A_i(x_0)$, we have $y(B_k(x_0)) = y$, $z(A_i(x_0)) = z$; thus $y = z$.

Let L be a lattice, $x_0 \in L$. The system of all prime intervals of L will be denoted by $\mathscr{P}$. Let us recall that in §1 we have shown that the following assertion is valid (cf. 1.7 and 1.8):

2.3. *If* $A, B \in F(x_0)$, *then* $C = A \cap B \in F(x_0)$ *and for any* $x \in L$, $x(C) = x(A)(B)$.

2.4. *Let* $[u, v] \in \mathscr{P}$, $A(x_0)$, $C(x_0) \in F(x_0)$ *and assume that*

$$u(A(x_0)) < v(A(x_0)), \quad u(C(x_0)) < v(C(x_0)).$$

Then $u(A(x_0))(C(x_0)) < v(A(x_0))(C(x_0))$.

Proof. For any $x_1 \in L$ we have

$$u(A(x_0)) < v(A(x_0)) \Leftrightarrow u(A(x_1)) < v(A(x_1)),$$

hence it suffices to prove our statement for the case $x_0 = u$. Under this assumption

$$u(A(x_0)) = u < v(A(x_0)) \leqq v$$

and thus (since $[u, v] \in \mathscr{P}$) $v(A(x_0)) = v$. Analogously $v(C(x_0)) = v$. Therefore

$$u(A(x_0))(C(x_0)) = u < v = v(A(x_0)(C(x_0)).$$

If $p = [u, v] \in \mathscr{P}$, $A \in F(x_0)$ and $u(A) < v(A)$, then A is said to be parallel to p. We denote by $F(x_0, p)$ the system of all $A \in F(x_0)$ that are parallel to p.

With respect to 2.3 the Lemma 2.4 can be formulated as follows:

2.4'. *If* $A(x_0)$, $B(x_0) \in F(x_0, p)$, *then* $A(x_0) \cap B(x_0) \in F(x_0, p)$.

Let us now suppose (in the whole §2) that L is a discrete lattice, card $L > 1$. Then L is conditionally complete. Moreover, L is compact in the following sense: if $u, v \in L$, $u \leqq v$ and $\{x_\alpha\} \subset [u, v]$, $\bigvee x_\alpha = v$, then there exists a finite subset $\{x_1, \ldots, x_n\} \subset \{x_\alpha\}$ such that $x_1 \vee \ldots \vee x_n = v$ and dually.

2.5. *Let* $p \in \mathscr{P}$. *For each* $x \in L$ *there is* $A^x(x_0) \in F(x_0, p)$ *with the property*

(*) *if* $A_i \in F(x_0, p)$, $A_i \subset A^x(x_0)$, *then* $x(A_i) = x(A^x(x_0))$.

The proof will consist of three steps.

(I) Let $x_1, x_2 \in L$, $x_1 \leqq x_2$. Let $R_i = R(A_i)$ and $R'_i = R'(A_i)$ have the same meaning as in §1. Let P be the system of all elements $y \in [x_1, x_2]$ such that

$$x_1 \equiv y(R(A_i)) \quad \text{for each} \quad A_i \in F(x_1, p).$$

Further let Q be the set of all elements $z \in [x_1, x_2]$ such that

$$x_1 \equiv z(R'(A_i)) \quad \text{for some} \quad A_i \in F(x_1, p).$$

Our aim now is to show that the sets P and Q have greatest elements (these will by denoted by p_0 and q_0. respectively) and that $p_0 \vee q_0 = x_2$.

Clearly P is a convex sublattice of L and if $x_1 \leqq t \leqq z \in Q$, then $t \in Q$. Let $z_1, z_2 \in \in Q$, hence $x_1 \equiv z_k(R'(A_{i_k}))$, $k = 1, 2$ for some $A_{i_1}, A_{i_2} \in F(x_1, p)$. In such case according to 2.4′ $A = A_{i_1} \cap A_{i_2} \in F(x_1, p)$, $R'(A) \geqq R'(A_{i_1})$, $R'(A) \geqq R'(A_{i_2})$. Hence $x_1 \equiv z_1 \vee z_2(R'(A))$, thus $z_1 \vee z_2 \in Q$. Therefore Q is a convex sublattice of L, too. Since L is conditionally complete and compact, P has a greatest element p_0 and analogously Q possesses a greatest element q_0. There is $A_{i_0} \in F(x_1, p)$ such that $x_1 \equiv \equiv q_0(R'(A_{i_0}))$ and clearly $x_1 \equiv p_0(R(A_{i_0}))$. Denote $p_0 \vee q_0 = v$ and assume that $v < x_2$. Let $v_1 \in L$, $v < v_1 \leqq x_2$ such that $[v, v_1]$ is a prime interval. Suppose, at first, that there exists $A_i \in F(x_1, p)$ such that v is not congruent to v_1 mod $R(A_i)$, thus $v \equiv v_1(R'(A_i))$. We have $q_0 \leqq v < v_1$, $q_0 \equiv v(R(A_i))$, $v \equiv v_1(R'(A_i))$ and thus according to 1.12 there is $t \in [q_0, v_1]$ such that $t \wedge v = q_0$, $t \vee v = v_1$, $q_0 \equiv \equiv t(R'(A_i))$. Clearly $q_0 < t$. Put $A_{i_0} \cap A_i = A$. According to 2.4′ $A \in F(x_1, p)$. Then $R'(A) \geqq R'(A_{i_0})$, $R'(A) \geqq R(A_i)$, therefore $x_1 \equiv t(R'(A))$ and so $t \in Q$; this is a contradiction with the maximality of q_0 in Q. This shows that we must have $v \equiv \equiv v_1(R(A_i))$ for each $A_i \in F(x_1, p)$. In particular, $v \equiv v_1(R(A_{i_0}))$. Clearly $p_0 \equiv \equiv v(R'(A_{i_0}))$. According to 1.12 there is $t \in L$ such that $t \wedge v = p_0$, $t \vee v = v_1$. The intervals $[p_0, t]$ and $[v, v_1]$ are projective, thus $p_0 \equiv t(R(A_i))$ for each $A_i \in \in F(x_1, p)$ and $p_0 < t$. But then $x_1 \equiv t(R(A_i))$ for each $A_i \in F(x_1, p)$, whence $t \in P$ and this is not possible, since p_0 is the greatest element of P. We have proved that $p_0 \vee q_0 = x_2$.

(II) Now let $x, x_0 \in L$, $x_1 = x \wedge x_0$, $x_2 = x \vee x_0$ and denote $p_1 = x \wedge p_0$, $q_1 = x \wedge q_0$. From (I) it follows that p_1 is the greatest element of the interval $[x_1, x]$ with the property that $x_1 \equiv p_1(R(A_i))$ for each $A_i \in F(x_1, p)$ and analogously q_1 is the greatest element of $[x_1, x]$ with the property $x_1 \equiv q_1(R'(A_i))$ for some $A_i \in \in F(x_1, p)$. Further according to (I) $x = p_1 \vee q_1$. Let $p_2 = x_0 \wedge p_0$, $q_2 = x_0 \wedge q_0$. For the elements x_0, p_2, q_2 we can obtain results analogical to those just proved for x, p_1, q_1; so we have $x_0 = p_2 \vee q_2$. Let A_{i_0} have the same meaning as in (I). Since $q_1, q_2 \in [x_1, q_0]$, $p_1, p_2 \in [x_1, p_0]$, the relations $q_1 \equiv q_2(R'(A_{i_0}))$, $p_1 \equiv p_2(R(A_i))$ for any $A_i \in F(x_0, p)$ are valid.

(III) Under the same denotations as in (I) and (II) put $x^* = p_1 \vee q_2$. Let $A_i \in \in F(x_0, p)$, $A_i \subset A_{i_0}$. Then

$$x = p_1 \vee q_1 \equiv p_1 \vee q_2 = x^*(R'(A_{i_0})).$$

Since $R'(A_{i_0}) \leqq R'(A_i)$,

(4) $$x \equiv x^*(R'(A_i)) .$$

Further we have

(4') $$x_0 = p_2 \vee q_2 \equiv p_1 \vee q_2(R(A_i)) .$$

From (4) and (4') it follows $x(A_i) = x^* = x(A_{i_0})$ for each $A_i \in F(x_0, p)$, $A_i \subset A_{i_0}$. We denote $A_{i_0} = A^x(x_0)$; the proof of the assertion 2.5 is complete.

Now let us denote $A^p = \{x(A^x(x_0)) : x \in L\}$.

2.6. *The set A^p is a sublattice of L and the mapping $\varphi_p : x \to x(A^x(x_0))$ is a homomorphism of the lattice L onto A^p. For any $x \in A^p$, $\varphi_p(x) = x$.*

Proof. Let $x_1, y_1 \in A^p$. There are elements $x, y \in L$ such that $\varphi_p(x) = x_1$, $\varphi_p(y) = y_1$. Then we have according to 2.4' and 2.5

$$A = A^x(x_0) \cap A^y(x_0) \cap A^{x \wedge y}(x_0) \in F(x_0, p) ,$$
$$x(A) = x_1 , \quad y(A) = y_1 , \quad (x \wedge y)(A) = (x \wedge y)(A^{x \wedge y}(x_0)) ,$$

thus $x_1 \wedge y_1 = (x \wedge y)(A^{x \wedge y}(x_0)) = \varphi_p(x \wedge y)$. An analogical result holds for $x \vee y$. Hence A^p is a sublattice of L and φ_p is a homomorphism of L onto A^p. If $x_1 \in A^p$, $x_1 = x(A^x(x_0))$, then for any $A \subset A^x(x_0))$ such that $A \in F(x_0, p)$ we have

$$x_1(A) = x(A^x(x_0))(A) = x(A)(A^x(x_0)) = x(A^x(x_0))(A^x(x_0)) = x(A^x(x_0)) = x_1 .$$

For $x_0 \in L$, $A(x_0) \in F(x_0)$ let $A^*(x_0)$ have the same meaning as in §1.

2.7. *Let $x, x_0 \in L$, $A(x_0) \in F(x_0)$, $B(x_0) \in F(x_0)$, $x(A(x_0)) = x(B(x_0))$. Then $x(A^*(x_0)) = x(B^*(x_0))$.*

Proof. Put $x_1 = x \wedge x_0$ and consider the lattice $[x_1, x]$. It is isomorphic to the direct product $D_1 \times D_2$, where $D_1 = A(x_1) \cap [x_1, x]$, $D_2 = A^*(x_1) \cap [x_1, x]$. The elements $x(A(x_1))$, $x(A^*(x_1))$ belong to the centre C_0 of the lattice $[x_1, x]$ (cf. [1], p. 28) and $x(A(x_1))$ is a complement of $x(A^*(x_1))$; the same holds for $x(B(x_1))$ and $x(B^*(x_1))$. Since C_0 is a Boolean algebra and $x(A(x_0)) = x(B(x_0))$ implies $x(A(x_1)) = x(B(x_1))$, we get $x(A^*(x_1)) = x(B^*(x_1))$; from this we obtain $x(A^*(x_0)) = x(B^*(x_0))$.

2.8. *For any $x \in L$ and $A_i \in F(x_0, p)$ from $A_i \subset A^x(x_0)$ it follows $x(A_i^*) = x(A^{x*}(x_0))$.*

This is an immediate consequence of 2.5 and 2.7.

Denote $A^{p*} = \{x(A^{x*}(x_0)) : x \in L\}$. Analogously as in 2.6 we can prove (by using 2.8 instead of 2.5) the proposition:

2.9. *The set A^{p*} is a sublattice of L and the mapping $\varphi_p^* : x \to x(A^{x*}(x_0))$ is a homomorphism of L onto A^p. For any $x \in A^{p*}$ we have $\varphi_p^*(x) = x$.*

For the sake of brevity we denote $x(A^x(x_0)) = x^1$, $x(A^{x*}(x_0)) = x^2$. Let us consider the mapping

$$(\alpha) \qquad x \to (x^1, x^2)$$

of the lattice L into $A^p \times A^{p*}$.

2.10. *The mapping α is one-to-one.*

Proof. Let $x, y \in L$, $x^1 = y^1$, $x^2 = y^2$ and denote $A^x(x_0) \cap A^y(x_0) = A$. Then $A \in F(x_0, p)$ and according to 2.6 and 2.9 we have

$$x(A) = x(A^x(x_0) = x^1\,, \quad x(A^*) = x(A^{x*}(x_0)) = x^2$$

and analogously for y^1, y^2. Therefore $x(A) = y(A)$, $x(A^*) = y(A^*)$; this implies $x = y$.

2.11. *The mapping α is an isomorphism of the lattice L onto $A^p \times A^{p*}$.*

Proof. It suffices to verify that the mapping φ is onto. Let $u \in A^p$, $v \in A^{p*}$. There exist $x, y \in L$ such that

$$u = x(A^x(x_0))\,, \quad v = y(A^{y*}(x_0))\,.$$

Put $A = A^x(x_0) \cap A^y(x_0)$. Then $u = x(A)$, $v = y(A^*)$. Thus there is an element $z \in L$ with the property

$$u = z(A)\,, \quad v = z(A^*)\,.$$

Let $B \in F(x_0, p)$, $B \subset A$. Then $z(B) \in A$ and therefore by using 1.7 we obtain

$$z(B) = z(B)\,(A) = z(A)\,(B) = u(B) = x(A)\,(B) = x(A \cap B) = u$$

and according to 2.7 $z(B^*) = v$. Thus $z(A^z(x_0)) = z(A)$, $z(A^{z*}(x_0)) = z(A^*)$. According to the definition of the mapping α this implies $z^1 = u$, $z^2 = v$, $\alpha(z) = (u, v)$.

2.12. $A^p \in F(x_0, p)$.

Proof. Consider the isomorphism $\alpha : L \to A^p \times A^{p*}$ and construct $A^p(x_0)$. According to 2.6 $A^p(x_0) = A^p$, thus it suffices to verify that A^p is parallel to the prime interval $p = [c, d]$. But this is equivalent to the assertion that $A^p(c)$ is parallel to p and thus we may assume that $c = x_0$. In such a case $d(A^p) = d > c = c(A^p)$. This shows that A^p is parallel to p.

2.12.1. $A^p \subset A$ *for each* $A \in F(x_0, p)$ *and* $x(A^p) = x(A^x(x_0))$ *for each* $x \in L$.

Proof. Let $y \in A^p$, $A \in F(x_0, p)$. Then $y = x(A^x(x_0))$ for some $x \in L$. By 2.5, $A^x(x_0)$ belongs to $F(x_0, p)$ and thus according to 2.4′ $B = A^x(x_0) \cap A \in F(x_0, p)$, therefore with respect to 2.5 $x(B) = y$. This implies $y \in B \subset A$, whence $A^p \subset A$. In particular, $A^p \subset A^x(x_0)$ for each $x \in L$, thus by 2.5 $x(A^p) = x(A^x(x_0))$.

2.13. *The lattice* A^p *is directly indecomposable.*

Proof. According to 2.12 card $A^p > 1$. Let $x_0 \in L$. Assume (by way of contradiction) at A^p is directly decomposable. Then there exist lattices C_1, C_2 with card $C_1 > 1$, card $C_2 > 1$ such that A^p is isomorphic to $C_1 \times C_2$; thus by 2.11 there is an isomorphism f of the lattice L onto $C_1 \times C_2 \times A^{p*}$ such that $A^p = C_1(x_0) \times C_2(x_0)$. Either $C_1(x_0)$ or $C_2(x_0)$ is paralell to p; we may suppose that $C_1(x_0)$ satisfies this condition. There exists $a \in C_2(x_0)$, $a \neq x_0$. Since $a \in A^p$, by 2.12.1 we have $a = a(A^p) = a(A^a(x_0))$ and therefore for any $D \in F(x_0, p)$

$$D \subset A^a(x_0) \Rightarrow a(D) = a\,.$$

Put $D = A^a(x_0) \cap C_1(x_0)$. Clearly $a(C_1(x_0)) = x_0$, whence $a(D) = a(A^a(x_0)) \,. \, (C_1(x_0)) = a(C_1(x_0)) = x_0 \neq a$, which is a contradiction. The proof is complete.

Let $\sim$ be an equivalence relation on the set $\mathscr{P}$ defined by

$$p_1 \sim p_2 \Leftrightarrow A^{p_1} = A^{p_2}\,.$$

Let $\mathscr{P}_1$ be a subset of $\mathscr{P}$ containing exactly one element from each equivalence class of the relation $\sim$. Consider the mapping $g : L \to \Pi A^p (p \in \mathscr{P}_1)$ defined by the rule

$$g(x)\,(p) = x(A^p)$$

($x \in L$). Clearly g is a homomorphism.

2.14. *The mapping* g *is one-to-one.*

Proof. Assume that there are elements $x, y \in L$ such that $x \neq y$, $g(x) = g(y) = t$; then there is a prime interval $p_1 = [u, v] \subset [x \wedge y, x \vee y]$ satisfying $g(u) = g(v) = t$. Hence $u(A^p) = v(A^p)$ for each $p \in \mathscr{P}_1$. But there exists $p_2 \in \mathscr{P}_1$ with $p_2 \sim p_1$ and $u(A^{p_2}) = u(A^{p_1}) < v(A^{p_1}) = v(A^{p_2})$ since $A^{p_1} \in F(x_0, p_1)$; we have a contradiction.

2.15. *The lattice* L *is a full subdirect product of lattices* A^p $(p \in \mathscr{P}_1)$.

Proof. According to 2.14 the mapping g is an isomorphism of L into ΠA^p $(p \in P_1)$. Since each A^p is a direct factor of L, $x_0 \in A^p$, the conditions (a) and (b) from §1 are fulfilled.

From 2.15 and 2.1 it follows:

2.16. Theorem. *Any discrete lattice is a weak product of directly indecomposable lattices.*

Since any two full subdirect decomposositions have a common refinement, the representation of a discrete lattice as a weak product of indecomposable lattices is unique.

3. ISOMORPHISMS OF UNORIENTED GRAPHS OF DISCRETE LATTICES

Let L be a discrete lattice and let $\mathcal{P}$ be the set of all prime intervals of L. We denote by $G(L)$ the unoriented graph such that the set of vertices of $G(L)$ equals L and two vertices x, y of $G(L)$ are assumed to be joined by an edge if and only if either $[x, y] \in \mathcal{P}$ or $[y, x] \in \mathcal{P}$. In [1] there is formulated the following problem (Problem 8): what discrete lattices L satisfy the condition that for each discrete lattice L' the implication

$$G(L) \sim G(L') \Rightarrow L \sim L' \tag{5}$$

is valid (where $\sim$ denotes the isomorphism of graphs or lattices, respectively). The answer to this problem for general lattices is unknown. In [4] a solution for the case of finite modular lattices was given. Now we shall prove that the result of [4] can be generalized for infinite modular lattices.

For any lattice L we denote by $\bar{L}$ the lattice that is dual to L. L is self-dual, if $L \sim \bar{L}$. The following propositions 3.1 and 3.2 are known [4]:

3.1. *Let L and L' be discrete modular lattices. Then the following conditions are equivalent:*

(i) $G(L) \sim G(L')$.

(ii) *There are lattices A, B such that $L \sim A \times B$, $L' \sim A \times \bar{B}$.*

3.2. *Let L be a finite modular lattice. Then the following conditions are equivalent:*

(i) *For any finite modular lattice L' the implication* (5) *is valid.*

(ii) *If $L \sim A \times B$, then $A \sim \bar{A}$* (i.e., *each direct factor of L is self-dual*).

Our aim now is to show that the assertion of 3.2 remains valid for infinite discrete modular lattices.

3.3. *Let L be a discrete lattice, $x_0 \in L$, and let L be a full subdirect product of lattices A_i $(i \in I)$. Let $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$,*

$$A = \{x \in L : i \in I_2 \Rightarrow x(A_i(x_0)) = x_0\},$$
$$B = \{x \in L : i \in I_1 \Rightarrow x(A_i(x_0)) = x_0\}.$$

Then $L \sim A \times B$.

Proof. Let $x \in L$ and for $i \in I$ let us write $x(i)$ rather than $x(A_i(x_0))$. According to 2.1 the set $\{i \in I : x(i) \neq x_0\}$ is finite and so it follows from §1, (b) and by induction that there is an element $x_1 \in A$ with the property $x(i) = x_1(i)$ for each $i \in I_1$. Analogously there exists $x_2 \in B$ satisfying $x(i) = x_2(i)$ for each $i \in I_2$. The mapping $\psi : x \to (x_1, x_2)$ is obviously an isomorphism of the lattice L into $A \times B$. Let $a \in A$, $b \in B$. Since the sets $\{i \in I : a(i) \neq x_0\}$, $\{i \in I : b(i) \neq x_0\}$ are finite and disjoint, with respect to (b) we can find $x \in L$ such that $x_1 = a$ and $x_2 = b$; this shows that ψ is onto.

The assertion of the proposition 3.3 need not hold for the case when L is not discrete.

3.4. *Let L be a discrete modular lattice such that each directly indecomposable direct factor of L is self-dual. Then for each discrete modular lattice L' the implication (5) holds.*

Proof. Let L' be a discrete modular lattice, $G(L) \sim G(L')$. According to 3.1 there exist lattices A, B such that $L \sim A \times B$, $L' \sim \bar{A} \times B$. Then A is isomorphic to a sublattice of L, whence A is discrete. By 2.16 A is a full subdirect product of directly indecomposable lattices A_i. Any A_i is isomorphic to a directly indecomposable direct factor of L and therefore $A_i \sim \bar{A}_i$. From this it follows $A \sim \bar{A}$ and hence $L \sim L'$.

3.5. *Let L be a discrete modular lattice such that for any discrete modular lattice L' the implication (5) is valid. Then each direct factor of L is self-dual.*

Proof. Since any direct factor A of L is discrete and thus A is a full subdirect product of directly indecomposable factors it suffices to prove that each directly indecomposable direct factor of L is self-dual. Let A_0 be a directly indecomposable direct factor of L and assume (by way of contradiction) that A_0 is not self-dual. Let $x_0 \in L$. We may assume that $A_0 \in F(x_0)$. There exists a full subdirect decomposition

$$L = (fs)\, \Pi A_i(x_0)\, (i \in I)\,,$$

where all factors $A_i(x_0)$ are directly indecomposable. Put

$$I_1 = \{i \in I : A_i(x_0) \sim A_0\}\,, \quad I_2 = I \setminus I_1$$

and let A, B have the same meaning as in 3.3. According to 3.3 $L \sim A \times B$, whence by 3.1 $G(L) \sim G(L')$, where $L' = \bar{A} \times B$. Thus if X is a directly indecomposable direct factor of L', then X is isomorphic to some $A_i(x_0)\,(i \in I_2)$ or to some $\overline{A_i(x_0)}$ $(i \in I_1)$; therefore X cannot be isomorphic to A_0. From this it follows that L' is not isomorphic to L, which is a contradiction.

3.6. *Let L be a full subdirect product of directly indecomposable lattices $A_i\,(i \in I)$. If all lattices A_i are self-dual, then all direct factors of L are self-dual.*

Proof. Let $x \in L$. Let L be isomorphic to a direct product $A \times B$. Then because any two full subdirect decompositions of L have a common refinement (Theorem 1.11) and since $A_i(x_0)$ are directly indecomposable, there is a subset $I_1 \subset I$ such that A is a full subdirect product of lattices $A_i(x_0)$ $(i \in I_1)$. Because $A_i(x_0)$ are self-dual, so is the lattice A.

By summarizing, we get from 3.4, 3.5 and 3.6:

3.7. Theorem. *Let L be a discrete modular lattice. Then the following conditions are equivalent*:

(i) *For any discrete modular lattice L' the implication* (5) *is fulfilled.*

(ii) *Each directly indecomposable direct factor of L is self-dual.*

(iii) *Each direct factor of L is self-dual.*

Let L and L' be finite lattices such that $G(L) \sim G(L')$. If G is modular, then so is L [4]. It remains as an open question whether this assertion is valid for infinite discrete lattices.

References

[1] *G. Birkhoff:* Lattice theory. Second Ed., Providence, 1948.

[2] *G. Grätzer:* Universal algebra, Princeton, 1968.

[3] *J. Hashimoto:* On direct product decomposition of partially ordered sets, Annals of Math. *54* (1951), 315–318.

[4] *J. Jakubík:* О графическом изоморфизме структур, Czech. Math. J. *4* (*79*) (1954), 131–142.

[5] *F. Šik:* Über subdirekte Summen geordneter Gruppen, Czech. Math. J. *10* (*85*) (1960), 400–424.

[6] *Tah-Kai Hu:* Weak products of simple universal algebras, Math. Nachr. *42* (1969), 157–171.

Author's address: Košice, Nám. Februárového víť. 9, ČSSR (Vysoká škola technická).