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APPROXIMATE DERIVATIVES AND BAIRE CLASSES

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Z. ZAHORSKI proved in [6] that if the approximate derivative of a function f exists at every point of an interval (a, b) then it is of Baire class 3. A. MATYSIAK [4] and K. KRZYŻEWSKI [3] independently proved the following stronger theorem.

Theorem A. *Let f be a finite function defined on the whole real line and let R be the set of all points x at which the approximate derivative f'_{ap} exists. If every point of R is a point of outer density of R, then*

(a) *there exists a countable set $Z \subset R$ such that f is of the first class of Baire with respect to $R - Z$,*

(b) *f'_{ap} is of Baire class 2 with respect to R.*

Z. Zahorski in [6] gives two examples of functions and shows that these functions have the approximate derivative and that this derivative is not of Baire class 1. It is not difficult to prove that these functions have not the approximate derivative in an uncountable set. (This follows from Theorem 2 because these functions have the upper approximate derivative which is nonnegative except for a countable set and the approximate derivative is equal to $-\infty$ at each point of a set which is dense in the Cantor set.)

A. M. BRUCKNER in [1] and T. ŚWIĄTKOWSKI in [5] independently proved the following theorem.

Theorem B. *Let f be a Darboux function of Baire class 1 which possesses the approximate derivative in (a, b), except perhaps on a denumerable set of points, and let $f'_{ap} \geqq 0$ almost everywhere in (a, b). Then f is nondecreasing and continuous in (a, b).*

A. M. Bruckner (1) and A. M. Bruckner, J. L. Leonard (2) show that it is not possible to omit the condition that f is of the first class of Baire in Theorem B. However, the functions constructed by them have not the approximate derivative in an uncountable set. The following Example 1 shows that indeed this condition in Theorem B must not be omitted, Example 2 shows that there exists a function f defined on $\langle 0, 1\rangle$ such that f'_{ap} exists on all $\langle 0, 1\rangle$, f is a Darboux function and f is not of Baire class 1.

Example 1. Let C be the Cantor set on $\langle 0, 1\rangle$, let φ be such a continuous function defined on $\langle 0, 1\rangle$ that $\varphi' < 0$ exists on all $\langle 0, 1\rangle$ and $\varphi'(x) = -\infty$ for $x \in C$, $\varphi(0) = 1$, $\varphi(1) = 0$. Such function exists in view of theorem 7 [7].

Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ be a sequence of all intervals contiguous to the Cantor set.

We put

$$f_1(x) = \varphi(x) + \sum_{x \leqq a_m} (\varphi(a_m) - \varphi(b_m)) \quad \text{for} \quad x \in C,$$

$$f_1(x) = \frac{f_1(a_m) + f_1(b_m)}{2} = \varphi(a_m) + \sum_{b_m \leqq a_n} (\varphi(a_n) - \varphi(b_n)) =$$

$$= \varphi(b_m) + \sum_{a_m \leqq a_n} (\varphi(a_n) - \varphi(b_n)) \quad \text{for} \quad x \in (a_m, b_m).$$

Then for $x_0 \in C$, $x \in C$ it is

$$\frac{f_1(x) - f_1(x_0)}{x - x_0} = \frac{\varphi(x) - \varphi(x_0)}{x - x_0} + \frac{1}{x - x_0} \Big(\sum_{x \leqq a_m} (\varphi(a_m) - \varphi(b_m)) -$$

$$- \sum_{x_0 \leqq a_n} (\varphi(a_n) - \varphi(b_n)) \leqq \frac{\varphi(x) - \varphi(x_0)}{x - x_0},$$

for $x_0 \in C$, $x \in (a_m, b_m)$, $x_0 < a_m$ it is

$$\frac{f_1(x) - f_1(x_0)}{x - x_0} = \frac{\varphi(b_m) - \varphi(x_0)}{x - x_0} - \frac{1}{x - x_0} \Big(\sum_{x_0 \leqq a_n < a_m} (\varphi(a_n) - \varphi(b_n)) \Big) \leqq$$

$$\leqq \frac{\varphi(x) - \varphi(x_0)}{x - x_0}$$

and for $x_0 \in C$, $x \in (a_m, b_m)$, $b_m < x_0$ it is

$$\frac{f_1(x) - f_1(x_0)}{x - x_0} = \frac{\varphi(a_m) - \varphi(x_0)}{x - x_0} + \frac{1}{x - x_0} \Big(\sum_{a_m \leqq a_n < x_0} (\varphi(a_n) - \varphi(b_n)) \Big) \leqq$$

$$\leqq \frac{\varphi(x) - \varphi(x_0)}{x - x_0}.$$

If $x_0 = a_n$ then $\lim_{x \to x_{0+}} f_1(x) < f_1(x_0)$ and therefore $f_1'^{+}(x_0) = -\infty$. Therefore for each $x \in C$ is $f_1'(x) = -\infty$.

Now we put

$$f(x) = f_1(x) \quad \text{for} \quad x \in C - \bigcup_{n=1}^{\infty} \{a_n, b_n\},$$

$$f(x) = f_1(x) \text{ for } x \in \left\langle a_n + \frac{b_n - a_n}{2^{n+1}}, \quad b_n - \frac{b_n - a_n}{2^{n+1}} \right\rangle,$$

$$f(a_n) = -1, \quad f(b_n) = 3.$$

On intervals $(a_n, a_n + (b_n - a_n)/2^{n+1})$ and $(b_n - (b_n - a_n)/2^{n+1}, b_n)$ we define f so that f is continuous and nondecreasing on $\langle a_n, b_n \rangle$ and f' exists on (a_n, b_n) (thus $f'(x) \geqq 0$ for $x \in (a_n, b_n)$).

We put

$$E_* = \bigcup_{n=1}^{\infty} \left\langle a_n, b_n - \frac{b_n - a_n}{2^{n+1}} \right\rangle, \quad E^* = \bigcup_{n=1}^{\infty} \left\langle a_n + \frac{b_n - a_n}{2^{n+1}}, b_n \right\rangle.$$

If $x_0 \in C$, $x_0 \neq a_n$, then

$$f'^{+}_{ap}(x_0) = \lim_{\substack{x \to x_0 + \\ x \in E_*}} \frac{f(x) - f(x_0)}{x - x_0} \leqq \lim_{\substack{x \to x_0 + \\ x \in E_*}} \frac{f_1(x) - f_1(x_0)}{x - x_0} = -\infty.$$

Similarly for $x_0 \in C$, $x_0 \neq b_n$, it is $f'^{-}_{ap}(x_0) = -\infty$. Therefore $f'_{ap}(x) = -\infty$ for $x \in C - \bigcup_{n=1}^{\infty} \{a_n, b_n\}$.

Obviously f is a Darboux function, $f'_{ap} \geqq 0$ almost everywhere and f'_{ap} exists at each point $x \in (0, 1)$, $x \neq a_n$, $x \neq b_n$.

Example 2. We construct the function f_1 as in Example 1. We put

$$f(x) = f_1(x) \quad \text{for} \quad x \in C - \bigcup_{n=1}^{\infty} \{a_n, b_n\},$$

$$f(x) = f_1(x) \quad \text{for} \quad x \in \left\langle a_n + \frac{b_n - a_n}{2^{n+1}}, \quad b_n - \frac{b_n - a_n}{2^{n+1}} \right\rangle,$$

$$f(a_n) = -1, \quad f(b_n) = 3, \quad f\left(a_n + \frac{b_n - a_n}{2^{n+2}}\right) = -2, \quad f\left(b_n - \frac{b_n - a_n}{2^{n+2}}\right) = 4.$$

On intervals $(a_n, a_n + (b_n - a_n)/2^{n+2})$, $(a_n + (b_n - a_n)/2^{n+2}, a_n + (b_n - a_n)/2^{n+1})$, $(b_n - (b_n - a_n)2^{n+1}, b_n - (b_n - a_n)/2^{n+2})$, $(b_n - (b_n - a_n)/2^{n+2}, b_n)$ we define f so that f is continuous on $\langle a_n, b_n \rangle$, f' exists on (a_n, b_n), $f'^{+}(a_n) = -\infty$, $f'^{-}(b_n) = -\infty$, f is nondecreasing on $\langle a_n + (b_n - a_n)/2^{n+2}, b_n - (b_n - a_n)/2^{n+2} \rangle$ and nonincreasing on $\langle a_n, a_n + (b_n - a_n)/2^{n+2} \rangle$, $\langle b_n - (b_n - a_n)/2^{n+2}, b_n \rangle$. Similarly as in Example 1 we prove that f is a Darboux function and that f'_{ap} exists on $(0, 1)$. This function f is not obviously of the first class of Baire.

In this paper we prove in Part I one characterization of functions of Baire class 1. In Part II functions are studied which have the approximate derivative on all interval (a, b). In this part we prove that every approximate derivative is of Baire class 1 (Theorem 3). Theorems 6 and 7 give a solution of the problems proposed by A. M. Bruckner in [1]. Bruckner's problem solved by Theorem 6 has been solved in paper [8] provided that f'_{ap} is of the first class of Baire. As remarked in [8], on base of theorem 3 of the present paper the problem is completely solved. In Part III it is proved that if R is the set of all points at which the approximate derivative of a function f (defined on (a, b)) exists then f'_{ap} is of Baire class 1 with respect to the set R and there exists a conhtable set $Z \subset R$ such that f is of Baire class 1 with respect to the set $R - Z$.

I

Theorem 1. *Let P be a topologically complete metric space, let f be a function defined on P. Then f is of Baire class 1 if and only if the following assertion is valid:*

For each closed set $F \subset P$ and for any real numbers $\alpha < \beta$ at most one of the sets $\{x \in F; f(x) \geqq \beta\}$, $\{x \in F, f(x) \leqq \alpha\}$ is dense in F.

Proof. a) If f is of Baire class 1 then the assertion is obviously valid.

b) Let the assertion be valid. We can assume that $f[P] \subset \langle 0, 1 \rangle$ and that P is a complete metric space. We prove the following proposition:

Let $\emptyset \neq A \subset P$ and let n be a natural number. Then there exist $x \in A$ and $\varepsilon > 0$ such that for each $y_1, y_2 \in A \cap U_\varepsilon(x)$ it is $|f(y_1) - f(y_2)| < 1/n$. (We denote $U_\varepsilon(x) = \{z \in P, \varrho(z, x) > \varepsilon\}$.)

We find the minimal integer k such that there exist $x_0 \in A$ and $\varepsilon_0 > 0$ such that $\{z \in A \cap U_{\varepsilon 0}(x_0), f(z) \geqq k/2n\} = \emptyset$. Then there exist $x \in A \cap U_{\varepsilon 0}(x_0)$ and $\varepsilon > 0$ such that $\{z \in A \cap U_\varepsilon(x), f(z) \leqq (k-2)/2n\} = \emptyset$ and $U_\varepsilon(x) \subset U_{\varepsilon 0}(x_0)$. (If we supposed that such x and ε do not exist then both sets $\{z \in A \cap U_{\varepsilon 0}(x_0), f(z) \geqq (k-1)/2n\}$, $\{z \in A \cap U_{\varepsilon 0}(x_0), f(z) \leqq (k-2)/2n\}$ would be dense in $\overline{A \cap U_{\varepsilon 0}(x_0)}$.)

Now let $F \subset P$, $F \neq \emptyset$ be a closed set. We choose $x_1 \in F$ and put $\varepsilon_1 = 1$. If the sequences $\{x_i\}$, $\{\varepsilon_i\}$ are defined for all $i \leqq j$ such that $x_i \in F$, $\varepsilon_i > 0$ then we put $A = F \cap U_{\varepsilon_j}(x_j)$, $n = j$ and by the proposition we find $x_{j+1} \in A$ and $\varepsilon_{j+1} > 0$ such that for each $y_1, y_2 \in A \cap U_{\varepsilon_{j+1}}(x_{j+1})$ it is $|f(y_1) - f(y_2)| < 1/n$ and $\overline{U_{\varepsilon_{j+1}}(x_{j+1})} \subset U_{\varepsilon_j}(x_j)$, $\varepsilon_{j+1} < \frac{1}{2}\varepsilon_j$.

Then there exists a point $x \in \bigcap_{j=1}^{\infty} U_{\varepsilon_j}(x_j)$ and the function $f|_F$ is obviously continuous in x. Hence it follows that f is a function of Baire class 1.

II

Theorem 2. *Let f be a function defined on (a, b). If $G \subset (a, b)$ is a nonempty G_δ set such that $\bar{f}'_{ap} < \alpha$ on a dense subset A of G then $\underline{f}'_{ap} \leqq \alpha$ at each point of a G_δ set H such that $A \subset H \subset G$ (consequently H is a residual set in G).*

Proof. For each $x \in A$ we find $1 > \delta_x > 0$ such that for each $h_1, h_2, 0 \leqq h_i \leqq \delta_x$ it holds

$$\mu_e \left\{ z \in (x - h_1, x + h_2); \frac{f(z) - f(x)}{z - x} \geqq \alpha \right\} \leqq \frac{h_1 + h_2}{4}.$$

We put $G_n = \bigcup_{x \in A} (x - \delta_x/4n, x + \delta_x/4n)$, $H = G \cap \bigcap_{n=1}^{\infty} G_n$. Then H is a G_δ set and $A \subset H \subset G$.

If $y \in H$, $\underline{f}'_{ap}(y) > \alpha$ then there exists $\delta > 0$ such that for each $h_1, h_2, 0 \leqq h_i \leqq \delta$ it is

$$\mu_e \left\{ z \in (y - h_1, y + h_2); \frac{f(z) - f(y)}{z - y} \leqq \alpha \right\} \leqq \frac{h_1 + h_2}{4}.$$

We find $x \in A$ such that $|x - y| \leqq \min \{\frac{1}{4}\delta, \frac{1}{4}\delta_x\}$, we can assume that $x \leqq y$ (in the case $y > x$ the proof is similar). We put $z_1 = \max \{x - \delta_x, y - \delta\}$, $z_2 = \min \{x + \delta_x, y + \delta\}$. Then $z_1 < x \leqq y < z_2$ and if we put

$$A_1 = \left\{ z \in (z_1, x); \frac{f(z) - f(x)}{z - x} \geqq \alpha \right\}, \quad A_2 = \left\{ z \in (z_1, x); \frac{f(z) - f(y)}{z - y} \leqq \alpha \right\}$$

then

$$\mu_e(A_1 \cup A_2) \leqq \frac{x - z_1}{4} + \frac{y - z_1}{4} < x - z_1.$$

Therefore there exists a point $t_1 \in (z_1, x)$ such that

$$\frac{f(t_1) - f(x)}{t_1 - x} < \alpha, \quad \frac{f(t_1) - f(y)}{t_1 - y} > \alpha.$$

Similarly we prove that there exists a point $t_2 \in (y, z_2)$ such that

$$\frac{f(t_2) - f(x)}{t_2 - x} < \alpha, \quad \frac{f(t_2) - f(y)}{t_2 - y} > \alpha.$$

But this implies that $(f(t_2) - f(t_1))/(t_2 - t_1) < \alpha$ and $(f(t_2) - f(t_1))/(t_2 - t_1) > \alpha$ which is a contradiction.

Theorem 3. *Let f be a function defined on an interval (a, b) which possesses the approximate derivative f'_{ap} at each point of (a, b). Then f'_{ap} is of the first class of Baire on (a, b).*

Proof. This theorem is an obvious consequence of Theorems 1 and 2.

Lemma 1. *Let f be a function defined on (a, b). Let $\underline{f}'_{ap} \geqq 0$ almost everywhere, $\underline{f}'_{ap} > -\infty$ except perhaps for a countable set E such that for each $x \in E$ and $\varepsilon > 0$ the sets $\{z \in (x - \varepsilon, x); f(z) < f(x) + \varepsilon\}$, $\{z \in (x, x + \varepsilon), f(z) > f(x) - \varepsilon\}$ are uncountable. Then f is nondecreasing on (a, b).*

Proof. Suppose that f is not nondecreasing on (a, b). Let φ be such a function that $\varphi(a) = 0$; $\varphi(b) = 1$, $\varphi'(x) = +\infty$ for all x such that $\underline{f}'_{ap}(x) < 0$ and $\varphi' > 0$ everywhere. Then there exists $\eta > 0$ such that the function $g(x) = f(x) + \eta\, \varphi(x)$ is not nondecreasing on (a, b).

Let $\mathscr{I}$ be a set of all open intervals (x, y) such that $g(x) > g(y)$. We put $G_n = \bigcup_{I \in \mathscr{I}, \mu(I) < 1/n} I$, $G = \bigcap_{n=1}^{\infty} G_n$, then G is a G_δ set. Let $(x, y) \in \mathscr{I}$. If $x \in E$ then there exists a point $x' \in \{z \in (x, y) - E,\ g(z) > g(y)\}$, i.e. $x < x' < y$, $(x', y) \in \mathscr{I}$, $x' \notin E$. If $x \notin E$, then $\underline{g}'_{ap}(x) \geqq 0$ and hence it follows that there exists x' such that $x < x' < y$, $(x', y) \in \mathscr{I}$, $x' \notin E$. Similarly we prove that there exists $y' \in (x', y)$ such that $(x', y') \in \mathscr{I}$, $y' \notin E$. Now it is easy to prove that for each $(x, y) \in \mathscr{I}$ and for each $\varepsilon > 0$ there exists $(x_1, y_1) \in \mathscr{I}$ such that $x < x_1 < y_1 < y$ and $y_1 - x_1 < \varepsilon$. This implies that for each $(x, y) \in \mathscr{I}$ it is $G \cap (x, y) \neq \emptyset$ and that G has no isolated point.

For each $(x, y) \in \mathscr{I}$ we denote by (x_0, y_0) the interval constructed as follows:

We find $(x_1, y_1) \subset (x, y)$, $(x_1, y_1) \in \mathscr{I}$ such that $x_1, y_1 \notin E$. Now we choose $y_0 \in (x_1, y_1)$ such that for each $z \in (x_1, y_0)$ it is $\mu_e\{t \in (x_1, z);\ g(t) < g(x_1)\} \leqq \frac{1}{4}(z - x_1)$ and $g(y_0) < g(x_1)$. Such y_0 exists; we put $y' = \inf\{t \in (x_1, y_1),\ g(t) < g(x_1)\}$ and if $x_1 < y'$ then it is sufficient to choose y_0 sufficiently close to y', if $x_1 = y'$ then the existence of y_0 follows from the fact that $\underline{g}'_{ap}(x_1) > 0$.

Similarly we choose $x_0 \in (x_1, y_0)$ such that for each $z \in (x_0, y_0)$ it is $\mu_e\{t \in (z, y_0),\ g(t) > g(y_0)\} \leqq \frac{1}{4}(y_0 - z)$ and $g(y_0) < g(x_0)$. Then for $z \in (x_0, y_0)$ it is either $\mu_e\{t \in (x_1, z);\ (g(t) - g(z))/(t - z) > 0\} \leqq \frac{1}{4}(z - x_1)$ or $\mu_e\{t \in (z, y_0);\ (g(t) - g(z) : (t - z) > 0\} \leqq \frac{1}{4}(y_0 - z)$.

Now we put $H_n = \bigcup_{(x,y) \in \mathscr{I}, y - x < 1/n} (x_0, y_0)$, $H = \bigcap_{n=1}^{\infty} Hn$. From the construction it follows that each $H_n \cap G$ is dense in G, thus H is a residual set in G and hence H is uncountable. But for each $x \in H$ it holds $\underline{g}'_{ap}(x) \leqq 0$, hence $H \subset E$ which is a contradiction.

Theorem 4. *Let f be a function defined on (a, b), let $\underline{f}'_{ap} \geqq 0$ almost everywhere in (a, b). Let at each point x of (a, b)*

$$\lim_{h \to 0+} f(x - h) \leqq f(x) \quad \textit{if} \quad \lim_{h \to 0+} f(x - h) \quad \textit{exists},$$

$$\lim_{h \to 0+} f(x + h) \leqq f(x) \quad \textit{if} \quad \lim_{h \to 0+} f(x + h) \quad \textit{exists}$$

and let E be a denumerable set such that if $x \notin E$, $\underline{f}'_{ap}(x) = -\infty$ then $f'_{ap}(x) = -\infty$. Let for each $x \in E$ and $\varepsilon > 0$ the sets $\{z \in (x - \varepsilon, x); f(z) < f(x) + \varepsilon\}$, $\{z \in (x, x +$

$+ \varepsilon); f(z) > f(x) - \varepsilon\}$ *be uncountable, and let* f *be of Baire class* 1 *with respect to the set* $\bar{E}$.

Then f *is nondecreasing on* (a, b).

Proof. We can assume that $\underline{f}'_{ap} \geqq \eta > 0$ almost everywhere. Suppose that f is not nondecreasing on $\langle \alpha, \beta \rangle \subset (a, b)$. From Theorem 2 it follows that $\{z \in (a, b), f'_{ap}(z) = -\infty\}$ is nowhere dense in (a, b), hence by Lemma 1 there exist maximal closed intervals of monotonicity of f on $\langle \alpha, \beta \rangle$. We denote these intervals $\langle a_n, b_n \rangle$ and we put $F = \overline{(\alpha, \beta) - \bigcup_n (a_n, b_n)}$. From Lemma 1 and Theorem 2 it follows that $\{z \in F, f'_{ap}(z) = -\infty\}$ is a residual set in F.

If we suppose that $F - \bar{E} \neq \emptyset$, $x_0 \in F$, $(x_0 - \delta, x_0 + \delta) \cap F \cap \bar{E} = \emptyset$ then for each $b_n \in (x_0 - \delta, x_0 + \delta)$ is $\underline{f}'_{ap}(b_n) > -\infty$ (because the case $\underline{f}'_{ap}(b_n) = -\infty$ is not possible). From the fact that $\underline{f}'_{ap} > 0$ a.e. it follows that $f(\frac{1}{2}(a_n + b_n)) < f(b_n)$ for each n. Since $\underline{f}'_{ap}(b_n) > -\infty$, it is $\liminf\text{ap}_{t \to b_n+} f(t) \geqq f(b_n)$ and therefore there exists δ_n such that $\delta_n < \min(\frac{1}{2}(b_n - a_n), 1)$ and for each $0 < h < \delta_n$ it is $\mu_e\{t \in (b_n, b_n + h); f(t) < f(\frac{1}{2}(b_n + a_n))\} < \frac{1}{4}h$. We put $G_k = \bigcup_{b_n \in (x_0 - \delta, x_0 + \delta)} (b_n, b_n + \delta_n/4k)$, $G = F \cap \bigcap_{k=1}^{\infty} G_k$. Then G is a residual set in $F \cap (x_0 - \delta, x_0 + \delta)$ and at each point $y \in G$ it is $\bar{f}'_{ap}(y) \geqq 0$, which is a contradiction. Thus $F \subset \bar{E}$ and f is of Baire class 1 with respect to F.

Now we prove the following proposition:

If $b_n \in (\alpha, \beta)$ *then for each* $\varepsilon, \delta > 0$ *there exists a point* $x \in (b_n, b_n + \delta) \cap (\alpha, \beta) \cap F$, $x \neq a_k, b_k$, $x \notin E$, $f(x) > f(b_n) - \varepsilon$.

We denote $B = \{b_k \in (b_n, b_n + \frac{1}{2}\delta); f(b_k) > f(b_n) - \frac{1}{2}\varepsilon\} \cap (\alpha, \beta)$. If $\bar{B}$ is a perfect set then we find a point x, of continuity of $f|_{\bar{B}}$, $x_0 \notin E$, $x_0 \neq a_k, b_k$. The point x_0 obviously satisfies the assertion of the proposition. If $\bar{B}$ is not a perfect set then there exists $b_k \in B$ such that for some δ_1, $0 < \delta_1 < b_n - b_k + \frac{1}{2}\delta$, $\delta_1 < \beta - b_k$ it is $(b_k, b_k + \delta_1) \cap \{x; f(x) > f(b_n) - \frac{1}{2}\varepsilon\} \cap \bigcup_n \langle a_n, b_n \rangle = \emptyset$. If we choose $x_0 \in (b_k, b_k + \delta_1)$ such that $f(x_0) > f(b_n) - \frac{1}{2}\varepsilon$, $x_0 \notin E$ then x_0 is the required point.

Now let k be a natural number, $b_n \in (\alpha, \beta)$, $b_n - a_n < 1/k$, $x_0 \in (b_n, b_n + \frac{1}{2}(b_n - a_n)) \cap F$; $x_0 \neq a_j, b_j$; $x_0 \notin E$; $f(x_0) > f(\frac{1}{2}(a_n + b_n))$. If $\underline{f}'_{ap}(x_0) > -\infty$ then we find $\delta > 0$ such that $\delta < b_n - a_n$ and $\mu_e\{t \in (x_0, x_0 + \delta); f(t) < f(\frac{1}{2}(a_n + b_n))\} < \frac{1}{4}\delta$. We put $x = x_0$, $b_n < y < x$, $x - y < \frac{1}{4}\delta$. If $f'_{ap}(x_0) = -\infty$ then we find $\delta > 0$, $\delta < x_0 - b_n$ such that $\mu_e\{t \in (x_0, x_0 - \delta), f(t) < f(\frac{1}{2}(a_n + b_n))\} < \frac{1}{4}\delta$ and we put $x = x_0 - \delta$, $b_n < y < x$, $x - y < \frac{1}{4}\delta$.

Let G_k be the sum of all intervals (y, x) which are constructed in this way. Then $G_k \cap F$ is dense in F, therefore $G = F \cap \bigcap_{k=1}^{\infty} G_k$ is a residual set in F. But at each point $x \in G$ it is $\bar{f}'_{ap}(x) \geqq 0$ and at each point of a residual set in F it is $f'_{ap}(x) = -\infty$; this is a contradiction.

Theorem 5. *Let f be defined on (a, b), let f possesses the approximate derivative at each point of (a, b). Let f be a Darboux function and let $f'_{ap} \geqq 0$ almost everywhere. Then f is continuous and nondecreasing on (a, b).*

Proof. This theorem is a consequence of Theorem 4 but it can be easily proved on the base of Theorem 3 and Lemma 1 as follows.

If x_0 is a point of continuity of f'_{ap}, then f is nondecreasing in some neighbourhood of x_0. Let $\langle a_n, b_n \rangle$ be the maximal intervals of monotonicity of f, $F = (a, b) - \bigcup_n (a_n, b_n)$, $f'_{ap}(a_n) \geqq 0$, $f'_{ap}(b_n) \geqq 0$. If x is a point of continuity of $f'_{ap}|_F$ then f is nondecreasing in some neighbourhood of x which is a contradiction.

Theorem 6. *Let f be defined on (a, b) and let f possesses the approximate derivative f'_{ap} on all (a, b). If at each point at which $\lim_{h \to 0+} f(x-h)$ exists it is $\lim_{h \to 0+} f(x-h) = f(x)$ and at each point at which $\lim_{h \to 0+} f(x+h)$ exists it is $\lim_{h \to 0+} f(x+h) = f(x)$ then f'_{ap} has the Denjoy property (i.e. for each $\alpha < \beta$ the set $f'^{-1}_{ap}[(\alpha, \beta)]$ is either empty or of positive measure), f'_{ap} is a Darboux function and f fulfils the Mean Value Theorem.*

Proof. Suppose that there exist $\alpha < \beta$ such that $f'^{-1}_{ap}[(\alpha, \beta)]$ is nonempty and of measure zero. In view of Theorems 2,4 there exist maximal intervals $\langle a_n, b_n \rangle$ such that $f(x) - \beta x$ is nondecreasing on $\langle a_n, b_n \rangle$ and maximal intervals $\langle \alpha_n, \beta_n \rangle$ such that $\alpha x - f(x)$ is nondecreasing on $\langle \alpha_n, \beta_n \rangle$ (one of the sequences $\{\langle a_n, b_n \rangle\}$, $\{\langle \alpha_n, \beta_n \rangle\}$ can be empty). Then the set $G = \bigcup_n (a_n, b_n) \cup \bigcup_n (\alpha_n, \beta_n)$ is dense in (a, b) and the set $\overline{(a, b) - G}$ is perfect. Let $x_0 \in (a, b) - G$ be a point of continuity of $f'_{ap}|_{\overline{(a,b)-G}}$. Then it is either $f'_{ap}(x_0) \leqq \alpha$ or $f'_{ap}(x_0) \geqq \beta$. In the first case, according to Lemma 1 the function $\alpha x - f(x)$ is nondecreasing in some neighbourhood of x which is a contradiction; the other case is similar.

Because f'_{ap} is of Baire class 1 and has the Denjoy property it is a Darboux function (see [7]). The third assertion can be easily proved on the base of this fact and Theorem 4.

Let $\mathscr{C}$ be a family of functions defined on $\langle a, b \rangle$. A subset E of $\langle a, b \rangle$ with the property that whenever $f \in \mathscr{C}$ is constant on E, then f must be constant on $\langle a, b \rangle$, is said to be a stationary set for $\mathscr{C}$.

Theorem 7. *A set E is a stationary set for the approximate derivatives of Darboux functions if and only if $\mu_i(\langle a, b \rangle - E) = 0$.*

Proof. a) If E is a stationary set for the approximate derivatives of Darboux functions, then it is $\mu_i(\langle a, b \rangle - E) = 0$ because if $\mu_i(\langle a, b \rangle - E) > 0$ then there would exist a finite (ordinary) derivative such that it is constant on E and is not constant on $\langle a, b \rangle$ (see [7]).

b) If $\mu_i(\langle a, b\rangle - E) = 0$ and if f is a Darboux function such that f'_{ap} exists on all $\langle a, b\rangle$ and $f'_{ap} = c$ on E then the set $E^* = \{z \in \langle a, b\rangle, f'_{ap}(z) = c\}$ is measurable and if we put $g(x) = f(x) - cx$ then $g'_{ap} = 0$ almost everywhere. According to Theorem 4 the function g is constant on $\langle a, b\rangle$, hence $f'_{ap} = c$ on all $\langle a, b\rangle$.

III

Theorem 8. *Let f be a function defined on the whole real line. Let R be the set of all points at which the approximate derivative f'_{ap} exists. Then f'_{ap} is of the first class of Baire with respect to R and there exists a countable set $Z \subset R$ such that f is of the first class of Baire with respect to the set $R - Z$.*

Proof. 1. Let α be a real number, $M = \{x \in R, f'_{ap}(x) \leqq \alpha\}$. According to Theorem 2 we can find for every natural n a G_δ set H_n such that $M \subset H_n \subset \overline{M}$ and $\underline{f}'_{ap}(x) \leqq \leqq \alpha + 1/n$ on H_n. Then $M = R \cap \bigcap_{n=1}^{\infty} H_n$, i.e. M is a G_δ set with respect to R. Similarly we prove that also $\{x \in R, f'_{ap}(x) \geqq \alpha\}$ is a G_δ set with respect to R.

2. Let α be a real number, $M = \{x \in R, f(x) \geqq \alpha\}$, $M_0 = M \cap \{x \in R, f'_{ap}(x) \geqq 0\}$. For each $x \in M_0$ and any natural n we can find δ^n_x, $0 < \delta^n_x < 1/n$ such that for each $y \in (x, x + \delta^n_x)$ it is

$$\mu_e\left\{z \in (x, y);\ f(z) \leqq \alpha - \frac{1}{n}\right\} < \tfrac{1}{4}(y - x)\,.$$

We put $G_n = \bigcup_{x \in M_0} (x, x + \frac{1}{4}\delta^n_x)$, then $M_0 - G_n$ is a countable set. Let $G = \bigcap_{n=1}^{\infty} G_n$. Then G is a G_δ set. Let $x \in G \cap R$. If $f'_{ap}(x) \geqq 0$, $\varepsilon > 0$, then we can choose $\delta > 0$ such that for each $0 < h < \delta$ it is

$$\mu_e\{z \in (x - h, x),\ f(z) \geqq f(x) + \tfrac{1}{2}\varepsilon\} < \tfrac{1}{4}h\,.$$

As $x \in G$ we can find $y \in M_0$ and $n > \max(1/\delta, 2/\varepsilon)$ such that $x \in (y, y + \frac{1}{4}\delta^n_y)$. Hence it follows that $y \in (x - \delta, x)$ and

$$\mu_e\left\{z \in (y, x),\ f(z) \leqq \alpha - \frac{1}{n}\right\} < \tfrac{1}{4}(y - x)\,,$$

$$\mu_e\{z \in (y, x), f(z) \geqq f(x) + \tfrac{1}{2}\varepsilon\} < \tfrac{1}{4}(y - x)\,.$$

Thus there exists $z_1 \in (y, x)$ such that $f(z_1) \geqq \alpha - 1/n$, $f(z_1) \leqq f(x) + \frac{1}{2}\varepsilon$ and therefore $f(x) \geqq \alpha - \varepsilon$. If $f'_{ap}(x) \leqq 0$, $\varepsilon > 0$, then we can choose $\delta > 0$ such that for each $0 < h < \delta$ it is

$$\mu_e\{z \in (x, x + h),\ f(z) \geqq f(x) + \tfrac{1}{2}\varepsilon\} < \tfrac{1}{4}h\,.$$

As $x \in G$ we can find $y \in M_0$ and $n > \max(1/\delta, 2/\varepsilon)$ such that $x \in (y, y + \frac{1}{4}\delta_y^n)$. Hence it follows that $y + \delta_y^n \in (x, x + \delta)$ and

$$\mu_e\{z \in (x, y + \delta_y^n), f(z) \geqq f(z) + \tfrac{1}{2}\varepsilon\} < \tfrac{1}{4}(y + \delta_y^n - x),$$

$$\mu_e\left\{z \in (x, y + \delta_y^n), f(z) \leqq \alpha - \frac{1}{n}\right\} \leqq \mu_e\left\{z \in (y, y + \delta_y^n), f(z) \leqq \alpha - \frac{1}{n}\right\} \leqq$$

$$\leqq \tfrac{1}{4}\delta_y^n \leqq \tfrac{1}{3}(y + \delta_y^n - x).$$

Thus there exists $z_1 \in (x, y + \delta_y^n)$ such that $f(z_1) \geqq \alpha - 1/n$, $f(z_1) \leqq f(x) + \frac{1}{2}\varepsilon$ and therefore $f(x) \geqq \alpha - \varepsilon$. Thus $M_0 \subset G^1 \cup S^1 \subset M$ where G^1 is a G_δ set with respect to R and S^1 is a countable set. Similarly we prove that

$$M \cap \{x \in R, f'_{ap}(x) \leqq 0\} \subset G^2 \cup S^2 \subset M$$

where G^2 is a G_δ set with respect to R and S^2 is a countable set.

This implies that $M = G_\alpha^+ \cup S_\alpha^+$ where G_α^+ is a G_δ set with respect to R and S_α^+ is a countable set. Similarly we prove that $\{x \in R; f(x) \leqq \alpha\} = G_\alpha^- \cup S_{\alpha+}^-$ where G_α^- is a G_δ set with respect to R and S_α^- is a countable set.

We put $Z = \bigcup_{r \text{ rational}} (S_r^+ \cup S_r^-)$. Then $\{x \in R; f(x) \geqq \alpha\} = \bigcap_{r \text{ rational}, r < \alpha} G_r^+ \cup S$ where $S \subset Z$ and similar result is true for $\{x \in R, f(x) \leqq \alpha\}$.

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