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Donald R. LaTorre

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ON SEMIGROUPS THAT ARE SEMILATTICES OF GROUPS

DONALD R. LATORRE, Clemson

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Let S be a semigroup. Following LAJOS [1], S is said to have property (M) if $L \cap R = LR$ for every left ideal L and every right ideal R of S; and S has property (L) [(R)] if $I_1 \cap I_2 = I_1 I_2$ for every pair of left [right] ideals I_1, I_2 of S.

Lajos proved that if S has property (M) then S is a semilattice of groups, and observed that a similar argument produced the same conclusion if S had both properties (L) and (R). In the following theorem it is shown that property (M) is equivalent to the conjunction of properties (L) and (R), and that each of these equivalent assertions is equivalent to S being a semilattice of groups.

Theorem. *The following conditions on a semigroup S are mutually equivalent.*

(A) *S has properties* (L) *and* (R).
(B) *All ideals of S are two-sided and $I_1 \cap I_2 = I_1 I_2$ for every pair of ideals I_1, I_2 of S.*
(C) *S has property* (M).
(D) *S is normal and regular.*
(E) *S is a semilattice of groups.*

Remark. SCHWARZ [2] calls S normal if $aS = Sa$ for all $a \in S$. In [3], Lajos proved that a normal semigroup is regular if and only if every left ideal of S is idempotent.

Proof. (A) implies (B) as follows. If L is a left ideal of S then $L = L \cap S = LS$ by property (L), so L is also a right ideal. Similarly, property (R) implies every right ideal is a two-sided ideal. The condition $I_1 \cap I_2 = I_1 I_2$ is clear.

It is obvious that (B) implies (C). Now (C) implies (D) by Theorems 2 and 3 of [1], and (D) implies (E) by the proof of Theorem 4 of [1].

We now show in turn that (E) implies (D), and (D) implies (A), which will complete the proof. Assuming (E), say $S = \bigcup_{\alpha \in \Omega} G_\alpha$ is a semilattice Ω of groups G_α $(\alpha \in \Omega)$, regularity of S is immediate. If $a, s \in S$, say $a \in G_\alpha$, $s \in G_\sigma$, then $as \in G_\alpha G_\sigma \subseteq G_{\alpha\sigma} =$

$= G_{\sigma\alpha}$ since Ω is a semilattice, and $sa \in G_\sigma G_\alpha \subseteq G_{\sigma\alpha}$. Since as, $sa \in G_{\sigma\alpha}$, $as = x(sa)$ for some $x \in G_{\sigma\alpha}$, whence $aS \subseteq Sa$ follows. Similarly, $Sa \subseteq aS$ and so S is normal. Finally, we show that (D) implies (A). If L is a left ideal of S, $a \in L$, and $s \in S$, then $as \in aS = Sa \subseteq SL \subseteq L$ using normality, whence L is a right ideal of S. Thus if L_1, L_2 are left ideals of S we have $L_1L_2 \subseteq L_1 \cap L_2$ since L_1 is a right ideal. If $a \in L_1 \cap$ $\cap L_2$, regularity gives $a = axa$ for some $x \in S$, so that $a = a(xa) \in L_1L_2$. Thus $L_1 \cap L_2 \subseteq L_1L_2$, which establishes property (L). Property (R) is proved similarly.

Added in Proof: Recently, S. LAJOS proved the equivalence of (C) and (E) [See Math. Reviews 40 #5757]. The Theorem was also observed by D. W. MILLER (unpublished).

References

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Author's address: Clemson University, Department of Mathematics, Clemson, South Carolina 29631, U.S.A.