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NEAR DOMAINS AS LINEAR PSEUDO TERNARIES

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H. KARZEL investigated in [2], §11 near domains with regard to sharply doubly transitive permutation groups. The purpose of the present Note is to characterize near domains as coordinatizing 3-groupoids of certain pseudo planes (pseudo planes were introduced by R. SANDLER in [3], p. 301). This topic is a generalization of the classical considerations of M. HALL presented in [1], chap. IV, §3.

By a *3-groupoid* we mean a non-void set S together with a ternary operation $\tau : S^3 \to S$. A 3-groupoid (S, τ) is called a *pseudo ternary* (cf. [3], p. 304) if two elements $0 \neq 1$ of S are distinguished such that $\tau(a, 0, b) = \tau(0, a, b) = b$, $\tau(1, a, 0) = \tau(a, 1, 0) = a$ $\forall a, b \in S$ and if to any

$$\begin{cases}(b, c, d) \in (S \setminus \{0\}) \times S^2 \\ (a, c, d) \in (S \setminus \{0\}) \times S^2 \\ (a, b, d) \in S^3\end{cases} \text{there exists just one} \begin{cases} a \in S \\ b \in S \\ c \in S\end{cases} \text{satisfying}$$

$\tau(a, b, c) = d$.

If there is given a pseudo ternary (S, τ) then define binary operations $+_\tau : S^2 \to S$, $\cdot_\tau : S^2 \to S$ by the rules $a +_\tau b := \tau(a, 1, b)$, $a \cdot_\tau b := \tau(a, b, 0)$ $\forall a, b \in S$. A pseudo ternary (S, τ) is said to be *linear* if $\tau(a, b, c) = (a \cdot_\tau b) +_\tau c$ $\forall a, b, c \in S$. If $T = (S, \tau)$ is a pseudo ternary then define for any $(u, v) \in (S \setminus \{0\}) \times S$ the permutation $\sigma_{u,v}$ of S by the rule $\sigma_{u,v}(x) = \tau(x, u, v)$ $\forall x \in S$. Further put $\Sigma_T := \{\sigma_{u,v} \mid (u, v) \in \in (S \setminus \{0\}) \times S\}$. Let us remark that $\sigma_{u_1,v_1} \neq \sigma_{u_2,v_2}$ if $(u_1, v_1) \neq (u_2, v_2)$. Finally let us introduce the notation $\leftarrow a$, $\rightarrow a$ for the solutions of $x + a = 0$ and $a + y = 0$ according to a given loop $(S, +)$ with neutral element 0

Begin with two simple assertions: *Let $T = (S, \tau)$ be a linear pseudo ternary. Then $(\Sigma_T, \circ)$ is a semigroup* (where $\circ$ is the usual composition of maps) *if and only if to any $(u_1, v_1), (u_2, v_2) \in (S \setminus \{0\}) \times S$ there exists a* (unique) $(u_3, v_3) \in (S \setminus \{0\}) \times \times S$ *such that*

$$(1) \qquad (((x \cdot_\tau u_1) +_\tau v_1) \cdot_\tau u_2) +_\tau v_2 = (x \cdot_\tau u_3) +_\tau v_3 \quad \forall x \in S\,.$$

(The proof is simple and will be omitted.)

If $T = (S, \tau)$ is a linear pseudo ternary such that $(\Sigma_T, \circ)$ is a semigroup then $(S \setminus \{0\}, \cdot_\tau)$ is a group.

Proof. Using (1) for $v_1 = v_2 = 0$ we get $(x \cdot_\tau u_1) \cdot_\tau u_2 = x \cdot_\tau u_3 +_\tau v_3$. Putting $x = 0$ we conclude $v_3 = 0$ whereas $x = 1$ yields $u_1 \cdot_\tau u_2 = u_3$. Thus $(x \cdot_\tau u_1) \cdot_\tau u_2 = x \cdot_\tau (u_1 \cdot_\tau u_2)$. But $(S \setminus \{0\}, \cdot_\tau)$ is a loop so that it is even a group. Q.E.D.

Recall that a *near domain* ([2], p. 123) is defined as a triple $(S, +, .)$ having the following properties

(i) $(S, +)$ is a loop with the neutral element 0,

(ii) $(S \setminus \{0\}, .)$ is a group with the neutral element 1,

(iii) $(a + b) . c = a . c + b . c \quad \forall a, b, c \in S$,

(iv) $a . 0 := 0, \ 0 . a := 0 \quad \forall a \in S$,

(v) $(a + b) + c = (a . d_{b,c}) + (b + c) \quad \forall a, b, c \in S$ where $d_{b,c}$ is the solution of $(1 + b) + c = x + (b + c)$,

(vi) $(1 + a) + (\rightarrow a) = 1 \ \forall a \in S$.

If $D = (S, +, .)$ is a near domain then denote by $\sigma_{u,v}$ the permutation of S determined by $\sigma_{u,v}(x) := (x . u) + v \ \forall x \in S$ for any given $(u, v) \in (S \setminus \{0\}) \times S$. Further put $\Sigma_D := \{\sigma_{u,v} \mid (u, v) \in (S \setminus \{0\}) \times S\}$.

Remark that for any near domain $(S, +, .)$, $\leftarrow a = \rightarrow a$ holds for all $a \in S$ so that we can use a simpler notation $-a$. Further it can be proved that $(-a) . b = a . (-b) = -(a . b) \ \forall a, b \in S$.

As there is shown in [2], pp. 124–125 for any near domain $D = (S, +, .)$, $(\Sigma_D, \circ)$ is a sharply doubly transitive permutation group on S and conversely, each sharply doubly transitive permutation group G on a set S (with at least two elements) determines a unique near domain D such that $(\Sigma_D, \circ) = G$.

Theorem 1. *If $T = (S, \tau)$ is a linear pseudo ternary such that $(\Sigma_T, \circ)$ is a semigroup and that for $+ := +_\tau, \cdot := \cdot_\tau$*

(2) $$\sigma^2_{\rightarrow 1, v} = id \quad \forall v \in S,$$

(3) $$\sigma^2_{u,0} = id \quad \text{for} \quad u \neq 1 \quad \text{implies} \quad u = \rightarrow 1,$$

then $(S, +, .)$ is a near domain.

Proof. Rewrite (2) as

(4) $$(((x . (\rightarrow 1)) + v) . (\rightarrow 1)) + v = x \quad \forall v, x \in S.$$

Putting here $v = x = 0$ we get $(\to 1) . (\to 1) = 1$. Similarly, for $v = x = 1$ we obtain $(((\to 1) + 1) . (\to 1)) + 1 = 1$ which implies $\to 1 = \leftarrow 1 = : -1$. Let $(u_1, v_1), (1, v_2) \in (S \setminus \{0\}) \times S$ so that there is a unique $u_3 \in S \setminus \{0\}$ such that

$$(5) \qquad ((x . u_1) + v_1) + v_2 = (x . u_3) + (v_1 + v_2) \quad \forall x \in S .$$

For $x = u_1^{-1}$ we obtain $(1 + v_1) + v_2 = (u_1^{-1} . u_3) + (v_1 + v_2)$, i.e., $u_3 = u_1 . d_{v_1,v_2}$ and (v) is fulfilled. If $1 \neq -1$ then for each $a \in S \setminus \{0\}$ we obtain $a . (-1) . a^{-1} \neq 1$ and $\sigma^2_{a\cdot(-1)\cdot a^{-1}} = id$ so that by (3) $a . (-1) . a^{-1} = -1$ and consequently $a . (-1) = (-1) . a$. This last equation is trivial for $a = 0$ and also for all $a \in S$ if $1 = -1$. Thus

$$(6) \qquad a . (-1) = (-1) . a \quad \forall a \in S .$$

By (4) for $x = v$ we obtain $(((v . (-1)) + v) . (-1)) + v = 1$ so that $v . (-1) = \leftarrow v$ for all $v \in S$. Consequently

$$(7) \qquad (\leftarrow a) . b = a . (\leftarrow b) = \leftarrow (a . b) \quad \forall a, b \in S .$$

Now let $(1, v_1), (u_2, 0) \in (S \setminus \{0\}) \times S$. So there is a unique $(u_3, v_3) \in (S \setminus \{0\}) \times S$ such that $(x + v_1) . u_2 = x . u_3 + v_3$ $\forall x \in S$. If we choose $x = 0$ then $v_3 = v_1 . u_2$ whereas $x = \leftarrow v_1$ yields $((\leftarrow v_1) . u_3) + (v_1 . u_2) = 0$, i.e., $(\leftarrow v_1) . u_3 = \leftarrow (v_1 . u_2)$. Therefore by (7) $(\leftarrow v_1) . u_3 = (\leftarrow v_1) . u_2$ and consequently $u_3 = u_2$. Thus the distributive law (iii) holds. More generally, the preceding investigations in connexion with (1) yield

$$(8) \qquad (((x . u_1) + v_1) . u_2) + v_2 = (x . (u_1 . u_2)) + (v_1 . u_2 + v_2) \quad \forall x, u_1, u_2, v_1, v_2 \in S .$$

Now $0 = 0 . (-1) = (a . (-1) + a) . (-1) = a + (\leftarrow a)$ so that $\leftarrow a = \to a := -a$ for all $a \in S$. Using (8) for $u_1 = u_2 = x = 1$, $v_2 = -v_1$ we verify (vi). Q.E.D.

If a linear pseudo ternary $T = (S, \tau)$ satisfies all the assumptions of Theorem 1 then by the results of Karzel mentioned above $(\Sigma_T, \circ)$ is a group and for any $(x_1, y_1), (x_2, y_2) \in (S \setminus \{0\}) \times S$ with $x_1 \neq x_2$ there is precisely one $(u, v) \in (S \setminus \{0\}) \times S$ satisfying $\tau(x_i, u, v) = y_i$, $i = 1, 2$.

Theorem 2. *For any near domain $D = (S, +, .)$ there is just one linear pseudo ternary (S, τ) such that $+ = +_\tau$, $\cdot = \cdot_\tau$, that $(\Sigma_D, \circ)$ is a semigroup and (2), (3) hold.*

Proof. Define $\tau : S^3 \to S$ by the rule $\tau(a, b, c) := (a . b) + c$ $\forall a, b, c \in S$. As immediate consequences of near domain properties (i) to (vi) we get that (S, τ) is a linear pseudo ternary such that $+_\tau = +$, $\cdot_\tau = \cdot$, that $(\Sigma_D, \circ)$ is a semigroup and that (2) is valid. The only non-trivial assertion is the validity of the remaining condition (3). This can be deduced as follows. By [2], pp. 126–128 $\{\{(x, y) \mid y = \sigma_{u,v}(x)\} \mid (u, v) \neq (1, 0), \sigma^2_{u,v} = id\}$ and $\{\{(x, y) \mid y = \sigma_{u,v}(x)\} \mid \sigma^2_{u,v} = id\}$ in case

$1 \neq -1$ or $1 = -1$, respectively are decompositions of S^2 into pairwise disjoint non-void subsets. But $\{\{(x, y) \mid y = x\,.\,(-1) + v\} \mid v \in S\}$ must be the same decomposition so that consequently $\{(x, y) \mid y = \sigma_{u,0}(x)\}$, $u \neq 1 = u^2$, is one term of it and therefore $u = -1$. The uniqueness of this (S, τ) already follows from the linearity property and from $+_\tau = +$, $\cdot_\tau = \cdot$. Q.E.D.

Now we are able to interpret simply Karzel's necessary and sufficient condition a) for a near domain $D = (S, +, \cdot)$ to be a near field (i.e. such that $(S, +)$ is a group), b) for a near field $D = (S, +, .)$ to be „projective" (i.e. such that the equation $x\,.\,a = (x\,.\,b) + c$ is uniquely solvable through $x \in S$ for all $a, b, c \in S$ with $a \neq b$).

In the first case the Karzel's condition ([2], p. 132) reads that for $J := \{\sigma_{u,v} \mid (u, v) \neq (1, 0), \sigma^2_{u,v} = id\}$, J^2 forms a subgroup in $(\Sigma_D, \circ)$. This means in our interpretation that $(S, +_\tau)$ is a group because of $\sigma_{-1,v_2} \circ \sigma_{-1,v_1} = \sigma_{1,-v_1+v_2}$ $\forall v_1, v_2 \in S$.

In the second case the Karzel's condition ([2], p. 135) reads that all $\sigma_{u,v} \in \Sigma_D$ fixing no elements belong to J^2. But this means in our interpretation that $\{(x, y) \mid y = \tau(x, u, v)\} \cap \{(x, y) \mid y = \tau(x, 1, 0)\} = \emptyset \Rightarrow u = 1$, i.e., $\{(x, y) \mid y = \tau(x, u, v)\} \cap \{(x, y) \mid y = \tau(x, 1, 0)\} \neq \emptyset$ for all $(u, v) \in (S \setminus \{0, 1\}) \times S$ and this gives already the statement that D is projective.

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