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DEFORMATIONS OF PLANE PSEUDOCONGRUENCES WITH PROJECTIVE CONNECTION

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Using basic ideas, conceptions and results introduced in [1], [2], [3], the elementary and projective deformations of pseudocongruences of planes with projective connection are studied and mutual relations among individual deformations are characterized.

1. A special König space $\mathscr{P}^3_{2,5}$ let be constructed according to [1], p. 71–72. Using the notation of GEJDELMAN ([4], p. 281), we shall call these spaces plane pseudocongruences with projective connection.

Let a plane speudocongruence $\mathscr{L}$ with projective connection be given by the equations

$$\nabla A_i = \sum_{j=1}^{6} \omega_{ij} A_j \tag{1.1}$$
$$\omega_{ij} = a^1_{ij}(u_1, u_2, u_3)\,\omega_1 + a^2_{ij}(u_1, u_2, u_3)\,\omega_2 + a^3_{ij}(u_1, u_2, u_3)\,\omega_3$$
$$\sum_{i=1}^{6} \omega_{ii} = 0 \quad (i, j = 1, 2 \ldots 6)$$

where $\omega_1, \omega_2, \omega_3$ are the Pfaff forms in the differentials du_1, du_2, du_3, $\omega_1 \wedge \omega_2 \wedge \wedge \omega_3 \neq 0$. The planes of the pseudocongruence are $P_2 = (A_1, A_2, A_3)$. We call the developable varieties $\mathscr{R}_3$ of $\mathscr{L}$ (corresponding to the curves of Ω_3) varieties with developable developments. The development of the variety $\mathscr{R}_3$ generated by planes $P_2(u_1, u_2, u_3)$ where $u_i = u_i(t)$, $(i = 1, 2, 3)$ is a developable variety if the dimension of their tangent spaces along a generating plane P_2 is less than five. In this case, each plane intersects each consecutive plane at least at the point which is called the focus. The equation of developable varieties of the speudocongruence $\mathscr{L}$ is ([2], p. 58)

$$[A_1, A_2, A_3, \nabla A_1, \nabla A_2, \nabla A_3] = 0\,. \tag{1.2}$$

The first term of (1.2) is a cubical form in du_i $(i = 1, 2, 3)$. We restrict ourselves to such pseudocongruences whose form mentioned above is the product of three independent forms in du_i. Let us denote them by $\omega_1, \omega_2, \omega_3$. The equation (1.2) reduces to

$$\omega_1\omega_2\omega_3 = 0 .$$

Let us introduce an important convention. If nothing other is mentioned then in all our considerations it will be always

$$s = i + 1 , \quad i + 2 \quad (i = 1, 2, 3) \tag{1.3}$$

and the indices $i, i + 1, \ldots, i + 5$ change according to the scheme

(1.4)

i	1	2	3
$i + 1$	2	3	1
$i + 2$	3	1	2
$i + 3$	4	5	6
$i + 4$	5	6	4
$i + 5$	6	4	5

where always $i = 1, 2, 3$.

We shall deal with such pseudocongruences $\mathscr{L}$ only where for $\omega_i = 0$ (ω_{i+1}, ω_{i+2} – arbitrary) there exists just one focus and the three foci considered do not lie on one straight line. Let us choose these foci to be the points A_1, A_2, A_3.

A point A_i to be a focus then

$$[(\nabla A_i)_{\omega_i=0}, A_1, A_2, A_3] = 0 ,$$

i.e.

$$a^s_{i,i+3} = a^s_{i,i+4} = a^s_{i,i+5} = 0 .$$

The fundamental equations are

$$\nabla A_i = \sum_{j=1}^{3} \omega_{ij} A_j + \sum_{j=4}^{6} a^i_{ij} A_j \omega_i .$$

Using the specialization

$$\sum_{j=4}^{6} a^i_{ij} A_j \to A_{i+3}$$

we obtain the fundamental equations in the form

$$\nabla A_i = \omega_i A_{i+3} + \sum_{j=1}^{3} \omega_{ij} A_j , \tag{1.5}$$

$$\nabla A_{i+3} = \sum_{j=1}^{6} \omega_{i+3,j} A_j .$$

If we substitute (in each local space of the pseudocongruence $\mathscr{L}$) the plane $P_2 = (A_1, A_2, A_3)$ by the point A_i, we obtain a variety with projective connection, the s.c. focal variety (A_i) of $\mathscr{L}$. Let A_i be a fixed point of the focal variety (A_i). The developments of all the curves of the focal variety into the focal space of A_i are curves with tangents lying in the plane (A_1, A_2, A_3, A_{i+3}), the s.c. tangent plane of the focal variety (A_i). This is the focal plane of $\mathscr{L}$.

2. Let $\mathscr{L}$ be a plane pseudocongruence with projective connection given by the equations (1.1). We restrict our consideration to the case when all three focal varieties are of the dimension three. After a suitable specialization of frames, we obtain (1.5). Without any loss of generality, we may assume

$$\omega_i = \mathrm{d}u_i$$

and we have the equations

$$\nabla A_i = \mathrm{d}u_i A_{i+3} + \sum_{j=1}^{3} \omega_{ij} A_j, \tag{2.1}$$

$$\omega_{ij} = a_{ij}^1 \,\mathrm{d}u_1 + a_{ij}^2 \,\mathrm{d}u_2 + a_{ij}^3 \,\mathrm{d}u_3 .$$

The variation of parameters and local frames is said to be compatible with some system of equations in ω_{ij} if the transformed formes $\bar{\omega}_{ij}$ satisfy the same system of equations.

The variations of parameters and local frames compatible with

$$\omega_{i,i+3} = \mathrm{d}u_i, \quad \omega_{i,s+3} = 0 \tag{2.2}$$

are given by

$$u_i = u_i(\bar{u}_i) \tag{2.3}$$

$$A_i = \mu_{ii} \bar{A}_i, \quad A_{i+3} = \sum_{j=1}^{6} \mu_{i+3,j} \bar{A}_j \tag{2.4}$$

where

$$\mu_{11}\mu_{22}\mu_{33} \det |\mu_{i+3,j}| = 1, \quad (j = 4, 5, 6).$$

Substituting into (2.1), we get

$$\mu_{ii} \nabla \bar{A}_i = \omega_{i,i+1}\mu_{i+1,i+1}\bar{A}_{i+1} + \omega_{i,i+2}\mu_{i+2,i+2}\bar{A}_{i+2} + $$
$$+ \mathrm{d}u_i(\mu_{i+3,i+1}\bar{A}_{i+1} + \mu_{i+3,i+2}\bar{A}_{i+2} + $$
$$+ \mu_{i+3,i+3}\bar{A}_{i+3} + \mu_{i+3,i+4}\bar{A}_{i+4} + \mu_{i+3,i+5}\bar{A}_{i+5}) \pmod{\bar{A}_i}$$

and

$$\bar{\omega}_{is} = \mu_{ii}^{-1}(\omega_{is}\mu_{ss} + \mathrm{d}u_i \mu_{i+3,s}), \tag{2.5}$$

$$\mu_{i+3,i+3} = \mu_{ii} \frac{\mathrm{d}\bar{u}_i}{\mathrm{d}u_i},$$

$$\mu_{i+3,s+3} = 0 .$$

Lemma 1. *The variations compatible with* (2.2) *are given by* (2.3) *and*

$$(2.6)\qquad A_i = \mu_{ii}\bar{A}_i\,, \quad A_{i+3} = \mu_{ii}\frac{d\bar{u}_i}{du_i}\bar{A}_{i+3} + \sum_{j=1}^{3}\mu_{i+3,j}\bar{A}_j$$

where with respect to (2.4), (2.5),

$$\mu_{11}^2\mu_{22}^2\mu_{33}^2\frac{d\bar{u}_1\,d\bar{u}_2\,d\bar{u}_3}{du_1\,du_2\,du_3} = 1\,.$$

From (2.5), we get

$$(2.7)\qquad \begin{aligned} \bar{a}^i_{i,i+1} &= \mu_{ii}^{-1}(\mu_{i+1,i+1}a^i_{i,i+1} + \mu_{i+3,i+1})\frac{du_i}{d\bar{u}_i}\,,\\ a^s_{is} &= \mu_{ii}^{-1}\mu_{ss}a^s_{is}\frac{du_s}{d\bar{u}_s}\,. \end{aligned}$$

Substituting (2.6) into $(1.5_{4,5,6})$, we get

$$\mu_{i+3,i}\nabla\bar{A}_i + \mu_{i+3,i+1}\nabla\bar{A}_{i+1} + \mu_{i+3,i+2}\nabla\bar{A}_{i+2} + \mu_{ii}\frac{d\bar{u}_i}{du_i}\nabla\bar{A}_{i+3} \equiv$$

$$\equiv \omega_{i+3,i+4}\mu_{i+1,i+1}\frac{d\bar{u}_{i+1}}{du_{i+1}}\bar{A}_{i+4} + \omega_{i+3,i+5}\mu_{i+2,i+2}\frac{d\bar{u}_{i+2}}{du_{i+2}}\bar{A}_{i+5}$$

$$(\mathrm{mod}\ \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_{i+3})$$

and

$$(2.8)\qquad \begin{aligned} \bar{a}^i_{i+3,i+4} &= \mu_{ii}^{-1}\mu_{i+1,i+1}\left(\frac{du_i}{d\bar{u}_i}\right)^2\frac{d\bar{u}_{i+1}}{du_{i+1}}a^i_{i+3,i+4}\\ \bar{a}^{i+1}_{i+3,i+4} &= \mu_{ii}^{-1}(\mu_{i+1,i+1}a^{i+1}_{i+3,i+4} - \mu_{i+3,i+1})\frac{du_i}{d\bar{u}_i}\\ \bar{a}^{i+2}_{i+3,i+4} &= \mu_{ii}^{-1}\mu_{i+1,i+1}\frac{du_i}{d\bar{u}_i}\frac{d\bar{u}_{i+1}}{du_{i+1}}\frac{du_{i+2}}{d\bar{u}_{i+2}}a^{i+2}_{i+3,i+4}\,. \end{aligned}$$

From (2.7) and (2.8), we obtain

$$\bar{a}^i_{is} - \bar{a}^s_{i+3,s+3} = \mu_{ii}^{-1}[\mu_{ss}(a^i_{is} - a^s_{i+3,s+3}) + 2\mu_{i+3,s}]\frac{du_i}{d\bar{u}_i}\,.$$

We may specialize the frames in such a way that

$$(2.9)\qquad a^i_{is} - a^s_{i+3,s+3} = 0$$

and

$$\mu_{i+3,s} = 0\,.$$

We obtain

$$(2.10)\qquad A_i = \mu_{ii}\bar{A}_i\,,\quad A_{i+3} = \mu_{i+3,i}\bar{A}_i + \mu_{ii}\frac{d\bar{u}_i}{du_i}\bar{A}_{i+3}\,.$$

Let us introduce the notation

$$(2.11)\qquad \begin{aligned} h_{is} &= a^i_{is} = a^s_{i+3,s+3})\,,\\ \nabla\alpha_{is} &= a^{i+1}_{is}\,du_{i+1} + a^{i+2}_{is}\,du_{i+2}\,,\\ \nabla\beta_{is} &= a^i_{i+3,s+3}\,du_i + a^s_{i+3,s+3}\,du_s\,. \end{aligned}$$

Lemma 2. *We may specialize the frames of a pseudocongruence $\mathscr{L}$ with projective connection in such a way that $\mathscr{L}$ is given by the equations*

$$(2.12)\qquad \begin{aligned} \nabla A_i &= \omega_{ii}A_i + \left(h_{i,i+1}\,du_i + \nabla\alpha_{i,i+1}\right)A_{i+1} +\\ &\quad + \left(h_{i,i+2}\,du_i + \nabla\alpha_{i,i+2}\right)A_{i+2} + du_iA_{i+3}\,,\\ \nabla A_{i+3} &= \omega_{i+3,1}A_1 + \omega_{i+3,2}A_2 + \omega_{i+3,3}A_3 + \omega_{i+3,i+3}A_{i+3} +\\ &\quad + \left(h_{i,i+1}\,du_{i+1} + \nabla\beta_{i,i+1}\right)A_{i+4} + \left(h_{i,i+2}\,du_{i+2} + \nabla\beta_{i,i+2}\right)A_{i+5}\,. \end{aligned}$$

The most general variation compatible with (2.2) and (2.9) is (2.3) and (2.10). After these variations we obtain

$$(2.13)\qquad \bar{h}_{is} = \mu_{ii}^{-1}\mu_{ss}\frac{du_i}{d\bar{u}_i}h_{is}$$

$$\nabla\bar{\alpha}_{is} = \mu_{ii}^{-1}\mu_{ss}\nabla\alpha_{is}\,,\quad \nabla\bar{\beta}_{is} = \mu_{ii}^{-1}\mu_{ss}\frac{du_i}{d\bar{u}_i}\frac{d\bar{u}_s}{du_s}\nabla\beta_{is}\,.$$

3. The dualization $\mathscr{L}^*$ of $\mathscr{L}$ is defined by the construction B ([1], p. 73). $\mathscr{L}^*$ is again a plane pseudocongruence with projective connection. Using the dual frames

$$(3.1)\qquad E^j = (-1)^{j+1}\left[A_1 \ldots A_{j-1}, A_{j+1} \ldots A_6\right],\quad (j = 1, 2 \ldots 6)\,,$$

the pseudocongruence $\mathscr{L}^*$ is formed by the planes $P_2^* = [E^4, E^5, E^6]$ (P_2^* being the local centers of $\mathscr{L}^*$) and the connection is given by the equations

$$(3.2)\qquad \begin{aligned} \nabla E^{i+3} &= -\,du_iE^i - \omega_{i+3,i+3}E^{i+3} - \left(h_{i+1,i}\,du_i + \nabla\beta_{i+1,i}\right)E^{i+4} -\\ &\quad - \left(h_{i+2,i}\,du_i + \nabla\beta_{i+2,i}\right)E^{i+5}\,,\\ \nabla E^i &= -\,\omega_{ii}E^i - \left(h_{i+1,1}\,du_{i+1} + \nabla\alpha_{i+1,i}\right)E^{i+1} -\\ &\quad - \left(h_{i+2,i}\,du_{i+2} + \nabla\alpha_{i+2,i}\right)E^{i+2} - \omega_{4,i}E^4 - \omega_{5,i}E^5 - \omega_{6,i}E^6\,. \end{aligned}$$

As a consequence of passing to the dualization, we obtain the following substitution

$$(3.3)\qquad \left|\begin{matrix} \mathscr{L} & A_i & A_{i+3} & E^i & E^{i+3} & du_i & h_{is} & \nabla\alpha_{is} & \nabla\beta_{is} \\ \mathscr{L}^* & E^{i+3} & E^i & A_{i+3} & A_i & -du_i & h_{si} & -\nabla\beta_{si} & -\nabla\alpha_{si} \end{matrix}\right\downarrow$$

$$\left|\begin{matrix} \mathscr{L} & \omega_{ii} & \omega_{i+3,i+3} & \omega_{i+3,i} & \omega_{i+4,i} \\ \mathscr{L}^* & \omega_{i+3,i+3} & \omega_{ii} & \omega_{i+3,i} & \omega_{i+3,i+1} \end{matrix}\right\downarrow$$

$$\left|\begin{matrix} \mathscr{L} & \omega_{i+5,i} & \omega_{i+3,i+1} & \omega_{i+3,i+2} \\ \mathscr{L}^* & \omega_{i+3,i+2} & \omega_{i+1,i} & \omega_{i+5,i} \end{matrix}\right\downarrow .$$

The natural correspondence $\mathscr{L} \to \mathscr{L}^*$ is hence developable.

Let us find the asymptotic curves of the focal varieties of the pseudocongruence $\mathscr{L}$ and $\mathscr{L}^*$. The asymptotic curves on the focal variety (A_i) are given by the equation

$$[A_1, A_2, A_3, \nabla^2 A_i] = 0$$

and they are

$$(3.4)\qquad (h_{is}\, du_i + \nabla\alpha_{is})\, du_s + (h_{is}\, du_{i+2} + \nabla\beta_{is})\, du_i = 0 .$$

The asymptotic lines on the focal variety (E^{i+3}) are

$$(3.5)\qquad (h_{si}\, du_s + \nabla\alpha_{si})\, du_i + (h_{si}\, du_i + \nabla\beta_{si})\, du_s = 0 .$$

On (A_i) let us choose a layer $du_{i+2} = 0$ or $du_{i+1} = 0$ and let us consider the bundle of nets determined by the nets $du_i\, du_{i+1} = 0$ or $du_i\, du_{i+2} = 0$ and (3.4_1) or (3.4_2) respectively. In this bundle, there exists a net apolar to the net $du_i\, du_{i+1} = 0$ or $du_i\, du_{i+2} = 0$ respectively. This net is given by the equations

$$(3.6)\qquad \nabla\alpha_{is}\, du_s + \nabla\beta_{is}\, du_i = 0 .$$

Let us call the curves determined by (3.6) pseudoasymptotic curves on the variety (A_i).

The pseudoasymptotic curves on the variety (E^{i+3}) will be given by the equations

$$(3.7)\qquad \nabla\alpha_{si}\, du_i + \nabla\beta_{si}\, du_s = 0 .$$

Using (2.13) we obtain invariant forms which are important for the study of deformations of pseudocongruences. They are: Point forms

$$(3.8)\qquad \varphi_i = \nabla\alpha_{i+1,i+2}\, \nabla\alpha_{i+2,i+1}$$

$$J_1 = \nabla\alpha_{12}\, \nabla\alpha_{23}\, \nabla\alpha_{31}\,, \quad J_2 = \nabla\alpha_{21}\, \nabla\alpha_{32}\, \nabla\alpha_{13}\,.$$

These forms are not independent. If we know any four of them, we may derive the fifth one. Their complex is called a point element of the pseudocongruence $\mathscr{L}$.

Hyperplanar forms

$$(3.9) \qquad \varphi_i^* = \nabla\beta_{i+1,i+2}\,\nabla\beta_{i+2,i+1}$$
$$J_1^* = \nabla\beta_{12}\,\nabla\beta_{23}\,\nabla\beta_{31}\,, \quad J_2^* = \nabla\beta_{21}\,\nabla\beta_{32}\,\nabla\beta_{13}\,.$$

These forms are dependent, too. Using any four of them, the fifth may be derived. Their complex is called a hyperplanar element of the pseudocongruence $\mathscr{L}$.

Focal forms

$$(3.10) \qquad F_{is} = \nabla\alpha_{is}\,\nabla\beta_{si}\,\frac{\mathrm{d}u_s}{\mathrm{d}u_i}\,,$$

pseudoasymptotic forms

$$(3.11) \qquad G_{is} = \frac{\nabla\alpha_{is}\,\mathrm{d}u_s}{\nabla\beta_{is}\,\mathrm{d}u_i}\,,$$

point and hyperplanar forms of the kind "i"

$$(3.12) \qquad g_{is} = \frac{h_{is}\,\mathrm{d}u_i}{\nabla\alpha_{is}}\,, \quad g_{is}^* = \frac{h_{is}\,\mathrm{d}u_s}{\nabla\beta_{is}}\,,$$

where

$$g_{is}^* = g_{is}G_{is}\,.$$

Finally let us add a group of invariant forms which are necessary for the study of projective deformations. Substituting from (2.6) into (2.12), we get

$$\mathrm{d}\mu_{ii}\bar{A}_i + \mu_{ii}\,\nabla\bar{A}_i \equiv (\mu_{ii}\omega_{ii} + \mu_{i+3,i}\,\mathrm{d}u_i)\,\bar{A}_i\,, \quad (\mathrm{mod}\ \bar{A}_{i+1}, \bar{A}_{i+2}, \bar{A}_{i+3})\,,$$

$$\mu_{i+3,i}\,\nabla\bar{A}_i + \mathrm{d}\left(\mu_{ii}\frac{\mathrm{d}\bar{u}_i}{\mathrm{d}u_i}\right)\bar{A}_{i+3} + \mu_{ii}\frac{\mathrm{d}\bar{u}_i}{\mathrm{d}u_i}\,\nabla\bar{A}_{i+3} \equiv$$
$$\equiv \omega_{i+3,i+3}\mu_{ii}\frac{\mathrm{d}\bar{u}_i}{\mathrm{d}u_i}\,\bar{A}_{i+3}\,, \quad (\mathrm{mod}\ \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_{i+4}, \bar{A}_{i+5})\,.$$

Hence

$$\bar{a}^s_{i+3,i+3} - \bar{a}^s_{ii} = (a^s_{i+3,i+3} - a^s_{ii})\,\frac{\mathrm{d}u_s}{\mathrm{d}\bar{u}_s}$$

and finally we obtain the invariant forms

$$(3.13) \qquad \psi_{is} = (a^s_{i+3,i+3} - a^s_{ii})\,\mathrm{d}u_s\,.$$

The substitution (3.3) will be completed by

$$(3.14) \qquad \left\downarrow \begin{matrix} \mathscr{L} & \varphi_i & \varphi_i^* & J_1 & J_2 & J_1 & J_2 & F_{is} & G_{is} & \psi_{is} \\ \mathscr{L}^* & \varphi_i^* & \varphi_i & -J_2 & -J_1 & -J_2 & -J_1 & F_{is} & 1/G_{is} & \psi_{is} \end{matrix} \right\downarrow .$$

4. Let $\mathscr{L}$ be a plane pseudocongruence with projective connection given by (2.12). Let $\tilde{\mathscr{L}}$ be another pseudocongruence; we denote all expressions connected with $\tilde{\mathscr{L}}$ by a tilde. Let the frames associated with $\tilde{\mathscr{L}}$ be specialized in the same way as those associated with $\mathscr{L}$.

Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a correspondence between $\mathscr{L}$ and $\tilde{\mathscr{L}}$ given by the equations

$$\mathrm{d}u_i = \sum_{j=1}^{3} m_{ij}\,\mathrm{d}u_j \tag{4.1}$$

where

$$\det |m_{ij}| \neq 0 .$$

The correspondence associates to a plane $P_2 \in \mathscr{L}$ a plane $\tilde{P}_2 \in \tilde{\mathscr{L}}$

$$CP_2 = \tilde{P}_2 .$$

The correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is called the projective deformation of order k if for each plane P_2 of the pseudocongruence $\mathscr{L}$ there exists a collineation $K : P_5 \to \tilde{P}_5$ such that the pseudocongruences $K\mathscr{L}$ and $\tilde{\mathscr{L}}$ have the analytic contact of order k along the plane $\tilde{P}_2 = CP_2$. We say that K realizes the projective deformation C of order k.

Now, we attend to the deformation of the first order. The conditions for the correspondence C to be a projective deformation of the first order consist in the existence of the collineation

$$K\tilde{A}_j = \sum_{r=1}^{6} c_{jr}A_r\,, \quad (j = 1, 2, \ldots, 6)$$

and such a form ϑ_1 that it holds

$$\begin{aligned} K[\tilde{A}_1, \tilde{A}_2, \tilde{A}_3] &= [A_1, A_2, A_3] \\ K\,\nabla[\tilde{A}_1, \tilde{A}_2, \tilde{A}_3] &= \nabla[A_1, A_2, A_3] + \vartheta_1[A_1, A_2, A_3] . \end{aligned} \tag{4.2}$$

From (4.2_1) we get

$$K\tilde{A}_i = \sum_{r=1}^{3} c_{ir}A_r\,,$$

$$\det |c_{ir}| = 1 .$$

From (4.2_2) it follows

$$\begin{aligned} \sum_{i=1}^{3}\sum_{r=4}^{6} A_iA_{i+1}A_r\{c_{i+3,r}(c_{i+1,i}c_{i+2,i+1} - c_{i+1,i+1}c_{i+2,i})\,\mathrm{d}\tilde{u}_i\,+ \\ + c_{i+4,r}(c_{i+2,i}c_{i,i+1} - c_{i,i}c_{i+2,i+1})\,\mathrm{d}\tilde{u}_{i+1}\,+ \\ + c_{i+5,r}(c_{i,i}c_{i+1,i+1} - c_{i,i+1}c_{i+1,i})\,\mathrm{d}\tilde{u}_{i+2}\} \equiv \sum_{i=1}^{3} A_{i+1}A_{i+2}A_{i+3}\,\mathrm{d}u_i\,. \end{aligned}$$

Hence
$$c_{i,s} = c_{i+3,s+3} = 0$$
$$du_i = c_{i+1,i+1}c_{i+2,i+2}c_{i+3,i+3}\, d\tilde{u}_i$$
and (4.1) may be reduced to
$$du_i = d\tilde{u}_i\,.$$

Proposition 1. *The correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is the projective deformation of the first order if and only if C is developable. The collineation realizing this deformation transforms the focal formations of the pseudocongruence $\mathscr{L}$ into the corresponding focal formations of the pseudocongruence $\tilde{\mathscr{L}}$.*

The tangent collineation K is of the form

$$(4.3)\qquad \begin{aligned} K\tilde{A}_i &= \varrho_i A_i \\ K\tilde{A}_{i+3} &= c_{i+3,i}A_i + c_{i+3,i+1}A_{i+1} + c_{i+3,i+2}A_{i+2} + \varrho_i A_{i+3} \end{aligned}$$

where

$$(4.4)\qquad \varrho_1\varrho_2\varrho_3 = 1$$

and

$$\tau_{ij} = \tilde{\omega}_{ij} - \omega_{ij}\,,$$
$$(4.5)\qquad \vartheta_1 = \sum_{i=1}^{3}(\tau_{ii} - \varrho_i^{-1}c_{i+3,i}\, du_i)\,.$$

The dual collineation $K^* : P_5^* \to \tilde{P}_5^*$ is given by

$$(4.6)\qquad \begin{aligned} K\tilde{E}^{*i+3} &= \varrho_i^{-1}E^{i+3}\,, \\ K\tilde{E}^{*i} &= -\varrho_i^{-2}c_{i+3,i}E^{i+3} - \varrho_{i+2}c_{i+4,i}E^{i+4} - \varrho_{i+1}c_{i+5,i}E^{i+5} + \varrho_i^{-1}E^i\,. \end{aligned}$$

This collineation is tangent to the correspondence $C : \mathscr{L}^* \to \tilde{\mathscr{L}}^*$.

With respect to Proposition 1 we shall suppose in further considerations that $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is a developable correspondence. Let it be given by

$$(4.7)\qquad d\tilde{u}_i = du_i\,.$$

The correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ induces correspondences between $\mathscr{L}$, $\mathscr{L}^*$, (A_i), (E^{i+3}) and $\tilde{\mathscr{L}}$, $\tilde{\mathscr{L}}^*$, (A_i^*), (E^{i+3*}). Let us denote them by C, too.

The tangent collineation of $C : \mathscr{L} \to \mathscr{L}$ or $C : \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ is determined by (4.3), (4.4), (4.6).

The collineations K_1, K_1^* of the form (4.3), (4.6) realizes the analytic contact of the first order of $(A_i) \to (\tilde{A}_i)$, $(E^{i+3}) \to \tilde{E}^{i+3})$, if

$$K_1\tilde{A}_i = \varrho_i A_i\,,\quad K_1\,\nabla\tilde{A}_i = \varrho_i\,\nabla A_i + \theta_i A_i$$
$$K_1^*\tilde{E}^{i+3} = \varrho_i^{-1}E^{i+3}\,,\quad K_1^*\,\nabla\tilde{E}^{i+3} = \varrho_i^{-1}\,\nabla E^{i+3} + \theta_i^* E^{i+3}\quad \text{respectively}\,.$$

Using (2.12), (2.11) or (4.3), (4.6), we get

$$K_1 \nabla \tilde{A}_i = \varrho_i \nabla A_i + (\varrho_i \tau_{ii} + c_{i+3,i}\, du_i)\, A_i +$$
$$+ [du_i(c_{i+3,i+1} + \varrho_{i+1} \tilde{h}_{i,i+1} - \varrho_i h_{i,i+1}) + \varrho_{i+1} \nabla \tilde{\alpha}_{i,i+1} - \varrho_i \nabla \alpha_{i,i+1}]\, A_{i+1} +$$
$$+ [du_i(c_{i+3,i+2} + \varrho_{i+2} \tilde{h}_{i,i+2} - \varrho_i h_{i,i+2}) + \varrho_{i+2} \nabla \tilde{\alpha}_{i,i+2} - \varrho_i \nabla \alpha_{i,i+2}]\, A_{i+2},$$

$$K_1^* \nabla \tilde{E}^{i+3} = \varrho_i^{-1} \nabla E^{i+3} + (-\varrho_i^{-1} \tau_{i+3,i+3} + \varrho_i^{-2} c_{i+3,i}\, du_i)\, E^{i+3} +$$
$$+ [du_i(\varrho_{i+2} c_{i+1,i} - \varrho_{i+1}^{-1} \tilde{h}_{i+1,i} + \varrho_i^{-1} h_{i+1,i}) +$$
$$+ \varrho_{i+1}^{-1} \nabla \tilde{\beta}_{i+1,i} - \varrho_i^{-1} \nabla \beta_{i+1,i}]\, E^{i+4} +$$
$$+ [du_i(\varrho_{i+1} c_{i+5,i} - \varrho_{i+2}^{-1} \tilde{h}_{i+2,i} + \varrho_i^{-1} h_{i+2,i}) +$$
$$+ \varrho_{i+2}^{-1} \nabla \tilde{\beta}_{i+2,i} - \varrho_i^{-1} \nabla \beta_{i+2,i}]\, E^{i+5}$$

respectively.

Lemma 3. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ or $C : \mathscr{L}^* \to \mathscr{L}^*$ be a developable correspondence. Tangent collineation $K_1 : \tilde{P}_5 \to P_5$ of the correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$, or $K_1^* : \tilde{P}_5^* \to P_5^*$ of the correspondence $C : \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ realizes the analytic contact of the first order of focal varieties $(A_i) \to (\tilde{A}_i)$ or $(E^{i+3}) \to (\tilde{E}^{i+3})$, if and only if it holds*

$$\varrho_s \nabla \tilde{\alpha}_{i,s} = \varrho_i \nabla \alpha_{i,s}, \quad c_{i+3,s} = \varrho_i h_{i,s} - \varrho_s \tilde{h}_{i,s}, \tag{4.7}$$

$$\varrho_i \nabla \tilde{\beta}_{s,i} = \varrho_s \nabla \beta_{s,i}, \quad c_{s+3,i} = \varrho_i \tilde{h}_{s,i} - \varrho_s h_{s,i} \tag{4.8}$$

respectively.

In a similar way, we find the conditions for the analytic contact of the first order of line complexes $[A_i A_{i+1}] \to [\tilde{A}_i \tilde{A}_{i+1}]$ and plane complexes $[E^{i+3} E^{i+4}] \to [\tilde{E}^{i+3} \tilde{E}^{i+4}]$. We obtain

Lemma 4. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ or $C : \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ be a developable correspondence. The tangent collineation $K_2 : \tilde{P}_5 \to P_5$ of the correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ or $K_2^* : \tilde{P}_5^* \to P_5^*$ of the correspondence $C : \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ realizes the analytic contact of the first order of line complexes $[A_i A_{i+1}] \to [\tilde{A}_i \tilde{A}_{i+1}]$ or plane complexes $[E^{i+3} E^{i+4}] \to [\tilde{E}^{i+3} \tilde{E}^{i+4}]$, if and only if it holds*

$$\begin{aligned} \varrho_{i+2} \nabla \tilde{\alpha}_{s-1,i+2} &= \varrho_{s-1} \nabla \alpha_{s-1,i+2} \\ c_{s+2,i+2} &= \varrho_{s-1} h_{s-1,i+2} - \varrho_{i+2} \tilde{h}_{s-1,i+2}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \varrho_{s-1} \nabla \tilde{\beta}_{i+2,s-1} &= \varrho_{i+2} \nabla \beta_{i+2,s-1} \\ c_{i+5,s-1} &= \varrho_{s-1} \tilde{h}_{i+2,s-1} - \varrho_{i+2} h_{i+2,s-1} \end{aligned} \tag{4.10}$$

respectively.

5. Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be plane pseudocongruences with projective connection. Let a one-to-one correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ (plane $\to$ plane) be given by the equations

$\tilde{u}_i = \tilde{u}_i(u_1, u_2, u_3)$. We shall say that C^b is the point extension of C if a collineation

$$\Pi = \Pi(u_1, u_2, u_3) : P_2(u_1, u_2, u_3) \rightarrow$$
$$\rightarrow \tilde{P}_2[\tilde{u}_1(u_1, u_2, u_3),\ \tilde{u}_2(u_1, u_2, u_3),\ \tilde{u}_3(u_1, u_2, u_3)]$$

is given for every pair of corresponding planes $P_2 \in \mathscr{L}$, $CP_2 = \tilde{P}_2 \in \tilde{\mathscr{L}}$.

The correspondence $C : \mathscr{L} \rightarrow \tilde{\mathscr{L}}$ is called a point deformation if and only if there exists a point extension C^b of C given by the collineation Π such that the following condition holds: Let $P_2^0 \in \mathscr{L}$ be a fixed plane, $\mathscr{R}$ an arbitrary plane variety in $\mathscr{L}$ passing through P_2^0. Let $A(R_2)$ be an arbitrarily chosen point in the plane $R_2 \in \mathscr{R}$. If R_2 runs through the variety $\mathscr{R}$, the points $A(R_2)$ describe a curve γ; let $\bar{\gamma}$ be its development into $P_5(P_2^0)$. The points $\tilde{A}(\tilde{R}_2) = \Pi A(R_2)$ describe a curve $\tilde{\gamma}$ on the plane variety $\tilde{\mathscr{R}} = C\mathscr{R}$; let $\bar{\tilde{\gamma}}$ be its development into the local space $\tilde{P}_5(CP_2^0)$. A collineation $H(P_2^0) : P_5(P_2^0) \rightarrow \tilde{P}_5(CP_2^0)$ exists for each $P_2^0 \in \mathscr{L}$ such that the curves $H(P_2^0)\,\bar{\gamma}, \bar{\tilde{\gamma}}$ have an analytic contact of the first order.

We shall say that H realizes the point deformation. Let a correspondence $C : \mathscr{L} \rightarrow \tilde{\mathscr{L}}$ be given by the equations (4.1) and let a point extension C^b of C be given by the collineation

$$\Pi \tilde{A}_i = \sum_{j=1}^{3} b_{ij} A_j\,, \quad \det |b_{ij}| \neq 0\,. \tag{5.1}$$

Suppose that C is a point deformation and C^b is the corresponding point extension of C. The collineation H realizing the point deformation should be of the form

$$H\tilde{A}_i = \sum_{j=1}^{3} b_{ij} A_j \tag{5.2}$$
$$H\tilde{A}_{i+3} = \sum_{j=1}^{6} b_{i+3,j} A_j\,, \quad \det |b_{rj}| \neq 0 \ (r, j = 1, 2, \ldots, 6)\,.$$

Let the curve $\tilde{\gamma}$ be described by the point

$$A = \sum_{i=1}^{3} x_i(t)\, A_i(u_1, u_2, u_3)\,; \quad u_i = u_i(t)\,.$$

Then

$$H\,\nabla\tilde{A} = \sum_{i=1}^{3} \sum_{s=1}^{3} \sum_{j=1}^{3} (\mathrm{d}x_i b_{ij} + x_i \tilde{\omega}_{is} b_{sj})\, A_j + \sum_{i=1}^{3} \sum_{j=1}^{6} x_i\, \mathrm{d}u_i b_{i+3,j} A_j$$
$$\nabla(C^b\tilde{A}) = \sum_{i=1}^{3} \sum_{s=1}^{3} \sum_{j=1}^{3} [(\mathrm{d}x_i b_{ij} + x_i\, \mathrm{d}b_j + x_i b_{is} \omega_{sj})\, A_j + x_i b_{ij}\, \mathrm{d}u_j A_{j+3}]\,.$$

If C is a point deformation, there are b_{ij}, λ_i satisfying

$$H\,\nabla\tilde{A} = \nabla(C^b\tilde{A}) + (\lambda_1\,\mathrm{d}u_1 + \lambda_2\,\mathrm{d}u_2 + \lambda_3\,\mathrm{d}u_3)\, C^b\tilde{A}$$

identically in x_i, $\mathrm{d}u_i$. Comparing the coefficients of $\mathrm{d}x_i A_{j+3}$, we get

(5.3) $$b_{i+3,j+3}\,\mathrm{d}\tilde{u}_i = b_{ij}\,\mathrm{d}u_j \quad (i, j = 1, 2, 3)$$

or

$$\det\left|b_{i+3,j+3}\right|\,\mathrm{d}\tilde{u}_1\,\mathrm{d}\tilde{u}_2\,\mathrm{d}\tilde{u}_3 = \det\left|b_{ij}\right|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 \quad (i, j = 1, 2, 3)\,.$$

Consequently the correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is developable in the sense that it transforms developable varieties into developable varieties again.

From (5.3) it follows that $m_{is} = 0$ and, without any loss of generality, we may suppose that the developable correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is given by

(5.4) $$\mathrm{d}\tilde{u}_i = \mathrm{d}u_i\,.$$

Further we obtain

(5.5) $$b_{ii} = b_{i+3,i+3},\; b_{is} = b_{i+3,s+3} = 0\,.$$

Let us denote $b_{ii} = \varrho_i$. Comparing the coefficient of $x_i A_j$ $(i, j = 1, 2, 3)$, we find

$$b_{i+3,i}\,\mathrm{d}u_i = \mathrm{d}\varrho_i - \varrho_i\tau_{ii} + \lambda\varrho_i \quad \text{where} \quad \lambda = \sum_{i=1}^{3}\lambda_i\,\mathrm{d}u_i\,,$$

and further

$$b_{i+3,s}\,\mathrm{d}u_i = \varrho_i\omega_{is} - \varrho_s\tilde{\omega}_{is}\,.$$

Comparing the coefficient of $\mathrm{d}u_i$, we obtain

(5.6) $$\varrho_s\,\nabla\tilde{\alpha}_{is} = \varrho_i\,\nabla\alpha_{is}\,, \quad b_{i+3,s} = \varrho_i h_{is} - \varrho_s\tilde{h}_{is}$$

(5.7) $$b_{i+3,i} = \varrho_i\left(\lambda_i + \frac{\partial \lg \varrho_i}{\partial u_i} - \tilde{a}^i_{ii} + a^i_{ii}\right)$$

(5.8) $$\lambda_i = \tilde{a}^i_{ss} - a^i_{ss} - \frac{\partial \lg \varrho_s}{\partial u_i}\,.$$

Eliminating ϱ_i from (5.6), we get

(5.9) $$\nabla\tilde{\alpha}_{i+1,i+2}\,\nabla\tilde{\alpha}_{i+2,i+1} = \nabla\alpha_{i+1,i+2}\,\nabla\alpha_{i+2,i+1}$$

and further

(5.10) $$\nabla\tilde{\alpha}_{12}\,\nabla\tilde{\alpha}_{23}\,\nabla\tilde{\alpha}_{31} = \nabla\alpha_{12}\,\nabla\alpha_{23}\,\nabla\alpha_{31}$$
$$\nabla\tilde{\alpha}_{21}\,\nabla\tilde{\alpha}_{32}\,\nabla\tilde{\alpha}_{13} = \nabla\alpha_{21}\,\nabla\alpha_{32}\,\nabla\alpha_{13}\,.$$

With respect to (3.8), the relations (5.9) and (5.10) assume the form

(5.11) $$\tilde{\varphi}_i = \varphi_i\,, \quad \tilde{J}_1 = J_1\,, \quad \tilde{J}_2 = J_2\,.$$

Conversely, let us suppose (5.9) and (5.10). From (5.9) it follows

$$\nabla\tilde{\alpha}_{i+1,i+2} = k_i \nabla\alpha_{i+1,i+2}\,; \quad \nabla\tilde{\alpha}_{i+2,i+1} = k_i^{-1} \nabla\alpha_{i+2,i+1}\,.$$

Substituting into (5.10), we obtain $k_1 k_2 k_3 = 1$; if we write $k_i = \varrho_{i+1}\varrho_{i+2}^{-1}$ the presumption of (5.9), (5.10) yields (5.6).

Proposition 2. *The correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is a point deformation if and only if C is developable and pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ have the same point elements. The collineation H realizing this deformation is determined uniquely by equations* (5.5), (5.6) *and* (5.7).

Comparing (4.7), (4.9) with the results (5.6), we obtain

Proposition 3. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a developable correspondence. The correspondence is a point deformation if and only if the induced correspondences $C : (A_i) \to (\tilde{A}_i)$ and $C : [A_i A_{i+1}] \to [\tilde{A}_i \tilde{A}_{i+1}]$ are simultaneously projective deformations of the first order and are realized by the same collineation.*

Let us carry out dual considerations and let us introduce s.c. hyperplanar deformation. Using (3.3) and (3.14), we derive the necessary and sufficient conditions for the hyperplanar deformation $C : \mathscr{L} \to \tilde{\mathscr{L}}$ in the form

$$b^*_{ii} = b^*_{i+3,i+3}\,, \quad b^*_{is} = b^*_{i+3,s+3} = 0\,, \tag{5.12}$$

$$\varrho_i \nabla\tilde{\beta}_{si} = \varrho_s \nabla\beta_{si}\,, \quad b^*_{s+3,i} = \varrho_i \tilde{h}_{si} - \varrho_s h_{si}\,, \tag{5.13}$$

$$b^*_{i+3,i} = \varrho_i \left(\lambda^*_i + \tilde{a}^i_{i+3,i+3} - a^i_{i+3,i+3} - \frac{\partial \lg \varrho_i}{\partial u_i} \right), \tag{5.14}$$

$$\lambda^*_i = a^i_{s+3,s+3} - \tilde{a}^i_{s+3,s+3} + \frac{\partial \lg \varrho_s}{\mathrm{d} u_i}\,, \tag{5.15}$$

and consenquently

$$\nabla\tilde{\beta}_{i+1,i+2} \nabla\tilde{\beta}_{i+2,i+1} = \nabla\beta_{i+1,i+2} \nabla\beta_{i+2,i+1}\,,$$
$$\nabla\tilde{\beta}_{12} \nabla\tilde{\beta}_{23} \nabla\tilde{\beta}_{31} = \nabla\beta_{12} \nabla\beta_{23} \nabla\beta_{31}\,,$$
$$\nabla\tilde{\beta}_{21} \nabla\tilde{\beta}_{32} \nabla\tilde{\beta}_{13} = \nabla\beta_{21} \nabla\beta_{32} \nabla\beta_{13}\,;$$

it means

$$\tilde{\varphi}^*_i = \varphi^*_i\,, \quad \tilde{J}^*_1 = J^*_1\,, \quad \tilde{J}^*_2 = J^*_2\,.$$

Proposition 4. *The correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is a hyperplanar deformation if and only if C is developable and pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ have the same hyperplanar elements. The collineation H^* realizing this deformation is uniquely determined by the equations* (5.12), (5.13), (5.14) *and* (5.15).

Comparing (4.8), (4.10) with the results (5.13), we get

Proposition 5. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a developable correspondence. The correspondence is a hyperplanar deformation if and only if the induced correspondences $C : (E^{i+3}) \to (\tilde{E}^{i+3})$, $C : [E^{i+3}E^{i+4}] \to [\tilde{E}^{i+3}\tilde{E}^{i+4}]$ are simultaneously projective deformations of the first order and are realized by the same collineation.*

6. Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a developable correspondence given by the equations (4.7) and let K of the form (4.3) be its tangent collineation. The correspondence C is said to be a focal deformation of the kind "i" if and only if the tangent collineation realizing this deformation realizes simultaneously the analytic contact of the first order of local varieties $(A_i) \to (\tilde{A}_i)$ and $(E^{i+3}) \to (\tilde{E}^{i+3})$.

From (4.7) and (4.8) we get

$$\varrho_s \nabla\tilde{\alpha}_{is} = \varrho_i \nabla\alpha_{is}\,, \quad \varrho_i \nabla\tilde{\beta}_{si} = \varrho_s \nabla\beta_{si} \tag{6.1}$$

$$c_{i+3,s} = \varrho_i h_{is} - \varrho_s \tilde{h}_{is}\,, \quad c_{s+3,i} = \varrho_i \tilde{h}_{si} - \varrho_s h_{si} \tag{6.2}$$

and consequently

$$\nabla\tilde{\alpha}_{is} \nabla\tilde{\beta}_{si} = \nabla\alpha_{is} \nabla\beta_{si} \tag{6.3}$$

i.e.

$$\tilde{F}_{is} = F_{is}\,.$$

The conditions are necessary and sufficient.

Proposition 6. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a developable correspondence. The correspondence is a focal deformation of the kind "i" if and only if pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ have the same focal forms $\tilde{F}_{is} = F_{is}$. The collineation realizing this deformation is determined by the equations* (6.2).

We shall say that a developable correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is quasifocal of the kind "i" if and only if its tangent collineation realizes simultaneously the analytic contact of the first order of $[A_{i+1}A_{i+2}] \to [\tilde{A}_{i+1}\tilde{A}_{i+2}]$, $[E^{i+4}E^{i+5}] \to [\tilde{E}^{i+4}\tilde{E}^{i+5}]$. From (4.9) and (4.10) we obtain

$$\varrho_i \nabla\tilde{\alpha}_{si} = \varrho_s \nabla\alpha_{si}\,, \quad \varrho_s \nabla\tilde{\beta}_{is} = \varrho_i \nabla\beta_{is} \tag{6.4}$$

$$c_{s+3,i} = \varrho_s h_{si} - \varrho_i \tilde{h}_{si}\,, \quad c_{i+3,s} = \varrho_s \tilde{h}_{is} - \varrho_i h_{is} \tag{6.5}$$

and

$$\nabla\tilde{\alpha}_{si} \nabla\tilde{\beta}_{is} = \nabla\alpha_{si} \nabla\beta_{is}\,, \quad \text{i.e.} \quad \tilde{F}_{si} = F_{si}\,. \tag{6.6}$$

Proposition 7. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a developable correspondence. The correspondence is a quasifocal deformation of the kind "i" if and only if pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ have the same focal forms $\tilde{F}_{si} = F_{si}$. The collineation realizing this deformation is given by the equations* (6.5).

Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be simultaneously a focal and quasifocal deformation of the kind "i". From (6.1), (6.4) and (3.11) we get

$$\frac{\nabla\tilde{\alpha}_{is}}{\nabla\tilde{\beta}_{is}} = \frac{\nabla\alpha_{is}}{\nabla\beta_{is}}, \quad \frac{\nabla\tilde{\alpha}_{si}}{\nabla\tilde{\beta}_{si}} = \frac{\nabla\alpha_{si}}{\nabla\beta_{si}}, \tag{6.7}$$

i.e.

$$\tilde{G}_{is} = G_{is}, \quad \tilde{G}_{si} = G_{si}.$$

From (6.2), (6.5) and (3.12) it follows

$$c_{s+3,i} = c_{i+3,s} = 0 \tag{6.8}$$

and consequently

$$\frac{\nabla\tilde{\alpha}_{is}}{\tilde{h}_{is}} = \frac{\nabla\alpha_{is}}{h_{is}}, \quad \frac{\nabla\tilde{\beta}_{si}}{\tilde{h}_{si}} = \frac{\nabla\beta_{si}}{h_{si}}, \tag{6.9}$$

$$\frac{\nabla\tilde{\alpha}_{si}}{\tilde{h}_{si}} = \frac{\nabla\alpha_{si}}{h_{si}}, \quad \frac{\nabla\tilde{\beta}_{is}}{\tilde{h}_{is}} = \frac{\nabla\beta_{is}}{h_{is}}, \tag{6.10}$$

i.e.

$$\tilde{g}_{is} = g_{is}, \quad \tilde{g}^*_{is} = g^*_{is}, \quad \tilde{g}_{si} = g_{si}, \quad \tilde{g}^*_{si} = g^*_{si}.$$

From (6.7), (3.4) and (3.5) it follows that the pseudoasymptotic curves of the varieties $(A_i) \to (\tilde{A}_i)$ and $(E^{i+3}) \to (\tilde{E}^{i+3})$ correspond to each other.

Using (3.6), (3.7), (6.10) and (6.9) we may see that also the asymptotic curves of the varieties $(A_i) \to (\tilde{A}_i)$ and $(E^{i+3}) \to (\tilde{E}^{i+3})$ correspond to each other.

Proposition 8. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be simultaneously a focal and quasifocal deformation of the kind "i". Then the pseudoasymptotic and asymptotic curves of focal varieties $(A_i) \to (\tilde{A}_i)$ and $(E^{i+3}) \to (\tilde{E}^{i+3})$ are corresponding to each other. The collineation realizing this deformation is given by* (6.8).

7. Now, let us deal with a projective deformation of the second order. The pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ let be given by (2.12), ($2.\tilde{1}\tilde{2}$) and let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a correspondence. C is a projective deformation of the second order if and only if there exists (for each plane $P_2 \in \mathscr{L}$) a tangent collineation K satisfying (4.2) and

$$K\,\nabla^2[\tilde{A}_1, \tilde{A}_2, \tilde{A}_3] = \nabla^2[A_1, A_2, A_3] + 2\vartheta_1\,\nabla[A_1, A_2, A_3] + (.)\,[A_1, A_2, A_3] \tag{7.1}$$

where

$$\vartheta_1 = \sum_{i=1}^{3}(\tau_{ii} - \varrho_i^{-1}c_{i+3,i}\,du_i).$$

With respect to Proposition 1, the projective deformation is a developable correspondence and we may suppose (4.7) and (4.3). There is

(7.2)

$$\nabla[A_1, A_2, A_3] = [A_1, A_2, A_3](\omega_{11} + \omega_{22} + \omega_{33}) + \\ + \sum_{i=1}^{3} [A_i, A_{i+1}, A_{i+5}]\, du_{i+2}\,,$$

$$\nabla[A_{i+1}, A_{i+2}, A_{i+3}] = [A_{i+1}, A_{i+2}, A_{i+3}](\omega_{i+1,i+1} + \omega_{i+2,i+2} + \omega_{i+3,i+3}) + \\ + [A_i, A_{i+2}, A_{i+3}](h_{i+1,i}\, du_{i+1} + \nabla\alpha_{i+1,i}) + \\ + [A_{i+1}, A_{i+2}, A_{i+4}](h_{i,i+1}\, du_{i+1} + \nabla\beta_{i,i+1}) + \\ + [A_{i+1}, A_i, A_{i+3}](h_{i+2,i}\, du_{i+2} + \nabla\alpha_{i+2,i}) + \\ + [A_{i+1}, A_{i+2}, A_{i+5}](h_{i,i+2}\, du_{i+2} + \nabla\beta_{i,i+2}) + \\ + [A_{i+2}, A_{i+3}, A_{i+4}]\, du_{i+1} + [A_{i+3}, A_{i+1}, A_{i+5}]\, du_{i+2} + \\ + [A_i, A_{i+1}, A_{i+2}]\, \omega_{i+3,i}$$

and consequently

(7.3)

$$\nabla^2[A_1, A_2, A_3] = (.)\,[A_1, A_2, A_3] + 2\sum_{i=1}^{3} du_i\, du_{i+1}[A_{i+2}, A_{i+3}, A_{i+4}] + \\ + \sum_{i=1}^{3} [A_{i+1}, A_i, A_{i+3}](\nabla\alpha_{i+2,i}\, du_i - \nabla\beta_{i+2,i}\, du_{i+2}) + \\ + \sum_{i=1}^{3} [A_{i+1}, A_i, A_{i+4}]\,. \\ .(\nabla\alpha_{i+2,i+1}\, du_{i+1} - \nabla\beta_{i+2,i+1}\, du_{i+2}) + \\ + \sum_{i=1}^{3} [A_i, A_{i+1}, A_{i+5}]\{d^2u_{i+2} + (2\omega_{ii} + 2\omega_{i+1,i+1} + \\ + \omega_{i+2,i+2} + \omega_{i+5,i+5})\, du_{i+2}\}\,.$$

From (7.3), an analogous equation for $\nabla^2[\tilde{A}_1, \tilde{A}_2, \tilde{A}_3]$ and (4.5) we obtain

(7.4)

$$K\,\nabla^2[\tilde{A}_1, \tilde{A}_2, \tilde{A}_3] = \nabla^2[A_1, A_2, A_3] + 2\vartheta_1\, \nabla[A_1, A_2, A_3] + (\cdot)\,[A_1, A_2, A_3] + \\ + \sum_{r=i}^{i+2} \sum_{i=1}^{3} \Phi^r_{i+1,i+2}[A_{i+1}, A_{i+2}, A_{r+3}]$$

where

(7.5)

$$\Phi^i_{i+1,i+2} = (\tau_{i+3,i+3} - \tau_{ii})\, du_i - 2\varrho_i^{-1} c_{i+3,i}\, du_i^2$$

$$\Phi^s_{i+1,i+2} = \nabla\alpha_{is}\, du_s - \nabla\beta_{is}\, du_i - \varrho_s\varrho_i^{-1}(\nabla\tilde{\alpha}_{is}\, du_s - \nabla\tilde{\beta}_{is}\, du_i) - \\ - 2\varrho_i^{-1} c_{i+3,s}\, du_i\, du_s\,.$$

If C is a projective deformation of the second order then there exist such functions $c_{i+3,i}, c_{i+3,s}$ that

$$\Phi^i_{i+1,i+2} = \Phi^s_{i+1,i+2} = 0 . \tag{7.6}$$

From (7.5) and (7.6) it follows

$$c_{i+3,s} = 0 \tag{7.7}$$

$$2c_{i+3,i} = \varrho_i(\tilde{a}^i_{i+3,i+3} - \tilde{a}^i_{ii} - a^i_{i+3,i+3} + a^i_{ii}) \tag{7.8}$$

$$\nabla\alpha_{is} = \varrho_i^{-1}\varrho_s \nabla\tilde{\alpha}_{is} , \quad \nabla\beta_{is} = \varrho_i^{-1}\varrho_s \nabla\tilde{\beta}_{is} \tag{7.9}$$

$$\tilde{a}^s_{i+3,i+3} - \tilde{a}^s_{ii} = a^s_{i+3,i+3} - a^s_{ii} . \tag{7.10}$$

Eliminating ϱ_i from (7.9), we get

$$\nabla\tilde{\alpha}_{is}\nabla\tilde{\alpha}_{si} = \nabla\alpha_{is}\nabla\alpha_{si} \tag{7.11}$$

$$\nabla\tilde{\alpha}_{12}\nabla\tilde{\alpha}_{23}\nabla\tilde{\alpha}_{31} = \nabla\alpha_{12}\nabla\alpha_{23}\nabla\alpha_{31} , \quad \nabla\tilde{\alpha}_{21}\nabla\tilde{\alpha}_{32}\nabla\tilde{\alpha}_{13} = \nabla\alpha_{21}\nabla\alpha_{32}\nabla\alpha_{13}$$

$$\nabla\tilde{\beta}_{is}\nabla\tilde{\beta}_{si} = \nabla\beta_{is}\nabla\beta_{si} \tag{7.12}$$

$$\nabla\tilde{\beta}_{12}\nabla\tilde{\beta}_{23}\nabla\tilde{\beta}_{31} = \nabla\beta_{12}\nabla\beta_{23}\nabla\beta_{31} , \quad \nabla\tilde{\beta}_{21}\nabla\tilde{\beta}_{32}\nabla\tilde{\beta}_{13} = \nabla\beta_{21}\nabla\beta_{32}\nabla\beta_{13}$$

$$\nabla\tilde{\alpha}_{is}\nabla\tilde{\beta}_{si} = \nabla\alpha_{is}\nabla\beta_{si} \tag{7.13}$$

$$\frac{\nabla\tilde{a}_{is}}{\nabla\tilde{\beta}_{is}} = \frac{\nabla\alpha_{is}}{\nabla\beta_{is}} . \tag{7.14}$$

With respect to (3.8)–(3.13) we have

$$\tilde{\varphi}_i = \varphi_i , \quad \tilde{J}_1 = J_1 , \quad \tilde{J}_2 = J_2 , \quad \tilde{\varphi}^*_i = \varphi^*_i , \quad \tilde{J}^*_1 = J^*_1 , \quad \tilde{J}^*_2 = J^*_2 , \tag{7.15}$$
$$\tilde{F}_{is} = F_{is} , \quad \tilde{G}_{is} = G_{is} , \quad \tilde{\psi}_{is} = \psi_{is} .$$

Conversely, let (7.15) hold. From (7.11) and (7.12) it follows

$$\nabla\tilde{\alpha}_{is} = k_i \nabla\alpha_{is} , \quad \nabla\tilde{\alpha}_{si} = k_i^{-1}\nabla\alpha_{is} , \quad \nabla\tilde{\beta}_{is} = g_i \nabla\beta_{is} , \quad \nabla\tilde{\beta}_{si} = g_i^{-1}\nabla\beta_{si} .$$

Substituting into (7.13) or (7.14), we get $k_i = g_i$ and using (7.12) we have $k_1k_2k_3 = 1$. If we put $k_i = \varrho_{i+1}\varrho_{i+2}^{-1}$, then the presumption of (7.15) yields (7.9).

Proposition 9. *The correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is a projective deformation of the second order if and only if*

$$\tilde{\varphi}_i = \varphi_i , \quad \tilde{J}_1 = J_1 , \quad \tilde{J}_2 = J_2 , \quad \tilde{\varphi}^*_i = \varphi^*_i , \quad \tilde{J}^*_1 = J^*_1 , \quad \tilde{J}^*_2 = J^*_2 ,$$
$$\tilde{F}_{is} = F_{is} , \quad \tilde{G}_{is} = G_{is} , \quad \tilde{\psi}_{is} = \psi_{is} .$$

Substitution (3.3) and (3.14) yields

Proposition 10. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order. The correspondence $C : \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ is also a projective deformation of the second order.*

8. Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order. According to (4.3), (4.4), (7.7) and (7.8), the osculating collineation realizing this deformation is

$$K\tilde{A}_i = \varrho_i A_i\,, \quad K\tilde{A}_{i+3} = c_{i+3,i}A_i + \varrho_i A_{i+3} \tag{8.1}$$

where $c_{i+3,i}$ is determined by (7.8).

The dualization $C : \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ is also a projective deformation of the second order and the osculating collineation realizing this deformation is

$$K\tilde{E}^{i+3} = \varrho_i^{-1}E^{i+3}\,, \quad K\tilde{E}^i = -\varrho_i^{-2}c_{i+3,i}E^{i+3} + \varrho_i^{-1}E^i \tag{8.2}$$

where $c_{i+3,i}$ are determined by (7.8).

If expressed in terms of points, relations (8.2) give (8.1).

Lemma 5. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order and (8.1) be its osculating collineation. The correspondence $C : \mathscr{L}^* \to \tilde{\mathscr{L}}^*$ has the same osculating collineation.*

According to Lemma 2, we may change the local frames using equations (2.10) and putting $d\bar{u}_i/du_i = 1$.

We get

$$K\tilde{A}_i = \varrho_i\mu_{ii}\bar{A}_i\,,$$
$$K\tilde{A}_{i+3} = [\tfrac{1}{2}\varrho_i(\tilde{a}^i_{i+3,i+3} - a^i_{i+3,i+3} - \tilde{a}^i_{ii} + a^i_{ii})\,\mu_{ii} + \varrho_i\mu_{i+3,i}]\,\bar{A}_i + \varrho_i\mu_{ii}\bar{A}_{i+3}\,.$$

By a suitable choice

$$\mu_{ii} = \varrho_i^{-1}\,, \quad \mu_{i+3,i} = \tfrac{1}{2}\varrho_i^{-i}(a^i_{i+3,i+3} - a^i_{ii} - \tilde{a}^i_{i+3,i+3} + \tilde{a}^i_{ii})$$

we obtain

Lemma 6. *If $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is a projective deformation of the second order, it is possible to attain by a suitable choice of local frames that*

$$K\tilde{A}_i = A_i\,, \quad K\tilde{A}_{i+3} = A_{i+3} \tag{8.3}$$

is the osculating collineation. In this case, we have (7.9) *and* (7.10) *and*

$$\varrho_i = 1\,, \quad \tilde{a}^i_{i+3,i+3} - \tilde{a}^i_{ii} = a^i_{i+3,i+3} - a^i_{ii}\,. \tag{8.4}$$

Let us suppose that $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is a projective deformation of the second order and that the osculating collineation realizing this deformation is determined by (8.3). $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is, of course, a point deformation and also a hyperplanar deformation. Let us determine collineations H and H^* realizing these deformations. Using (8.4), (7.8), (7.9), (7.10) and substituting into (5.6) and (5.7), we get

Lemma 7. If $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is a projective deformation of the second order and (8.3) is the osculating collineation, then the osculating collineations H, H^* realizing the point deformation and the hyperplanar deformation respectively are given by

$$\begin{aligned} H\tilde{A}_i &= A_i\,, \\ H\tilde{A}_{i+3} &= c_{i+3,i}A_i + (h_{i,i+1} - \tilde{h}_{i,i+1})\,A_{i+1} + \\ &\quad + (h_{i,i+2} - \tilde{h}_{i,i+2})\,A_{i+2} + A_{i+3}\,, \end{aligned} \tag{8.5}$$

$$\begin{aligned} H^*\tilde{A}_i &= A_i\,, \\ H^*\tilde{A}_{i+3} &= -c_{i+3,i}A_i + (\tilde{h}_{i,i+1} - h_{i,i+1})\,A_{i+1} + \\ &\quad + (\tilde{h}_{i,i+2} - h_{i,i+2})\,A_{i+2} + A_{i+3} \end{aligned} \tag{8.6}$$

where

$$\begin{aligned} c_{i+3,i} &= \tilde{a}^i_{i+2,i+2} - \tilde{a}^i_{ii} - a^i_{i+2,i+2} - a^i_{ii} = \\ &= \tilde{a}^i_{i+1,i+1} - \tilde{a}^i_{ii} - a^i_{i+1,i+1} - a^i_{ii}\,. \end{aligned}$$

Collineations K, H, H^* are mutually different. If any two of them coincide, then all three are coinciding. Using (8.7), we have from (8.5) and (8.6)

Proposition 11. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order. Pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are simultaneously subjected to the point and hyperplanar deformation. All these three deformations are realized by the same collineation, if and only if $\tilde{h}_{is} = h_{is}$, $\tilde{a}^i_{ss} - \tilde{a}^i_{ii} = a^i_{ss} - a^i_{ii}$.*

9. Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order; suppose that (7.9), (7.10) and (8.4) hold. The osculating collineation K is (8.3). We shall say that C is 1) weakly singular, 2) singular, 3) strongly singular of the kind "i", if $C : (A_i) \to (\tilde{A}_i)$ is a projective deformation of order 1) one, 2) two, 3) three and it is possible to realize the deformation C by the same collineation K. If C is (weakly, strongly) singular of all three kinds simultaneously, C is said to be (weakly, strongly) singular.

There is

$$\begin{aligned} K\tilde{A}_i = A_i\,,\quad K\,\nabla\tilde{A}_i &= \nabla A_i + \tau_{ii}A_i + (\tilde{h}_{i,i+1} - h_{i,i+1})\,\mathrm{d}u_iA_{i+1} + \\ &\quad + (\tilde{h}_{i,i+2} - h_{i,i+2})\,\mathrm{d}u_iA_{i+2}\,. \end{aligned}$$

The deformation C is weakly singular of the kind "i", if and only if

$$\tilde{h}_{is} = h_{is} . \tag{9.1}$$

Let $\tilde{\sigma}_i = [\tilde{A}_i, \tilde{A}_{i+1}, \tilde{A}_{i+2}] + \lambda_i \tilde{A}_i, \tilde{A}_{i+1}, \tilde{A}_{i+3}] + \mu_i[\tilde{A}_i, \tilde{A}_{i+2}, \tilde{A}_{i+3}]$ be an arbitrary tangent plane of the variety $(\tilde{A}_i)$. Then

$$\begin{aligned} K\tilde{\sigma}_i &= [A_i, A_{i+1}, A_{i+2}] + \lambda_i[A_i, A_{i+1}, A_{i+3}] + \mu_i[A_i, A_{i+2}, A_{i+3}] \\ H\tilde{\sigma}_i &= \tilde{H}\tilde{\sigma}_i + [\lambda_i(h_{i,i+2} - \tilde{h}_{i,i+2}) + \mu_i(h_{i,i+1} - \tilde{h}_{i,i+1})] [A_i, A_{i+1}, A_{i+2}] \\ H^*\tilde{\sigma}_i &= K\tilde{\sigma}_i + [\lambda_i(\tilde{h}_{i,i+2} - h_{i,i+2}) + \mu_i(\tilde{h}_{i,i+1} - h_{i,i+1})] [A_i, A_{i+1}, A_{i+2}] . \end{aligned}$$

We obtain

Proposition 12. *Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be a projective deformation of the second order. The collineations K, H, H^* induce the same collineation of the bundle of tangent planes of the focal variety (A_i), if and only if C is weakly singular of the kind "i".*

Let $C : \mathscr{L} \to \tilde{\mathscr{L}}$ be weakly singular. We get

$$\begin{aligned} \nabla^2 A_i = \sum_{r=i}^{i+2} & A_r(d\omega_{ir} + \omega_{ii}\omega_{ir} + \omega_{i,i+1}\omega_{i+1,r} + \omega_{i,i+2}\omega_{i+2,r} + du_i\omega_{i+3,r}) + \\ & + A_{i+3}(d^2u_i + \omega_{ii}\, du_i + \omega_{i+3,i+3}\, du_i) + \\ & + A_{i+4}(\omega_{i,i+1}\, du_{i+1} + \omega_{i+3,i+4}\, du_i) + \\ & + A_{i+5}(\omega_{i,i+2}\, du_{i+2} + \omega_{i+3,i+5}\, du_i) . \end{aligned}$$

$$K\tilde{A}_i = A_i , \quad K\,\nabla\tilde{A}_i = \nabla A_i + \tau_{ii}A_i ,$$

$$\begin{aligned} K\,\nabla^2\tilde{A}_i = \nabla^2 A_i &+ 2\tau_{ii}A_i + (.)\, A_i + \\ &+ [(c_{i+3,i}\, du_i - c_{i+4,i+1}\, du_{i+1})\, \omega_{i,i+1} + \tau_{i+3,i+1}\, du_i]\, A_{i+1} + \\ &+ [(c_{i+3,i}\, du_i - c_{i+5,i+2}\, du_{i+2})\, \omega_{i,i+2} + \tau_{i+3,i+2}\, du_i]\, A_{i+2} . \end{aligned}$$

The correspondence $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is singular of the kind "i" if and only if

$$\begin{aligned} c_{s+3,s} = 0 , \quad h_{is}c_{i+3,i} &+ \tilde{a}^i_{i+3,s} - a^i_{i+3,s} = 0 \\ a^{i+1}_{is} c_{i+3,i} &+ \tilde{a}^{i+1}_{i+3,s} - a^{i+1}_{i+3,s} = 0 \\ a^{i+2}_{is} c_{i+3,i} &+ \tilde{a}^{i+2}_{i+3,s} - a^{i+2}_{i+3,s} = 0 \end{aligned} \tag{9.2}$$

where $c_{i+3,i}$ is given by (8.7).

If $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is singular, we get from (7.9), (7.10), (8.4) and (8.7), $\tilde{h}_{is} = h_{is}$, (9.2) the following relations:

$$\tau_{is} = 0 , \quad \tau_{i+3,s} = 0 , \quad \tau_{ii} = 0 , \quad \tau_{i+3,i+3} = 0 .$$

Further, we get

$$K\tilde{A}_i = A_i\,,\quad K\,\nabla\tilde{A}_i = \nabla A_i\,,\quad K\,\nabla^2\tilde{A}_i = \nabla^2 A_i + \mathrm{d}u_i\tau_{i+3,i}A_i\,,$$

$$K\,\nabla^3 A_i = \nabla^3 A_i + 3\mathrm{d}u_i\tau_{i+3,i}A_i + (.)\,A_i + $$
$$+ A_{i+1}\{-2\mathrm{d}u_i\omega_{i,i+1}\tau_{i+3,i} + (\omega_{i,i+1}\,\mathrm{d}u_{i+1} + \omega_{i+3,i+4}\,\mathrm{d}u_i)\,\tau_{i+4,i+1}\} +$$
$$+ A_{i+2}\{-2\mathrm{d}u_i\omega_{i,i+2}\tau_{i+3,i} + (\omega_{i,i+2}\,\mathrm{d}u_{i+2} + \omega_{i+3,i+5}\,\mathrm{d}u_i)\,\tau_{i+5,i+2}\}\,.$$

Let C be strongly singular e.g. of the first kind. Then

(9.3) $$\tau_{41} = 0\,,\quad (\omega_{12}\,\mathrm{d}u_2 + \omega_{45}\,\mathrm{d}u_1)\,\tau_{52} = 0\,,\quad (\omega_{13}\,\mathrm{d}u_3 + \omega_{46}\,\mathrm{d}u_1)\,\tau_{63} = 0\,.$$

Equations $\omega_{12}\,\mathrm{d}u_2 + \omega_{45}\,\mathrm{d}u_1 = 0$, $\omega_{13}\,\mathrm{d}u_3 + \omega_{46}\,\mathrm{d}u_1 = 0$ are equations of the asymptotic curves on the variety (A_1) and hence are not satisfied identically. Therefore, we obtain from (9.3)

(9.4) $$\tau_{52} = \tau_{63} = 0\,.$$

In this case, we have $\tau_{ij} = 0$ for all $i, j = 1, 2, \ldots, 6$. The same result follows when we begin with the variety (A_2) or (A_3).

Proposition 13. *If $C : \mathscr{L} \to \tilde{\mathscr{L}}$ is a strongly singular projective deformation of the kind "i", pseudocongruences $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are identical.*

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