Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 1, 99–108

Persistent URL: http://dml.cz/dmlcz/101005

CENTER-PROJECTIVE CONNECTIONS

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(Received October 29, 1969)

I. Center-projective bundles. Let M be a differentiable manifold of the dimension $n \geqq 2$. Let $\mathscr{F}$ denote the module of differentiable scalar functions on M and $\mathscr{S}$ the set of densities on M. We notice that every density s of the weight q may be represented locally by $s = |\det f_j^{(i)}|^q$ where $(f^{(1)}, \ldots, f^{(n)})$ are elements of $\mathscr{F}$ and $f_j^{(i)}$ is the j-th partial derivative with respect to the local map in question. We denote by R^n the n-dimensional Cartesian space, i.e. the space of n-tuples of real numbers provided with the usual topology based on cubes.

Let p be a fixed point of M and (O, x) some local map which covers p, O being an open set and $x : O \to R^n$ a parametrisation. Then we define differential operators $\mathbf{x}_1(p), \ldots, \mathbf{x}_n(p)$ by putting for any $f \in \mathscr{F}$

$$\mathbf{x}_i(p) f = (\partial_i f \circ x^{-1}) (x(p))\ ^1) . \tag{1}$$

The sequence $(\mathbf{x}_i(p))_{i \leqq n}$ is just a natural linear frame at p defined by the parametrisation x. Instead of $\mathbf{x}_i(-)$ we shall write simply $\mathbf{x}_i$. If (U, y) is another map covering p, then it defines the parameters

$$A_i^k(p, x/y) = \mathbf{y}_i(p)\, x^k\,, \quad A_i^k(p, y/x) = \mathbf{x}_i(p)\, y^k\,. \tag{2}$$

(x^k or y^k is the k-th component of the corresponding parametrisation.) The following relations follow from the theorem on differentiation of composed mappings: $\mathbf{x}_i = A_i^j(-, y/x)\, \mathbf{y}_j$. The two parametrisations x and y define the same linear frame at p if and only if we have $A_j^i(p, x/y) = \delta_j^i$ (of Kronecker). In other words: if and only if x and y define the same jet of the first order at p. [1]. Then we define the frames of the second order by putting

$$\mathbf{x}_{ij}(p) f = (\partial_{ij} f \circ x^{-1}) (x(p))\,, \quad A_{hl}^k(p, x/y) = \mathbf{y}_{hl}(p)\, x^k\,. \tag{3}$$

Then the sequence $(\mathbf{x}_1(p), \ldots, \mathbf{x}_n(p), \ldots, \mathbf{x}_{kh}(p), \ldots)$ is a frame of the second order

[1]) These symbols for partial derivatives were proposed by W. WALISZEWSKI.

at p, defined by x. Just as above, the two maps x and y, both covering p, define the same frame of the second order if and only if $A^i_j(p, x/y) = \delta^i_j$ and $A^i_{jl}(p, x/y) = 0$, i.e. if and only if they define the same jet of the second order.

Two functions f and $g \in \mathcal{F}$ define the same jet of the first order at p if it is $f(p) = g(p)$ and if for any local map, say (O, x), we have $\mathbf{x}_i(p)(f - g) = 0$ for $i = 1, \ldots, n$. We denote by $\mathcal{F}'$ the module of fields of the first order jets of scalars.

If a frame $(x_i(p))_{i \leq n}$ is given then each vector $\mathbf{v}$ tangent to M at p may be represented in the form $\mathbf{v} = \mathbf{v}^i \mathbf{x}_i(p)$. If we have two representations of the same vector, say $\mathbf{v} = v^i \mathbf{x}_i(p) = u^i \mathbf{y}_i(p)$, then the following relations hold: $v^i = A^i_j(p, x/y)\, u^j$ and $u^i = A^i_j(p, y/x)\, v^j$.

Then we compute the first prolongation of the vector field $\mathbf{v}$ which is to be represented in any coordinate neighbourhood of p. We put $v^i_j(p) = \mathbf{x}_j(p)\, v^i$ and $\mathbf{v}_* = (v^i \mathbf{x}_{ji} + v^i_j \mathbf{x}_i)_{j=1,\ldots,n}$. Thus the pair $(\mathbf{v}, \mathbf{v}_*)$ is the representations of the first prolongation of $\mathbf{v}$. The following transformation rules may be easily obtained from (1), (2), (3) by computation: if $\mathbf{v}_* = (v^i \mathbf{x}_{ji} + v^i_j \mathbf{x}_i)_j = (u^i \mathbf{y}_{ji} + u^i_j \mathbf{y}_i)_{j \leq n}$ then we have

(4) $$\mathbf{x}_{ij} = A^k_{ij}(-, y/x)\, \mathbf{y}_k + A^k_{ij}(-, y/x)\, A^l_k(-, x/y)\, \mathbf{y}_{lk}\,,$$

(5) $$v^i_j = u^k_l A^l_j(-, y/x)\, A^i_k(-, x/y) + u^k A^i_{kl}(-, x/y)\, A^l_j(-, \mathbf{y}/x)\,.$$

(6) **Proposition.** *A vector field may be viewed as the Lie derivative of the elements of $\mathcal{F}$. The first prolongation of the vector field* $\mathbf{v}$ *may be viewed as the Lie derivative of the elements of $\mathcal{F}'$.*

It follows from the well known expressions in local coordinates that

$$(\mathfrak{L}_v f)(p) = v^i\, \mathbf{x}_i(p) f\,,$$
$$(\mathfrak{L}_v(\mathbf{x}_i f))(p) = (v^j\, \mathbf{x}_{ji}(p) + v^j_i\, \mathbf{x}_j(p)) f\,.$$

Let us write the formula for the Lie derivative of a density s of the weight q. We have

(7) $$(\mathfrak{L}_v s)(p) = v^i\, \mathbf{x}_i(p)\, s + (\sum_j v^j_j)\, qs\,.$$

We introduce the new operator $\mathbf{x}_0(p)$ which corresponds to any local map (O, x) by putting

(8) $$\mathbf{x}_0(p)\, z = (\text{weight of } z)\, z(p)$$

for every $z \in \mathcal{S}$. Thus formula (7) may be presented in a compact form

(9) $$(\mathfrak{L}_v s)(p) = \sum_{J=0}^{n} v^J\, \mathbf{x}_J(p)\, s$$

where by definition $v^0 = \sum_i v^i_i$. *From now on capital indices vary from* 0 *to* n. We have to examine yet what is the rule of transformation of the Lie derivatives of the form $v^J \mathbf{x}_J$ when changing the local parametrisation.

(10) **Proposition.** *If* $\mathfrak{L}_v = v^J \mathbf{x}_J = u^J \mathbf{y}_J$ *then we have the following relations*

$$u^0 = v^0 + A^0_k(p, y/x)\, v^k\,, \quad u^i = A^i_k(p, y/x)\, v^k$$

where $A^0_k(p, y/x) = A^h_l(p, x/y)\, A^l_{hk}(p, y/x)$.

They may be obtained directly from (5) and from the identity $A^l_j(-, x/y)\, A^j_k(-, y/x) = \delta^l_k$.

(11) **Proposition.** *We have the following transformation rule for the components of* $(\mathbf{x}_J)$

$$\mathbf{x}_0 = \mathbf{y}_0\,, \quad \mathbf{y}_i = A^K_i(-, x/y)\, \mathbf{x}_k\,.$$

This follows from the previous proposition and from the invariancy of the Lie derivative.

The transformation rules in both propositions above may be written briefly as follows: $u^J = A^J_H v^H$, $\mathbf{y}_J = {}^{\sim}A^L_J \mathbf{x}_L$ where both matrices A and ${}^{\sim}A$ are of the form

$$\begin{bmatrix} 1, & A^0_k \\ 0, & A^i_k \end{bmatrix}. \tag{12}$$

We notice that it is *a matrix of a center-projective transformation* which leaves invariant that point of the projective n-space which has the uniform coordinates $(1, 0, \ldots, 0)$. This provides a reason to propose the following

(13) **Definition.** A center-projective frame at $p \in M$ defined by a local map (O, x) is the $(n + 1)$-tuple of operators $(\mathbf{x}_0(p), \mathbf{x}_1(p), \ldots, \mathbf{x}_n(p))$ (see (1) and (8)).

(14) **Proposition.** *The two local parametrisations* x *and* y *of a neighbourhood of* p *define the same center-projective frame if and only if the matrix* (12) *is the unit matrix, i.e. if* $A^0_k(p, x/y) = 0$ *and* $A^i_k(p, x/y) = \delta^i_k$.

It follows directly from Proposition (11).

(15) **Proposition.** *The condition of Proposition* (14) *may be reformulated as follows:*

$$A^i_k(p, x/y) = \delta^i_k \quad \text{and} \quad \mathbf{y}_k(p)\,(\det A^h_l(p, x/y)) = 0\,.$$

Proof. We put $A(p, x/y) = \det A^h_l(p, x/y)$. We compute

$$\mathbf{y}_k(p)\, A(-, x/y) = \sum_{p,q} (\mathbf{y}_k(p)\, A^p_q(-, x/y))\,.\, \text{minor}\,(A^p_q(p, x/y)) =$$
$$= A\,.\, A^q_p(p, y/x)\, A^p_{qk}(p, x/y) = A\,.\, A^0_k(p, x/y)\,,$$

which yields our proposition.

We introduce the following notions and notation:

The principal fibre bundle of linear frames on M will be denoted by HM; the bundle of the frames of the second order will be denoted by H_2M.

The bundles of vectors tangent to M at p and of their first prolongations will be denoted by TM and T_2M respectively. Their restrictions to the fibre over p (i.e. the tangent spaces to M at p of the first and of the second order) will be denoted by $(TM)_p$ and $(T_2M)_p$ respectively.

(17) **Definition.** The divergence space tangent to M at p is the linear space of all derivatives of densities. It will be denoted by $(KM)_p$ and the corresponding bundle will be denoted by KM.

(18) The principal bundle associated with KM is the bundle of frames of the form $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$. It is called the bundle of centro-projective frames and it will be denoted by PM.

(19) L^n, L^n_2 and $Ł^n$ denote the structural groups of HM, H_2M and of PM respectively. The are called the linear group, the prolongated linear group and the center-projective group.

PM and $Ł^n$ may be described in the terms of jets as follows: We consider local diffeomorphisms of neighbourhoods of $0 \in R^n$ into R^n which transform 0 to some point a. We say that two such mappings h and k are pj-equivalent if and only if $A^i_j(0, h/k) = \delta^i_j$ and, moreover, $\partial_j \det A^p_q(0, g/k) = 0$ for $i, j = 1, \ldots, n$.

(20) **Definition.** A class of pj-equivalence of diffeomorphisms will be named a projective jet of the first order. If h is a representing diffeomorphism then the related projective jet will be denoted by $(pj\ h)_{0,a}$.

A generalization of the notion of the projective jet onto such jets of local diffeomorphisms of R^n into the manifold M is obvious.

Now we are able to formulate the main notions in the terms of projective jets: $Ł^n$ is the set of projective jets of the form $(pj\ -)_{0,0}$ provided with an operation of group multiplication as follows:

$$(pj\ k)_{0,0} \cdot (pj\ h)_{0,0} = (pj\ k \circ h)_{0,0} \,.$$

A center-projective frame at $p \in M$ is a projective jet of the form $(pj\ x^{-1})_{x(p),p}$ where x is some parametrisation of a neighbourhood of p. (By using a suitable translation on R^n we may always assume that $x(p) = 0$.) The right action by $g \in Ł$ on a center-projective frame $\mathbf{x} = (pj\ x^{-1})_{0,p}$ may be performed as follows: If $g = (pj\ g)_{0,0}$ then we have

$$\mathbf{x} \cdot g = (pj\ x^{-1} \circ g)_{0,0} \,.$$

One may examine easily that the two parametrisations x and y of a neighbourhood of p define the same center-projective frame if and only if we have $(pj\; x \circ y^{-1}) = (pj\; \iota)$ where ι is the identity mapping of R^n.

We denote by τ (by π) the natural projections of the second order jets (of the projective jets) to the jets of the first order. Then we introduce the following equivalence relations ξ in the set of jets of the second order:

$$\xi(j^2 f)_{a,f(a)} = \xi(j^2 g)_{a,f(a)} \quad \text{if and only if} \quad (pj\, f \circ g^{-1})_{f(a),f(a)} = (pj\; \nu)_{f(a),f(a)} \,.$$

Using the same notation ξ for the mapping of L_2^n (and of H_2M respectively) onto the classes of the ξ-equivalence we have

(21) **Theorem.** *The mapping ξ preserves the group operations.*

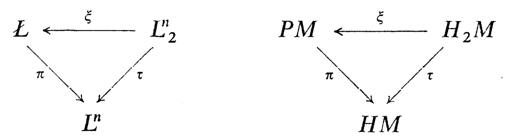
Proof. We have to show that if $\mathbf{x}$ and $\mathbf{y}$ are the frames of the second order such that $\xi(\mathbf{x}) = \xi(\mathbf{y})$ then for each $g \in L_2^n$ we have $\xi(\mathbf{x}g) = \xi(\mathbf{y}g)$. In fact, let $g = (j_2\gamma)_{0,0}$. Thus we have $(\mathrm{pj}\,(xg) \circ (yg)^{-1}) = (pj\; x \circ y^{-1}) = (pj\; \iota)$, i.e. $\xi(xg) = \xi(yg)$. We have to examine the same for the group elements. Let a_1, a_1^*, a_2, a_2^* be elements of L_2^n such that $a_\alpha = (j^2\, \varphi_\alpha)_{0,0}$, $a_\alpha^* = (j^2\, \varphi_\alpha^*)_{0,0}$ for $\alpha = 1, 2$. We have to show that if $\xi(a_\alpha) = \xi(a_\alpha^*)$ then we have $\xi(a_1 \,.\, a_2) = \xi(a_1^* \,.\, a_2^*)$. In fact we have

$$\begin{aligned}(pj\; \varphi_1 \circ \varphi_2 \circ (\varphi_1^* \circ \varphi_2^*)^{-1}) &= (pj\; \varphi_1 \circ (\varphi_2 \circ \varphi_2^{*-1}) \circ \varphi_1^{*-1}) = \\ &= (pj\; \varphi_1) \,.\, (pj\; \varphi_2 \circ \varphi_2^{*-1}) \,.\, (pj\; \varphi_1^{*-1}) = \\ &= (pj\; \varphi_1) \,.\, (pj\; \iota) \,.\, (pj\; \varphi_1^{*-1}) = (pj\; \nu) \,.\end{aligned}$$

This means that the relation ξ holds between $a_1 \,.\, a_2$ and $a_1^* \,.\, a_2^*$, q.e.d.

It follows from the above theorem that the following diagrams of homomorphisms are commutative:

(22)



Now we have to compute the mapping ξ of L_2^n in the coordinates. Notice that the parameters $A_k^i(p, y/x)$, $A_{kh}^i(p, y/x)$ are natural coordinates of an element $a \in L_2^n$, namely of that one which transforms the frame $(y_i, y_{ij})_{i,j \leq n}$ onto the frame $(x_i, x_{ij})_{i,j \leq n}$. It follows from proposition (14) that a does not change the corresponding center-projective frame $(y_J)_{J=0,\dots,n}$ if and only if $A_k^i(p, x/y) = \delta_k^i$ and $A_k^0(p, x/y) = A_l^h(p, y/x)\, A_{hk}^l(p, x/y) = 0$. Thus we have the formula

$$\xi((A_k^i, A_{kh}^i)_{i,k,h \leq n}) = (A_k^J)_{\substack{J=0,\dots,n \\ k=1,\dots,n}} \quad \text{where} \quad A_k^0 = (A^{-1})_l^h\, A_{hk}^l \,.$$

(23) **Definition.** If v and w are two fields of divergences (i.e. the local cross sections of KM) both being defined in some open set $U \subset M$, then we define their generalized Poisson bracket by

$$\mathfrak{L}_{[v,w]}s = (\mathfrak{L}_v \circ \mathfrak{L}_w - \mathfrak{L}_w \circ \mathfrak{L}_v)\, s$$

for every density s.

We shall compute v, w in the local coordinates. If we have $v = v^J \mathbf{x}_J$ and $w = w^J \mathbf{x}_J$ then for any $s \in \mathscr{S}$ of the weight q we have

$$\mathfrak{L}_v \circ \mathfrak{L}_w = \mathfrak{L}_v(q\, w^0 s + w^i x_i s) =$$
$$= v^0 \mathbf{x}_0(q\, w^0 s + w^i \mathbf{x}_i s) + v^k \mathbf{x}_k(q\, w^0 s + w^i \mathbf{x}_i s) =$$
$$= q^2 v^0 w^0 s + q\, v^0 w^i \mathbf{x}_i s + q\, v^k(\mathbf{x}_k w^0)\, s + q\, v^k w^0 \mathbf{x}_k s + v^k(\mathbf{x}_k w^i)(\mathbf{x}_i s) + v^i w^k(\mathbf{x}_{ik} s)\,.$$

Interchanging v and w we obtain

$$\mathfrak{L}_{[v,w]}s = (v^k(\mathbf{x}_k w^0) - w^k(\mathbf{x}_k v^0))\, qs + (v^k(\mathbf{x}_k w^j) - w^k(\mathbf{x}_k v^j))\, \mathbf{x}_j s\,.$$

Thus we have $[v, w] = [v, w]^J\, \mathbf{x}_J$ where

$$[v, w]^J = v^k \mathbf{x}_k w^J - w^k \mathbf{x}_k v^J\,.$$

(24) **Definition.** A proportionality class of divergences is named a punctor. (Cf. [3].)

A geometrical sense of a punctor is simple. We map the linear space $(KM)_p$ onto a center-projective space denoted by $(\Pi M)_p$. Then every divergence $v^J \mathbf{x}_J$ is mapped onto a punctor whose homogeneous coordinates are $(v^0, v^1, \ldots, v^n)$. If $v^0 \neq 0$ then this punctor may be provided with local coordinates $(z^i)_{i=1,\ldots,n}$ where $z^i = v^i/v^0$. The transformation rule of a punctor written in these coordinates is

$$z^i \to \frac{A^i_k(-, x/y)\, z^k}{1 + A^0_k(-, x/y)\, z^k}\,.$$

(25) **Definition.** The center-projective space $(\Pi M)_p$ whose elements are punctors at a fixed point $p \in M$ will be named the center-projective space.

II. Center-projective connections. We consider Lie algebras $\mathbf{L}^n$, $\mathbf{Ł}^n$, $\mathbf{L}^n_2$ of the groups L^n, $Ł^n$, L^n_2 respectively. We interpret them as vector spaces which are tangent to the corresponding group manifolds at the unit element. Formulas are known for the commutator in L^n and L^n_2 [4]. Namely, if $(\mathbf{l}^j_i)_{i,j \leq n}$ and $(\mathbf{l}^j_i, \mathbf{l}^{jk}_i)_{i,j,k \leq n}$ are natural bases in $\mathbf{L}^n$ and $\mathbf{L}^n_2$ respectively (in the traditional notation $\mathbf{l}^j_i = \partial/\partial g^i_j$, $\mathbf{l}^{jk}_i = \partial/\partial g^i_{jk}$) then we have

$$[\mathbf{l}^j_i, \mathbf{l}^k_l] = \mathbf{l}^j_l \delta^k_i - \mathbf{l}^k_i \delta^j_l\,,$$

(26) $$[\mathbf{l}^j_i, \mathbf{l}^{kh}_l] = \mathbf{l}^{jk}_l \delta^h_i + \mathbf{l}^{jh}_l \delta^k_i - \mathbf{l}^{kh}_i \delta^j_l\,,$$

(26bis) $$[\mathbf{l}^{jf}_i, \mathbf{l}^{kh}_l] = 2\mathbf{l}^{jf\,(k}_l \delta^{h)}_i - 2\mathbf{l}^{kh\,(j}_i \delta^{f)}_l\,.$$

(27) **Theorem.** *There exist homomorphisms T, Π, Ξ such that the following diagram is commutative*:

(28)
$$\begin{array}{ccc} \mathbf{Ł}^n & \xleftarrow{\ \Xi\ } & \mathbf{L}_2^n \\ & {\scriptstyle \Pi}\searrow \quad \swarrow{\scriptstyle T} & \\ & \mathbf{L}^n & \end{array}$$

Proof. We recall the first diagram (22) and put $T = \tau'(e)$, $\Pi = \pi'(e)$, $\Xi = \xi'(e)$ where denotes a tangential mapping and e is the unit element. The group $Ł^n$ is a subgroup of L^{n+1} so that we may obtain formulas for $[.,.]$ in $\mathbf{Ł}^n$ from (25) taking into account that $\mathbf{l}_0^J = 0$. We have then $[\mathbf{l}_I^J, \mathbf{l}_L^K] = \mathbf{l}_L^J\delta_I^K - \mathbf{l}_I^K\delta_L^J$. Hence we obtain formulas

(29)
$$[\mathbf{l}_i^j, \mathbf{l}_0^k] = \mathbf{l}_0^j\delta_i^k, \quad [\mathbf{l}_0^j, \mathbf{l}_0^k] = 0$$

which together with (26) yield the Lie structure of $\mathbf{Ł}^n$.

Then T is a mapping which maps $(\mathbf{l}_i^j, \mathbf{l}_i^{jk})_{i,j,k \leqq n}$ to $(\mathbf{l}_i^j)_{i,j \leqq n}$ while Π maps $(\mathbf{l}_i^j, \mathbf{l}_0^j)$ to $(\mathbf{l}_i^j)_{i,j \leqq n}$. In order to compute Ξ we differentiate ξ at the point e. In view of $\xi_0^e(A_k^i, A_{kh}^i) \to ({}^*A_j^i A_{ik}^j)$ we obtain

(30)
$$\Xi((\mathbf{l}_i^j, \mathbf{l}_i^{jk})_{i,j,k \leqq n}) = (\mathbf{l}_i^j, \mathbf{l}_0^j)$$

where $\mathbf{l}_0^j = \mathbf{l}_k^{kj}$. In order to prove that Ξ is a homomorphism with respect to $[.,.]$ we have to perform a contraction of indices in (26). Then we obtain formula (29) which expresses the Lie algebra $\mathbf{Ł}^n$. After this the commutativity of the diagram is evident, q.e.d.

Ξ may be written also in the form $\Xi(\mathbf{l}_h^{kj}) = \delta_h^k \mathbf{l}_0^j$.

We notice that the following splitting sequence

$$0 \xrightarrow{\varepsilon} \mathbf{R}^n \xrightarrow{\eta} \mathbf{Ł}^n \xrightarrow{\pi} \mathbf{L}^n \to 0$$

is exact. η denotes a mapping which transforms $(a^1, \ldots, a^n) \in R^n$ to $\sum_j a^j \mathbf{l}_0^j \in \mathbf{Ł}^n$.

Let H_2M be a principal bundle of the third order frame over M. Let p_2 be a multiplicative structure of all frames of the second order on R^n. Thus p obeys a distinguished element θ, namely a jet of the identical mapping having its source and its target at $0 \in R^n$. We denote by T_θ the space which is tangent to p_2 at θ and by $(TH_2M)_u$ the vector space tangent to H_2M at its arbitrary point u. Let $\tilde{u} \in H_3M$ and let u be its projection in H_2M. *Thus there exists an invariant form $\omega(u)$ which maps any vector $X \in (TH_2M)_u$ to some vector $\langle\omega(u) \mid X\rangle \in T_\theta$.* We refer to an intrinsic definition of ω, cf. [1]. If $\tilde{u} = (j_3 f)_{0,x}$ then $u = (j^2 f)_{0,x}$. X being a vector from $(TH_2M)_u$ there exists in H_2M a curve $R \ni \tau \to u_\tau$ such that $u_\tau \in H_2M$, $u_0 = u$ and X is tangent to this curve at u. Thus there exists a one-parameter family of mappings $\tau \to f_\tau$ such that we have $u_\tau = (j^2 f_\tau)_{0,x_\tau}$. We may assume that $f_0 = f$ and $u_0 = u$. We take into consideration the composed mapping $f^{-1} \circ f_\tau$ if τ is near to 0. The mapping $\tau \to$ $\to (j_2 f^{-1} \circ f_\tau)_{0,}$, is a curve in R^n passing through θ. We set $\langle \omega(u) \mid X \rangle$ to be equal to the vector which is tangent to this curve. If we write the decomposition

$$\omega = \omega^j \otimes \boldsymbol{l}_j + \omega^j_i \otimes \boldsymbol{l}^i_j + \omega^j_{ik} \otimes \boldsymbol{l}^{ik}_j$$

then we have the following recurrent formulas for computing of $\omega - s$ (cf. [3])

(31)
$$\begin{aligned} \mathrm{d}y^i &= a^i_j \omega^j , \\ \mathrm{d}a^i_j &= a^i_{jk}\omega^k + a^i_k \omega^k_j , \\ \mathrm{d}a^i_{hj} &= a^i_{hjk}\omega^k + a^i_{hk}\omega^k_j + a^i_{jk}\omega^k_h + a^i_k \omega^k_{hj} . \end{aligned}$$

Here y^i, a^i_j, a^i_{jk}, a^i_{hkj} are the coordinates of the jet $\tilde{u}$ which are computed with respect to some local map which maps the basic point x to $(y^1, \ldots, y^n)$. Let $(\tilde{a}^i_j)$ be reciprocal to (a^i_j). In view of proposition (10) we have $a^0_h = \tilde{a}^k_j a^j_{hk}$. Thus (a^i_j, a^0_j) are local coordinates of a projective jet which is a map of u.

(32) **Definition.** We define the center-projective frame of the r-th order at $x \in M$ to be a class of the following equivalence ϱ of local diffeomorphisms from R^n to M

$$\varrho f = \varrho g \Leftrightarrow j^r f = j^r g \quad \text{and} \quad j^r(\det j^1 f) = j^r(\det j^1 g) \quad \text{at} \quad x$$

(33) **Proposition.** *The set of center-projective frames of the r-th order at the point $0 \in R^n$ obeys the structure of a group.*

Proof is almost obvious. We name that group *the center-projective group of the r-th order.*

In particular we shall deal with the second order projective frames. Let g be a diffeomorphism of R^n into itself such that $g(0) = 0$. We have

(34) **Proposition.** *If $(g^i_j, g^i_{jk}, g^i_{jkh})$ are the coordinates of the corresponding element of L^n_3, i.e. of $(j^3 g)_{0,0}$ and $(\tilde{g}^i_j, \tilde{g}^i_{jk}, \tilde{g}^i_{jkl})$ are the coordinates of its inverse, then the coordinates of the center-projective jet of g are $(g^L_j, g^L_{jk})_{L=0,1,\ldots,n}$ where we have*

$$g^0_j = \tilde{g}^k_h g^h_{kj} , \quad g^0_{jk} = \tilde{g}^i_h g^h_{ijk} - \tilde{g}^i_l \tilde{g}^l_{rk} \tilde{g}^r_s g^s_{jk} .$$

Proof. The first formula was given in proposition (10). The second one will be obtained by some elementary operations with det $(\partial_i g^j)$, $\partial_k(\det \partial_i g^j)$ and $\partial_h \partial_k(\det \partial_i g^j)$.

If we take $a^i_j, \ldots$ instead of $g^i_j, \ldots$; $a^i_j, \ldots, a^i_{jkl}$ being local coordinates of a frame at $x \in M$, then the formula of proposition (34) yields the projection of the third order linear frames to the corresponding center-projective frames of the second order.

Now let us compute $\mathrm{d}a^0_j$ in complementary to (31). We have

(35)
$$\mathrm{d}a^0_j = \mathrm{d}(\tilde{a}^p_q a^q_{pj}) = (\mathrm{d}\tilde{a}^p_q)\, a^q_{pj} + \tilde{a}^p_q \,\mathrm{d}a^q_{pj} .$$

We use (31) and the following identities

$$(\mathrm{d}\tilde{a}^p_q)\, a^q_s = -\tilde{a}^p_q\, \mathrm{d}a^q_s = -\tilde{a}^p_q(a^q_{sl}\omega^l + a^q_l\omega^l_s)\,.$$

After some simplifications we obtain from (35)

(36) $$\mathrm{d}a^0_j = a^0_{jk}\omega^k + \omega^0_j + a^0_k\omega^k_j \quad \text{where} \quad \omega^0_j = \omega^k_{kj}\,.$$

In view of the equality $a^0_0 = 1$ we may write (31) and (36) together

(37) $$\mathrm{d}a^L_j = a^L_{jk}\omega^k + a^L_J\omega^J_j$$

and hence

(38) $$\omega^L_i = \tilde{a}^L_H(\mathrm{d}a^H_i - a^H_{ik}\omega^k)$$

where $(\tilde{a}^L_J) = (a^J_L)^{-1}$.

Now we formulate

(39) **Proposition.** The contraction $(\omega^i_{kl}) \to (\omega^0_l)$ maps the components of the canonical form on H_3M to the components of the canonical forms on the center-projective bundle.

We assume now that an infinitesimal connection on H_2M is given. We denote by γ the corresponding form of this connection. Then we write the decomposition

$$\gamma = \gamma^i_j \otimes \boldsymbol{I}^j_i + \gamma^i_{jk} \otimes \boldsymbol{I}^{jk}_i\,.$$

The canonical form ω differs from γ only by a linear combination of the forms ω^i. Thus there exists an object of connection Γ, $H_3M \ni u \to \Gamma(u)$. We have the decomposition $\Gamma = \omega^L \otimes (\Gamma^i_{jl}\boldsymbol{I}^j_i + \Gamma^i_{jkl}\boldsymbol{I}^{jk}_i)$, cf. [1]. The components of Γ are provided with the following transformation rule: if $g \in L^n_3$ then we have (cf. [5])

(40) $$\Gamma^i_{jk}(u\,.\,g) = \tilde{g}^i_s g^r_j \Gamma^s_{rk}(u) - g^s_j\tilde{g}^i_{sk}\,,$$
$$g^s_p\Gamma^k_{shi}(u\,.\,g) + g^l_h\tilde{g}^s_{pl}\Gamma^k_{si}(u\,.\,g) - \tilde{g}^k_{pl}g^l_s\Gamma^s_{hi}(u\,.\,g) =$$
$$= \tilde{g}^k_s g^r_h \Gamma^s_{pqi}(u) + \tilde{g}^k_{sl}g^l_h\Gamma^s_{pi}(u) - (\tilde{g}^k_{pl}g^l_h)_{|i}$$

where the last term is to be computed from the decomposition

$$\mathrm{d}(\tilde{g}^k_{pl}g^l_h) = (\tilde{g}^k_{pl}g^l_h)_{|i}\,\omega^i(u)\,.$$

We have then

$$\gamma^i_j = \omega^i_j + \Gamma^i_{jh}\omega^h\,, \quad \gamma^i_{jk} = \omega^i_{jk} + \Gamma^i_{jkh}\omega^h\,.$$

If we perform a contraction with respect to the indices k and h in (40), taking into account the symmetry of Γ with respect to the first two lower indices, then we obtain

$$g^j_i\tilde{g}^s_p\, \Gamma^0_{sj}(\bar{u}\,.\,\bar{g}) + (\tilde{g}^0_p)_{|i} = \Gamma^0_{pi}(u) + \tilde{g}^0_s\, \Gamma^s_{pi}(u)$$

where $\Gamma^0_{sj} = \sum \Gamma^k_{skj}$. Hence we have

(41) $$\Gamma^0_{jh}(\bar{u}\,,\,\bar{g}) = \tilde{g}^0_L\Gamma^L_{ph}g^p_j - g^p_j\tilde{g}^0_{ph}\,.$$

Here we have denoted by $\bar{u}$ (by $\bar{g}$) the center-projective frame (the element of the center-projective group) which is a canonical map of u (of g). Formula (41) may be written together with (40) in the following unified form

$$\Gamma^K_{jh}(\bar{u} \cdot \bar{g}) = g^K_L \, \Gamma^L_{ph}(\bar{u}) \, g^p_j - g^s_j \tilde{g}^K_{sh} . \tag{42}$$

Thus we have obtained the following

(43) **Theorem.** *Given any second order bundle with a connection (H_2M, γ), then there exists a projection $(H_2M, \gamma) \to (PM, \bar{\gamma})$. γ is here a connection form on PM and the corresponding object $\bar{\Gamma}$ of this connection obeys the transformulation rule* (42).

The covariant differentials of the prolongated vector field v_* and of the corresponding divergence $\bar{v}$ (Def. (17)) have the following local expressions

$$\nabla v_* = \left(\mathrm{d}v^i + v^j \gamma^i_j,\ \mathrm{d}v^i_k - v^i_j \gamma^j_k + v^j_k \gamma^i_j + v^j \gamma^i_{jk}\right)_{i,j,k=1,\dots,n} ,$$
$$\nabla \bar{v} = \left(\mathrm{d}v^L + v^j \gamma^L_j\right)_{L=0,1,\dots,n} .$$

Then the both above formulas imply easily the following

(44) **Theorem.** *The following diagram of operations is commutative*:

$$\begin{array}{ccc} v_* & \longrightarrow & \bar{v} \\ \downarrow & & \downarrow \\ \nabla v_* & \longrightarrow & \nabla \bar{v} \end{array}$$

Thus the mappings indicated above as "horizontal" are to be performed by contraction, and those indicated as "vertical" are covariant differentiations with respect to γ and to γ_* respectively.

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