Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 1, 76–98

Persistent URL: http://dml.cz/dmlcz/101004

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

ON THE MODIFIED LOGARITHMIC POTENTIAL

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(Received September 29, 1969)

Introduction. Let K be a simple oriented path of finite length in the complex plane R^2. Given a continuous real-valued function F on K, consider the corresponding integral of the Cauchy type

$$\int_K \frac{F(\xi)}{\xi - z} \, d\xi \tag{1}$$

as well as its real part

$$P_K F(z) = \operatorname{Re} \int_K \frac{F(\xi)}{\xi - z} \, d\xi \tag{2}$$

(called the modified logaritmic potential with density F) and imaginary part

$$W_K F(z) = \operatorname{Im} \int_K \frac{F(\xi)}{\xi - z} \, d\xi \tag{3}$$

(which is the double layer logarithmic potential with density F). Investigation of the behavior of $P_K F(z)$ and $W_K F(z)$ as $z \notin K$ approaches K is of importance for a number of applications (see [1], [9]). Under additional assumptions on K (like smoothness and Ljapunov condition) and F (like Hölder continuity) the integral

$$\int_K \frac{F(\xi) - F(\eta)}{\xi - z} \, d\xi$$

is well known to possess angular limits as z tends to a fixed point $\eta \in K$. Necessary and sufficient conditions on K guaranteeing the existence of angular limits of $W_K F(z)$ at η for arbitrary continuous F have been established in [3]. The present paper deals with angular limits of $P_K F$. We fix a bounded lower-semicontinuous function $Q \geqq 0$ and consider the class $\Omega_Q(\eta)$ of all continuous real-valued functions F satisfying

$$F(\xi) - F(\eta) = o(Q(\xi)) \quad \text{as} \quad \xi \to \eta . \tag{4}$$

Our main objective is to determine necessary and sufficient conditions on K (whose end-point and initial-point are denoted by β and α, respectively) guaranteeing, for any $F \in \Omega_Q(\eta)$, the existence of angular limits of

$$P_K F(z) - F(\eta) \log \frac{|\beta - z|}{|\alpha - z|}$$

at η. For this purpose it is useful to associate with K the following simple geometric quantities generalizing those introduced in [2]. Let us form the sum

$$U_K^Q(\varrho, \eta) = \sum_\xi Q(\xi), \quad \xi \in K \cap \{\xi; |\xi - \eta| = \varrho\}$$

counting, with the weight $Q(\xi)$, the points ξ in the intersection of K and the circumference of center η and radius ϱ. Then $U_K^Q(\varrho, \eta)$ is a Lebesgue measurable function of the variable $\varrho > 0$ and we may put

$$U_K^Q(\eta) = \int_0^\infty \varrho^{-1} U_K^Q(\varrho, \eta) \, \mathrm{d}\varrho .$$

Consider also, for each $\gamma \in \langle 0, 2\pi)$ and $r > 0$, the segment $S_r^\gamma(\eta) = \{\eta + \varrho e^{i\gamma}; 0 < \varrho < r\}$ and introduce the sum

$$V_{Kr}^Q(\gamma, \eta) = \sum_\xi |\xi - \eta| \, Q(\xi), \quad \xi \in K \cap S_r^\gamma(\eta),$$

counting, with the weight $|\xi - \eta| \, Q(\xi)$, the points ξ in the intersection of K and $S_r^\gamma(\eta)$. Since $V_{Kr}^Q(\gamma, \eta)$ is a Lebesgue measurable function of the variable $\gamma \in \langle 0, 2\pi)$, we are justified to define

$$V_{Kr}^Q(\eta) = \int_0^{2\pi} V_{Kr}^Q(\gamma, \eta) \, \mathrm{d}\gamma .$$

With this notation we are now in position to formulate the following typical corollary of main results (some of whose have been announced without proofs in [4], [7]) established below.

Theorem. *Let $S \subset R^2 \setminus K$ be a connected set whose closure meets K at η only. Suppose that the contingent of S at η (in the sense of* [11], *chap.* IX, § 2 – *see also theorem 9 below) together with its reflection in η is disjoint from the contingent of K at η.*

If

$$\limsup_{\substack{z \to \eta \\ z \in S}} \left| P_K F(z) - F(\eta) \log \frac{|\beta - z|}{|\alpha - z|} \right| < \infty$$

for any $F \in \Omega_Q(\eta)$, then

(5) $$U_K^Q(\eta) + \sup_{r > 0} r^{-1} V_{Kr}^Q(\eta) < \infty .$$

Conversely, suppose that (5) *holds. Let* $\theta(r) \geqq 0$ *be a bounded continuous non-decreasing function of the variable* $r \geqq 0$, $\theta \not\equiv 0$. *If* F *is a bounded Baire function on* K *satisfying*

$$F(\xi) - F(\eta) = O(\theta(|\xi - \eta|)\, Q(\xi)) \quad as \quad \xi \to \eta\,,$$

then the integral

$$\int_K \frac{F(\xi) - F(\eta)}{|\xi - \eta|}\, \mathrm{d}|\xi - \eta| = P_K^0 F(\eta)$$

converges and for $z \in S$ *the following estimate holds*

$$(6) \qquad P_K F(z) - P_K^0 F(\eta) - F(\eta) \log \frac{|\beta - z|}{|\alpha - z|} = O\left(|z - \eta| \int_{|z-\eta|}^{\infty} r^{-2}\theta(r)\, \mathrm{d}r\right)$$

as $z \to \eta$.

If F *satisfies*

$$F(\xi) - F(\eta) = o(\theta(|\xi - \eta|)\, Q(\xi)) \quad as \quad \xi \to \eta$$

then the right-hand side in (6) *can be replaced by*

$$o\left(|z - \eta| \int_{|z-\eta|}^{\infty} r^{-2}\, \theta(r)\, \mathrm{d}r\right)$$

or

$$O(|z - \eta|)$$

according as the integral $\int_0^\infty r^{-2}\, \theta(r)\, \mathrm{d}r$ *diverges or converges.*

1. Notation. If f is a complex- or real-valued function defined on an interval $J \subset R^1$ then, for each set $G \subset J$ which is open in J, $\operatorname{var} f(G)$ will denote the variation of f on G; thus $\operatorname{var} f(\emptyset) = 0$ and, for $G \neq \emptyset$, $\operatorname{var} f(G)$ is the least upper bound of all the sums

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)|\,,$$

where $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle$ are non-overlapping compact intervals contained in G. For any $M \subset J$ we let

$$\operatorname{var} f(M) = \inf_G \operatorname{var} f(G)\,,$$

where G runs over all sets $G \subset J$ that are open in J and contain M. If necessary, we shall also use the more explicit notation like $\operatorname{var}_t [f(t); M]$ to denote $\operatorname{var} f(M)$. As it is well known, $\operatorname{var} f(\ldots)$ is a Carathéodory outer measure; its restriction to var f-measurable subsets of J is a measure. The integral (over $M \subset J$) of an extended real-valued function F with respect to this measure will be denoted by the symbols

$$\int_M F \, \mathrm{d} \operatorname{var} f , \quad \int_M F(t) \, \mathrm{d} \operatorname{var} f(t) , \quad \text{etc.}$$

We shall say that f has locally finite variation on J provided $\operatorname{var} f(I) < \infty$ for every compact interval $I \subset J$. For such f the integral

$$\int_M F \, \mathrm{d}f \quad \left(= \int_M F(u) \, \mathrm{d} f(u) \right)$$

is always to be understood in the sense of Lebesgue-Stieltjes.

We shall now recall several known basic lemmas to be used below.

2. Lemma. *Let f be a continuous real-valued function of bounded variation on $\langle a, b \rangle$ and let p be a function on $f(\langle a, b \rangle)$. Suppose that p has a continuous derivative on $f(\langle a, b \rangle)$ and put $h = p \circ f$ (= the composite of f and p). Then h has bounded variation on $\langle a, b \rangle$ and, for each lower-semicontinuous (extended real-valued) function $F \geqq 0$ on $\langle a, b \rangle$,*

$$\int_a^b F \, \mathrm{d} \operatorname{var} h = \int_a^b |p'(f(t))| \, F(t) \, \mathrm{d} \operatorname{var} f(t) .$$

3. Lemma. *Let f, g be continuous functions having locally finite variation on an interval J. Then, for each lower-semicontinuous function $F \geqq 0$ on J,*

$$\int_J F \, \mathrm{d} \operatorname{var} (f \, . \, g) \leqq \int_J F|f| \, \mathrm{d} \operatorname{var} g + \int_J F|g| \, \mathrm{d} \operatorname{var} f .$$

4. Lemma. *Let f be a continuous real-valued function having locally finite variation on an interval J. Suppose that $F \geqq 0$ is a lower-semicontinuous function on J and denote, for each $u \in R^1$, by $\sigma(u; F)$ the sum*

$$\sum_t F(t) , \quad f(t) = u ,$$

which is extended over all $t \in J$ with $f(t) = u$ (so that $\sigma(u; F) = 0$ provided $u \notin f(J)$ and $\sigma(u; F) = +\infty$ whenever $F(t) > 0$ for uncountably many $t \in J$ with $f(t) = u$). Then $\sigma(u; F)$ is a Lebesgue measurable function of the variable $u \in R^1$ and

$$\int_{-\infty}^{+\infty} \sigma(u; F) \, \mathrm{d}u = \int_J F \, \mathrm{d} \operatorname{var} f .$$

5. Lemma. *Let f be a continuous (real- or complex-valued) function having locally finite variation on an interval J. Then* $\operatorname{var} f(M) = 0$ *for each $M \subset J$ with countable $f(M)$.*

For continuous F, elementary proofs of lemmas 2–4 may be found in [5] (see theorems 6.22, 6.21, 6.17); their extension to the case of a lower-semicontinuous F is immediate since such an F is a limit of a non-decreasing sequence of continuous functions. Let us note here that proof of lemma 4 is based on Banach's theorem on variation of a continuous function (see also [8], [10]). For the proof of lemma 5 (which is, in fact, an easy consequence of lemma 4) see, e.g., [3], lemma 3.4.

6. Notation. In what follows we shall always assume that ψ is a continuous complex-valued function of bounded variation on $\langle a, b\rangle$ and $q \geqq 0$ is a bounded lower-semicontinuous function on $\langle a, b\rangle$. R^2 will denote the Euclidean plane whose points will be identified with complex numbers. Given $z \in R^2$ and $\varrho > 0$ we put

$$\Omega_\varrho(z) = \{\xi \in R^2;\ |\xi - z| < \varrho\}\,,$$

$$u^q_\psi(\varrho, z) = \sum_t q(t)\,, \quad |\psi(t) - z| = \varrho\,,$$

the last sum being extended over all $t \in \langle a, b\rangle$ with $|\psi(t) - z| = \varrho$; similarly, put for any $\gamma \in \langle 0, 2\pi\rangle$

$$v^q_{\psi\varrho}(\gamma, z) = \sum_t q(t)\,|\psi(t) - z|\,,$$

where now the sum is extended over all $t \in \langle a, b\rangle$ satisfying

$$\varrho > |\psi(t) - z| > 0\,, \quad \psi(t) - z = |\psi(t) - z|\,\mathrm{e}^{i\gamma}\,.$$

If f is a function in R^1, then $\operatorname{spt} f$ will denote the support of f.

7. Lemma. *For fixed $z \in R^2$, $u^q_\psi(\varrho, z)$ is a Lebesgue measurable function of the variable ϱ and, for fixed $\varrho > 0$, $v^q_{\psi\varrho}(\gamma, z)$ is a Lebesgue measurable function of the variable $\gamma \in \langle 0, 2\pi\rangle$. The integrals*

$$u^q_\psi(z) = \int_0^\infty \varrho^{-1}\, u^q_\psi(\varrho, z)\,\mathrm{d}\varrho\,, \quad v^q_{\psi\varrho}(z) = \int_0^{2\pi} v^q_{\psi\varrho}(\gamma, z)\,\mathrm{d}\gamma \tag{7}$$

are lower-semicontinuous functions of the variable $z \in R^2$.

Proof. Given $z \in R^2$ and $\varrho > 0$ we denote by $\mathscr{S}_\varrho(z)$ the system of all components of $\{t \in \langle a, b\rangle;\ 0 < |\psi(t) - z| < \varrho\}$. With each $J \in \mathscr{S}_\varrho(z)$ we associate a continuous argument $\vartheta_z(t; J)$ of $\psi(t) - z$ on J. Denote by F_J the restriction of

$$F(t) = q(t)\,|\psi(t) - z| \tag{8}$$

to J.

Employing lemma 4 we get

$$\int_J F(t)\,\mathrm{d}\,\mathrm{var}_t\,\vartheta_z(t; J) = \sum_{n=-\infty}^{+\infty} \int_0^{2\pi} \sigma(\gamma + 2n\pi; F_J)\,\mathrm{d}\gamma\,.$$

Let us agree to write briefly $\sum_J \ldots$ for the sum extended over all $J \in \mathscr{S}_\varrho(z)$. Since

$$\sum_J \sum_{n=-\infty}^{+\infty} \sigma(\gamma + 2n\pi; F_J) = v^q_{\psi\varrho}(\gamma, z)\,, \quad \gamma \in \langle 0, 2\pi \rangle\,,$$

we conclude that $v^q_{\psi\varrho}(\gamma, z)$ is a Lebesgue measurable function of the variable $\gamma \in \in \langle 0, 2\pi \rangle$ and

$$v^q_{\psi\varrho}(z) = \sum_J \int_J F(t)\,\mathrm{d}\,\mathrm{var}_t\,\vartheta_z(t; J)\,. \tag{9}$$

Let $\mathscr{F}_\varrho(z)$ be the class of all continuous real-valued functions f in R^1 such that $|f| \leqq 1$ and

$$\operatorname{spt} f \subset \{t \in \langle a, b \rangle;\ 0 < |\psi(t) - z| < \varrho\}\,.$$

We shall first observe that

$$v^q_{\psi\varrho}(z) = \sup\left\{\operatorname{Im} \int_a^b F(t) \frac{f(t)}{\psi(t) - z}\,\mathrm{d}\psi(t); f \in \mathscr{F}_\varrho(z)\right\} \tag{10}$$

where, of course $f(t)/(\psi(t) - z)$ means 0 outside $\operatorname{spt} f \subset \{t; \psi(t) \neq z\}$. Indeed, if $f \in \mathscr{F}_\varrho(z)$, then there is a finite number of components $J_1, \ldots, J_s \in \mathscr{S}_\varrho(z)$ such that

$$\operatorname{spt} f \subset \bigcup_{j=1}^{s} J_j\,.$$

Fix $t_j \in J_j$. One easily verifies that, for $t \in J_j$,

$$\operatorname{Im} \int_{t_j}^{t} \frac{\mathrm{d}\psi(u)}{\psi(u) - z}$$

differs only by an additive constant from $\vartheta_z(t; J_j)$ (see 7.43 in [5]). Consequently,

$$\operatorname{Im} \int_a^b F(t) f(t) \frac{\mathrm{d}\psi(t)}{\psi(t) - z} = \sum_{j=1}^{s} \int_{J_j} F(t) f(t)\,\mathrm{d}_t\,\vartheta_z(t; J_j) \leqq v^q_{\psi\varrho}(z)$$

on account of (9). Fix now an arbitrary $k < v^q_{\psi\varrho}(z)$. Then there is a finite number of components $J_1, \ldots, J_n \in \mathscr{S}_\varrho(z)$ such that

$$\sum_{j=1}^{n} \int_{J_j} F(t)\,\mathrm{d}\,\mathrm{var}_t\,\vartheta_z(t; J_j) > k\,.$$

For each j $(= 1, \ldots, n)$ choose a $k_j < \int_{J_j} F(t) \, \mathrm{d} \operatorname{var}_t \vartheta_z(t; J_j)$ and an $f_j \in \mathscr{F}_\varrho(z)$ with $\operatorname{spt} f_j \subset J_j$ such that

$$\sum_{j=1}^{n} k_j = k$$

and

$$\int_{J_j} f_j(t) \, F(t) \, \mathrm{d}_t \, \vartheta_z(t; J_j) > k_j \,, \quad 1 \leqq j \leqq n \,.$$

Defining $f = \sum_{j=1}^{n} f_j$ we get $f \in \mathscr{F}_\varrho(z)$ and

$$\operatorname{Im} \int_a^b F(t) f(t) \frac{\mathrm{d}\psi(t)}{\psi(t) - z} = \sum_{j=1}^{n} \int_{J_j} f_j(t) \, F(t) \, \mathrm{d}_t \, \vartheta_z(t; J_j) > \sum_{j=1}^{n} k_j = k \,.$$

Thus (10) is established.

Given $f \in \mathscr{F}_\varrho(z)$, there is an $\varepsilon > 0$ such that $f \in \mathscr{F}_\varrho(\xi)$ for any $\xi \in \Omega_\varepsilon(z)$. Since

$$\int_a^b F(t) f(t) \frac{\mathrm{d}\, \psi(t)}{\psi(t) - \xi}$$

is a continuous function of $\xi \in \Omega_\varepsilon(z)$, we conclude from (10) that $v^q_{\psi\varrho}(\ldots)$ is lower-semicontinuous at z.

Let now J run over $\mathscr{S}_\infty(z)$. Employing lemma 4 one easily obtains that $u^q_\psi(\varrho, z)$ is a Lebesgue measurable function of the variable ϱ and

$$u^q_\psi(z) = \sum_J \int_J \frac{q(t)}{|\psi(t) - z|} \, \mathrm{d} \operatorname{var}_t |\psi(t) - z| \,. \tag{11}$$

Hence

$$u^q_\psi(z) = \sup \left\{ \operatorname{Re} \int_a^b f(t) \, q(t) \, \frac{\mathrm{d}\psi(t)}{\psi(t) - z} \,; \quad f \in \mathscr{F}_\infty(z) \right\}$$

and the lower-semicontinuity of $u^q_\psi(\ldots)$ at z follows.

8. Notation. If f is a bounded Baire function on $\langle a, b \rangle$ we define for $z \in R^2 \setminus \psi(\langle a, b \rangle)$

$$p_\psi f(z) = \int_a^b \frac{f(t)|}{\psi(t) - z|} \, \mathrm{d}_t |\psi(t) - z| \left(= \operatorname{Re} \int_a^b \frac{f(t)}{\psi(t) - z} \, \mathrm{d}\psi(t) \right).$$

Given $S \subset R^2$ and $\eta \in R^2$ we denote by

$$S \odot \eta = S \cup \{2\eta - \xi; \; \xi \in S\}$$

the union of S and its reflection in η.

9. Theorem. *Let $S \subset R^2 \setminus \psi(\langle a, b\rangle)$ be a connected set whose closure meets $\psi(\langle a, b\rangle)$ at η only.*

Suppose that

$$\limsup_{\substack{z \to \eta \\ z \in S}} |p_\psi f(z)| < \infty \tag{12}$$

for each continuous function f on $\langle a, b\rangle$ satisfying

$$f(t) = o(q(t)) \quad \text{as} \quad \psi(t) \to \eta\,. \tag{13}$$

Then

$$u_\psi^q(\eta) < \infty\,.$$

If, besides that, the contingent[1] *of $\psi(\langle a, b\rangle)$ at η does not meet the contingent of $S \odot \eta$ at η, then*

$$\sup_{r>0} r^{-1}\, v_{\psi r}^q(\eta) < \infty\,.$$

Proof. Consider the class $\mathscr{C}_q$ of all continuous functions f on $\langle a, b\rangle$ vanishing on $\{t \in \langle a, b\rangle;\ \psi(t) = \eta\}$ and satisfying (13) as well as

$$|f| \leqq c_f q \tag{14}$$

for suitable constant c_f (depending on f). Defining $\|f\|$ as the least upper bound of all c_f satisfying (14) we get a norm on $\mathscr{C}_q$ which turns $\mathscr{C}_q$ into a Banach space. Note that, for $f \in \mathscr{C}_q$ and $z \notin \psi(\langle a, b\rangle)$,

$$|p_\psi f(z)| \leqq \|f\| \int_a^b q \,\mathrm{d}\operatorname{var}\psi / \operatorname{dist}(\psi(\langle a, b\rangle), z)\,, \tag{15}$$

where $\operatorname{dist}(\psi(\langle a, b\rangle), z) = \inf\{|\psi(t) - z|;\ t \in \langle a, b\rangle\}$. Combining (15) with the assumption (12) we conclude that

$$f \in \mathscr{C}_q \Rightarrow \sup_{z \in S} |p_\psi f(z)| < \infty\,. \tag{16}$$

With each $z \in S$ we associate the functional L_z defined by

$$L_z(f) = p_\psi f(z)\,, \quad f \in \mathscr{C}_q\,.$$

Clearly, each L_z is a bounded linear functional on $\mathscr{C}_q$ whose norm is given by

$$\|L_z\| = \int_a^b \frac{q(t)}{|\psi(t) - z|} \,\mathrm{d}\operatorname{var}_t |\psi(t) - z|\,. \tag{17}$$

[1]) cf. [11], chap. IX, § 2; let us recall that a half-line $H \subset R^2$ issuing at η belongs to the contingent of $K \subset R^2$ at η provided there are points $z_n \in K \setminus \{\eta\}$ $(n = 1, 2, \ldots)$ tending to η such that the half-lines $\{\eta + r(z_n - \eta);\ r \geqq 0\}$ converge (in the natural sense) to H.

Since, in view of (16), all the functionals in $\{L_z\}_{z \in S}$ are pointwise bounded on $\mathscr{C}_q$, we conclude by the principle of uniform boundedness that

(18) $$\sup_{z \in S} \|L_z\| = c < \infty .$$

According to (11) we have for $z \in S$ $(\subset R^2 \setminus \psi(\langle a, b \rangle))$

(19) $$u_\psi^q(z) = \int_a^b \frac{q(t)}{|\psi(t) - z|} \,\mathrm{d}\,\mathrm{var}_t\, |\psi(t) - z| .$$

Combining this with (17), (18) we arrive at

(20) $$\sup_{z \in S} u_\psi^q(z) = c < \infty ,$$

which implies

(21) $$u_\psi^q(\eta) \leqq c$$

by the lower-semicontinuity of $u_\psi^q(\ldots)$ established in lemma 7. Suppose now that the contingent of $\psi(\langle a, b \rangle)$ at η is disjoint from the contigent of $S \odot \eta$ at η. Given $r > 0$ denote by $\mathscr{S}_r(\eta)$ the system of all components of $\{t \in \langle a, b \rangle;\ 0 < |\psi(t) - \eta| < r\}$. With each $I \in \mathscr{S}_\infty(\eta)$ and $z \in R^2 \setminus \{\eta\}$ we associate a continuous argument $\omega_z(t; I)$ of $(\psi(t) - \eta)/(z - \eta)$ on I. It is easily seen that there is an $R_0 > 0$ such that for $t \in I \in$ $\in \mathscr{S}_\infty(\eta)$ and $z \in S$

(22) $$(|z - \eta| < R_0,\ |\psi(t) - \eta| < R_0) \Rightarrow |\sin \omega_z(t; I)| \geqq R_0 .$$

We may assume R_0 to be small enough to guarantee

$$S \cap \{z; |z - \eta| = R_0\} \neq \emptyset$$

(note that S is connected and η belongs to the closure of S). Consider now an arbitrary r with $0 < r \leqq R_0$ and choose a $z \in S$ with $|z - \eta| = r$. Consider a $J \in \mathscr{S}_r(\eta)$. There is a uniquely determined $I_J \in \mathscr{S}_\infty(\eta)$ containing J and we put

$$\omega_z(t; J) = \omega_z(t; I_J) , \quad t \in J .$$

For the sake of brevity, we shall also write

$$\varrho_\xi(t) = |\psi(t) - \xi| , \quad t \in \langle a, b \rangle , \quad \xi \in R^2 .$$

With this notation we have for $t \in J$ $(\in \mathscr{S}_r(\eta))$

$$\varrho_z(t) \leqq \varrho_\eta(t) + |z - \eta| \leqq 2r ,$$

$$\varrho_z^2(t) = \varrho_\eta^2(t) + r^2 - 2r\, \varrho_\eta(t) \cos \omega_z(t; J) ,$$

whence

$$\int_J q(t)\,\varrho_z^{-2}(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_z^2(t) \geqq (2r)^{-2}\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_\eta^2(t) - 2r\,\varrho_\eta(t)\cos\omega_z(t;J)] \geqq$$

$$\geqq (2r)^{-1}\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_\eta(t)\cos\omega_z(t;J)] -$$

$$-(2r)^{-2}\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_\eta^2(t) \geqq (2r)^{-1}\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_\eta(t)\cos\omega_z(t;J)] -$$

$$-\frac{1}{4}\int_J q(t)\,\varrho_\eta^{-2}(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_\eta^2(t) = (\text{see lemma 2}) =$$

$$= (2r)^{-1}\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_\eta(t)\cos\omega_z(t;J)] - \frac{1}{2}\int_J q(t)\,\varrho_\eta^{-1}(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_\eta(t)\,.$$

Hence we get by (19), (20) and lemma 2 letting J run over $\mathscr{S}_r(\eta)$

$$c \geqq u_\psi^q(z) = \int_a^b q(t)\,\varrho_z^{-1}(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_z(t) =$$

$$= \frac{1}{2}\int_a^b q(t)\,\varrho_z^{-2}(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_z^2(t) \geqq (4r)^{-1}\sum_J\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_\eta(t)\cos\omega_z(t;J)] -$$

$$-\frac{1}{4}\sum_J\int_J q(t)\,\varrho_\eta^{-1}(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_\eta(t) =$$

$$= (\text{employ (11) with } z \text{ replaced by } \eta) = (4r)^{-1}\sum_J\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_\eta(t)\cos\omega_z(t;J)] -$$

$$-\frac{1}{4}\,u_\psi^q(\eta) \geqq (\text{see (21)}) \geqq$$

$$\geqq \frac{1}{4}\left\{r^{-1}\sum_J\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_\eta(t)\cos\omega_z(t;J)] - c\right\}.$$

Consequently,

$$r^{-1}\sum_J\int_J q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_\eta(t)\cos\omega_z(t;J)] \leqq 5c\,. \tag{23}$$

Using lemmas 2 and 3 we obtain

$$\sum_J\int_J \varrho_\eta(t)\,q(t)\,\mathrm{d}\,\mathrm{var}_t\,[\cos\omega_z(t;J)] =$$

$$= \sum_J\int_J \varrho_\eta(t)\,q(t)\,\mathrm{d}\,\mathrm{var}_t\,\frac{\varrho_\eta(t)\cos\omega_z(t;J)}{\varrho_\eta(t)} \leqq$$

$$\leqq \sum_J \int_J q(t)\, \mathrm{d}\, \mathrm{var}_t\, [\varrho_\eta(t) \cos \omega_z(t; J)] + \sum_J \int_J q(t)\, |\cos \omega_z(t; J)|\, \mathrm{d}\, \mathrm{var}_t\, \varrho_\eta(t) \leqq$$

$$\leqq (\text{see } (23)) \leqq 5cr + \sum_J r \int_J \varrho_\eta^{-1}(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \varrho_\eta(t) =$$

$$= 5cr + r\, u_\psi^q(\eta) \leqq (\text{see } (21)) \leqq 6cr\,.$$

On the other hand, lemma 2 together with (22) yield for any $J \in \mathscr{S}_r(\eta)$

$$\int_J \varrho_\eta(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, [\cos \omega_z(t; J)] = \int_J \varrho_\eta(t)\, q(t)\, |\sin \omega_z(t; J)|\, \mathrm{d}\, \mathrm{var}_t\, \omega_z(t; J) \geqq$$

$$\geqq R_0 \int_J \varrho_\eta(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \omega_z(t; J)\,,$$

whence we conclude

$$\sum_J \int_J \varrho_\eta(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \omega_z(t; J) \leqq 6crR_0^{-1}\,.$$

Noting that $\omega_z(t; J)$ differs only by an additive constant from a continuous argument of $\psi(t) - \eta$ on J we have by (9), (8)

$$\sum_J \int_J \varrho_\eta(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \omega_z(t; J) = v_{\psi r}^q(\eta)\,.$$

We have thus shown that

$$0 < r \leqq R_0 \Rightarrow r^{-1}\, v_{\psi r}^q(\eta) \leqq 6cR_0^{-1}\,.$$

Consider now an arbitrary $r > R_0$. It follows easily from (10), (8) that

$$r^{-1}\, v_{\psi r}^q(\eta) \leqq R_0^{-1} \int_a^b q(t)\, \mathrm{d}\, \mathrm{var}\, \psi(t)\,,$$

so that

$$\sup_{r>0} r^{-1}\, v_{\psi r}^q(\eta) \leqq R_0^{-1} \max \left\{6c, \int_a^b q\, \mathrm{d}\, \mathrm{var}\, \psi\right\}$$

and the proof is complete.

10. Remark. *If $S \subset R^2 \backslash \psi(\langle a, b\rangle)$ is a connected set whose closure meets $\psi(\langle a, b\rangle)$ at a single point η such that the contingent of $S \odot \eta$ at η is disjoint from the contingent of $\psi(\langle a, b\rangle)$ at η, then*

(24) $$\sup_{z \in S} u_\psi^q(z) < \infty$$

implies

$$(25) \qquad u^q_\psi(\eta) + \sup_{r>0} r^{-1}\, v^q_{\psi r}(\eta) < \infty \,.$$

This has been established in the course of the above proof. The converse of this assertion is also valid as shown in proposition 12 which will be needed below. Before going into its proof we shall recall the following known lemma (which, as shown in [6], may be used as a basis for development of the Lebesgue theory of integration).

11. Lemma. *Let $\mu \geqq 0$ be a measure defined on Borel subsets of an interval I and suppose that $F \geqq 0$ is an extended real-valued Baire function on I. Given $\tau > 0$ let*

$$F_\tau = \{t \in I;\ F(t) > \tau\}\,.$$

Then

$$\int_I F \,\mathrm{d}\mu = \int_0^\infty \mu(F_\tau)\,\mathrm{d}\tau\,.$$

Now we are in position to prove the following

12. Proposition. *Let $S \subset R^2 \backslash \psi(\langle a, b\rangle)$ be a set whose closure meets $\psi(\langle a, b\rangle)$ at a single point η. Suppose that the contingent of $\psi(\langle a, b\rangle)$ at η is disjoint from the contingent of $S \odot \eta$ at η.*

If

$$u^q_\psi(\eta) + \sup_{r>0} r^{-1}\, v^q_{\psi r}(\eta) < \infty$$

then

$$\sup_{z\in \bar{S}} u^q_\psi(z) < \infty\,.$$

Proof. In accordance with the notation introduced earlier we shall write $\mathscr{S}_\infty(\eta)$ for the system of all components of $\{t \in \langle a, b\rangle;\ |\psi(t) - \eta| > 0\}$ and, for each $I \in \mathscr{S}_\infty(\eta)$, we fix a continuous argument $\vartheta_\eta(t; I)$ of $\psi(t) - \eta$ on I; given $z \in R^2 \backslash \{\eta\}$ we denote by $\omega_z(t; I)$ a continuous argument of $[\psi(t) - \eta]/z - \eta$ (so that $\vartheta_\eta(\ldots; I) - \omega_z(\ldots; I)$ is constant on I). For the sake of brevity we put for $z \in R^2$

$$(26) \qquad \varrho_z(t) = |\psi(t) - z|\,, \quad t \in \langle a, b\rangle\,.$$

We agree to use I as a generic notation for elements of $\mathscr{S}_\infty(\eta)$ and set for $r > 0$

$$I_r = \{t \in I;\ \varrho_\eta(t) < r\}\,, \quad I^r = I \backslash I_r\,.$$

Let

$$k = \sup_{r>0} r^{-1}\, v^q_{\psi r}(\eta)\,.$$

In view of (9), (8) we have for any $r > 0$

(27) $$v^q_{\psi r}(\eta) = \sum_I \int_{I_r} \varrho_\eta(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \vartheta_\eta(t; I) \leqq kr\,.$$

Consider now the sum

(28) $$s(r) = \sum_I \int_{I^r} \varrho_\eta^{-1}(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \vartheta_\eta(t; I)\,.$$

Put for $\tau > 0$

$$I^r_\tau = \{t \in I;\ r \leqq \varrho_\eta(t) < \tau^{-1/2}\}\,,$$

so that

$$I^r_\tau = \{t \in I^r;\ \varrho_\eta^{-2}(t) > \tau\}\,.$$

Defining

$$\mu(B) = \int_B \varrho_\eta(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \vartheta_\eta(t; I)$$

for Borel sets $B \subset I$ and employing lemma 11 one easily obtains

$$\int_{I^r} \varrho_\eta^{-1}(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \vartheta_\eta(t; I) = \int_{I^r} \varrho_\eta^{-2}(t)\, \mathrm{d}\, \mu(t) = \int_0^{r^{-2}} \mu(I^r_\tau)\, \mathrm{d}\tau\,.$$

Noting that $I^r_\tau \subset I_x$ with $x = \tau^{-1/2}$ we conclude from (27) that

$$\sum_I \mu(I^r_\tau) \leqq \sum_I \mu(I_x) \leqq kx = k\tau^{-1/2}\,,$$

whence

$$s(r) \leqq \int_0^{r^{-2}} k\tau^{-1/2}\, \mathrm{d}\tau = 2kr^{-1}\,.$$

Consequently,

(29) $$\sup_{r>0} r\, s(r) \leqq 2k\,.$$

Denote by T the union of $\mathscr{S}_\infty(\eta)$. Since

$$\psi(\langle a, b\rangle \backslash T) = \{\eta\}\,,$$

we infer from lemma 5

$$\mathrm{var}\, \psi(\langle a, b\rangle \backslash T) = 0\,.$$

Let now z be an arbitrary point in R^2. Since $\mathrm{var}\, \psi = \mathrm{var}\, (\psi - z)$ dominates $\mathrm{var}\, \varrho_z$, we have also

(30) $$\mathrm{var}\, \varrho_z(\langle a, b\rangle \backslash T) = 0$$

and (19) yields

$$u_\psi^q(z) = \sum_I \int_I \varrho_z^{-1}(t)\, q(t)\, \mathrm{d}\, \mathrm{var}_t\, \varrho_z(t)\,. \tag{31}$$

In view of our assumptions concerning the contingents of $S \odot \eta$ and $\psi(\langle a, b \rangle)$ at η, there is an $R_0 > 0$ such that (22) holds for $z \in S$ and $t \in I$ $(\in \mathscr{S}_\infty(\eta))$. Let $z \in S$, put $|z - \eta| = r$ and assume $r < R_0$. Then

$$\frac{\varrho_n(t)}{\varrho_z(t)} \leqq \frac{1}{\sin \omega_z(t; I)} \leqq R_0^{-1}\,, \tag{32}$$

$$\frac{r}{\varrho_z(t)} \leqq \frac{1}{\sin \omega_z(t; I)} \leqq R_0^{-1} \tag{33}$$

and

$$\varrho_z^2(t) = r^2 + \varrho_\eta^2(t) - 2r\, \varrho_\eta(t) \cos \omega_z(t; I)\,. \tag{34}$$

Hence we obtain using lemmas 2,3

$$\int_I q(t)\, \varrho_z^{-1}(t)\, \mathrm{d}\, \mathrm{var}_t\, \varrho_z(t) = \frac{1}{2} \int_I q(t)\, \varrho_z^{-2}(t)\, \mathrm{d}\, \mathrm{var}_t\, [\varrho_\eta^2(t) - 2r\, \varrho_\eta(t) \cos \omega_z(t; I)] \leqq$$

$$\leqq \frac{1}{2} \int_I q(t)\, \varrho_z^{-2}(t)\, \mathrm{d}\, \mathrm{var}_t\, \varrho_\eta^2(t) + r \int_I q(t)\, \varrho_z^{-2}(t)\, \mathrm{d}\, \mathrm{var}_t\, [\varrho_\eta(t) \cos \omega_z(t; I)] \leqq$$

$$\leqq \frac{1}{2} R_0^{-2} \int_I q(t)\, \varrho_\eta^{-2}(t)\, \mathrm{d}\, \mathrm{var}\, \varrho_\eta^2(t) + r \int_{I_r} q(t)\, \varrho_z^{-2}(t)\, \varrho_\eta(t)\, \mathrm{d}\, \mathrm{var}\, \omega_z(t; I) +$$

$$+ r \int_{I_r} q(t)\, \varrho_z^{-2}(t)\, \mathrm{d}\, \mathrm{var}\, \varrho_\eta(t) +$$

$$+ r \int_{I^r} q(t)\, \varrho_z^{-2}(t)\, \varrho_\eta(t)\, \mathrm{d}\, \mathrm{var}\, \omega_z(t; I) + r \int_{I^r} q(t)\, \varrho_z^{-2}(t)\, \mathrm{d}\, \mathrm{var}\, \varrho_\eta(t) \leqq$$

$$\leqq R_0^{-2} \Big[\int_I q(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\, \mathrm{var}\, \varrho_\eta(t) + r^{-1} \int_{I_r} q(t)\, \varrho_\eta(t)\, \mathrm{d}\, \mathrm{var}\, \omega_z(t; I) +$$

$$+ \int_{I_r} q(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\, \mathrm{var}\, \varrho_\eta(t) + r \int_{I^r} q(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\, \mathrm{var}\, \omega_z(t; I) +$$

$$+ \int_{I_r} q(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\, \mathrm{var}\, \varrho_\eta(t) \Big] = R_0^{-2} \Big[2 \int_I q(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\, \mathrm{var}\, \varrho_\eta(t) +$$

$$+ r^{-1} \int_{I_r} q(t)\, \varrho_\eta(t)\, \mathrm{d}\, \mathrm{var}\, \omega_z(t; I) + r \int_{I_r} q(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\, \mathrm{var}\, \omega_z(t; I) \Big]\,.$$

Making use of (31), (27), (28) and (29) we get

$$u_\psi^q(z) \leqq R_0^{-2}[2u_\psi^q(\eta) + 3k]\,.$$

Since η is the only point in $\psi(\langle a, b \rangle)$ belonging to the closure of S, we have

$$\inf \{\varrho_z(t);\ |z - \eta| \geqq R_0,\ t \in \langle a, b \rangle\} = \delta > 0\,,$$

whence it follows for any $z \in S$ with $|z - \eta| \geqq R_0$

$$u_\psi^q(z) = \int_a^b q(t)\, \varrho_z^{-1}(t)\, \mathrm{d} \operatorname{var} \varrho_z(t) \leqq \delta^{-1} \int_a^b q(t)\, \mathrm{d} \operatorname{var} \psi(t)\,.$$

We conclude that

$$\sup_{z \in S} u_\psi^q(z) \leqq \max \left\{\delta^{-1} \int_a^b q\, \mathrm{d} \operatorname{var} \psi,\ R_0^{-2}[2u_\psi^q(\eta) + 3k]\right\}.$$

13. Remark. The above proposition together with remark 10 form an alternative to the inequalities concerning so-called cyclic and radial variation as established in [2].

Now we are able to show that the converse of theorem 9 is also valid. We shall derive a more precise result.

14. Theorem. *Let $S \subset R^2 \backslash \psi(\langle a, b \rangle)$ be a set whose closure meets $\psi(\langle a, b \rangle)$ at a single point η. Let the contingent of $S \odot \eta$ at η be disjoint from the contingent of $\psi(\langle a, b \rangle)$ at η and assume (25). Let $\theta(r) \geqq 0$ be a continuous non-decreasing function of the variable $r \geqq 0$, $\theta \not\equiv 0$. If $\varkappa \in R^1$ and f is a continuous function on $\langle a, b \rangle$ satisfying*

$$|f(t) - \varkappa| = O(\theta(|\psi(t) - \eta|)\, q(t)) \quad \text{as} \quad \psi(t) \to \eta\,, \tag{35}$$

then for $z \in S$

$$\begin{aligned} p_\psi f(z) - \varkappa \log \frac{|\psi(b) - z|}{|\psi(a) - z|} - \int_a^b \frac{f(t) - \varkappa}{|\psi(t) - \eta|}\, \mathrm{d}_t\, |\psi(t) - \eta| = \\ = O\left(|z - \eta| \int_{|z-\eta|}^\infty \theta(x)\, x^{-2}\, \mathrm{d}x\right) \quad \text{as} \quad z \to \eta\,. \end{aligned} \tag{36}$$

If f satisfies (35) with O replaced by o, then the right-hand side in (36) can be replaced by

$$o\left(|z - \eta| \int_{|z-\eta|}^\infty \theta(x)\, x^{-2}\, \mathrm{d}x\right) + O(|z - \eta|)\,.$$

Proof. For the sake of brevity, we put

$$f_\varkappa(t) = f(t) - \varkappa\,, \quad t \in \langle a, b \rangle\,,$$

and adopt the notation introduced in (26). Then

$$p_\psi f(z) = \varkappa \log \frac{|\psi(b) - z|}{|\psi(a) - z|} + \int_a^b f_\varkappa(t)\, \varrho_z^{-1}(t)\, \mathrm{d}\, \varrho_z(t)\,, \quad z \in S\,. \tag{37}$$

Fix now constants $\varepsilon > 0$, $R_0 > 0$ such that, for $t \in \langle a, b \rangle$,

$$0 < \varrho_\eta(t) < R_0 \Rightarrow |f_\varkappa(t)| \leqq \varepsilon\theta(\varrho_\eta(t))\, q(t)\,. \tag{38}$$

Let $z \in S$, $|z - \eta| = r$, and put

$$M_r = \{t \in \langle a, b \rangle;\ 0 < \varrho_\eta(t) < r\}\,.$$

Then

$$\left|\int_{M_r} f_\varkappa(t)\, \varrho_z^{-1}(t)\, \mathrm{d}\varrho_z(t)\right| \leqq \varepsilon\theta(r) \int_{M_r} \varrho_z^{-1}(t)\, \mathrm{d}\ \mathrm{var}\ \varrho_z(t) \leqq \varepsilon\theta(r)\, u_\psi^q(z) \tag{39}$$

by (19). Similarly, (11) implies

$$\left|\int_{M_r} f_\varkappa(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\varrho_\eta(t)\right| \leqq \varepsilon\theta(r)\, u_\psi^q(\eta)\,. \tag{40}$$

Next consider the set

$$L^r = M_{R_0} \setminus M_r = \{t \in \langle a, b \rangle;\ r \leqq \varrho_\eta(t) < R_0\}\,.$$

We have

$$\int_{L^r} f_\varkappa(t)\, \varrho_z^{-1}(t)\, \mathrm{d}\varrho_z(t) - \int_{L^r} f_\varkappa(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\varrho_\eta(t) = A_1 + A_2\,, \tag{41}$$

where

$$A_1 = \int_{L^r} f_\varkappa(t)\, [\varrho_z^{-1}(t) - \varrho_\eta^{-1}(t)]\, \mathrm{d}\varrho_z(t)\,,$$

$$A_2 = \int_{L^r} f_\varkappa(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}_t[\varrho_z(t) - \varrho_\eta(t)]\,.$$

Defining

$$A = \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_z^{-1}(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\ \mathrm{var}\ \varrho_z(t) \tag{42}$$

we get

$$|A_1| \leqq \varepsilon \int_{rL} \theta(\varrho_\eta(t))\, q(t)\, |\varrho_z(t) - \varrho_\eta(t)|\, \varrho_z^{-1}(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\ \mathrm{var}\ \varrho_z(t) \leqq \varepsilon r A\,. \tag{43}$$

Define the measure ν on Borel sets $B \subset \langle a, b \rangle$ by

$$\nu(B) = \int_{B \cap L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_z^{-1}(t)\, \mathrm{d}\ \mathrm{var}\ \varrho_z(t)\,, \tag{44}$$

so that

$$A = \int_a^b \varrho_\eta^{-1}(t)\, \mathrm{d}\nu(t)\,.$$

Applying lemma 11 and noting that

$$L^r_u = \{t \in L^r;\ \varrho_\eta^{-1}(t) > u\} = \{t \in \langle a, b\rangle;\ r \leqq \varrho_\eta(t) < u^{-1}\}$$

equals $\emptyset$ or L^r according as $u > r^{-1}$ or $u \leqq R_0^{-1}$, one easily obtains

$$A = \nu(L^r)\, R_0^{-1} + \int_{R_0^{-1}}^{r^{-1}} \nu(L^r_u)\, \mathrm{d}u\,.$$

Introducing the variable $\tau = u^{-1}$ we have for $R_0^{-1} < u \leqq r^{-1}$

$$L^r_u = M_\tau \backslash M_r$$

and the last integral transforms into

$$\int_r^{R_0} \tau^{-2} \nu(M_\tau \backslash M_r)\, \mathrm{d}\tau\,.$$

Using the estimates (compare (44), (19))

$$\nu(M_\tau \backslash M_r) \leqq \nu(M_\tau) \leqq \theta(\tau)\, u^q_\psi(z)\,,$$
$$\nu(L^r) \leqq \nu(M_{R_0}) \leqq \theta(R_0)\, u^q_\psi(z)\,,$$

we get

$$A \leqq u^q_\psi(z) \left[\theta(R_0)\, R_0^{-1} + \int_r^{R_0} \theta(\tau)\, \tau^{-2}\, \mathrm{d}\tau\right]. \tag{45}$$

Next consider

$$|A_2| \leqq \varepsilon \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\,\mathrm{var}_t \frac{\varrho_z^2(t) - \varrho_\eta^2(t)}{\varrho_z(t) + \varrho_\eta(t)} \leqq$$
$$\leqq \varepsilon \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t) \frac{\varrho_z^2(t) - \varrho_\eta^2(t)}{[\varrho_z(t) + \varrho_\eta(t)]^2}\, \mathrm{d}\,\mathrm{var}_t\, [\varrho_z(t) + \varrho_\eta(t)] +$$
$$+ \varepsilon \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t)\, [\varrho_z(t) + \varrho_\eta(t)]^{-1}\, \mathrm{d}\,\mathrm{var}_t\, [\varrho_z^2(t) - \varrho_\eta^2(t)]\,.$$

We have thus

$$|A_2| \leqq \varepsilon(rC_1 + rC_2 + C_3)\,, \tag{46}$$

where

$$C_1 = \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t)\, [\varrho_z(t) + \varrho_\eta(t)]^{-1}\, \mathrm{d}\operatorname{var} \varrho_z(t)\,,$$

$$C_2 = \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t)\, [\varrho_z(t) + \varrho_\eta(t)]^{-1}\, \mathrm{d}\operatorname{var} \varrho_\eta(t)\,,$$

$$C_3 = \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t)\, [\varrho_z(t) + \varrho_\eta(t)]^{-1}\, \mathrm{d}\operatorname{var} [\varrho_z^2(t) - \varrho_\eta^2(t)]\,.$$

As before, we associate with each component I of $\{t \in \langle a, b\rangle;\ \varrho_\eta(t) > 0\} = M_\infty$ a continuous argument $\omega_z(t; I)$ of $[\psi(t) - \eta]/z - \eta$ on I. We may clearly assume that $R_0 > 0$ has been chosen small enough to guarantee (22) for $z \in S$ and $t \in I \in \in \mathscr{S}_\infty(\eta)$ (= the system of all components of M_∞). Noting that, for $t \in L^r$,

$$\varrho_z(t) \leqq r + \varrho_\eta(t) \leqq 2\varrho_\eta(t)$$

and assuming $|z - \eta| = r < R_0$ we infer from (32)

$$t \in L^r \Rightarrow \tfrac{1}{2} \leqq \varrho_\eta(t)\, \varrho_z^{-1}(t) \leqq R_0^{-1}\,. \tag{47}$$

This permits us to derive the following estimates

(48)

$$C_1 \leqq \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t)\, \varrho_z^{-1}(t)\, (1 + \tfrac{1}{2})^{-1}\, \mathrm{d}\operatorname{var} \varrho_z(t) \leqq \tfrac{2}{3}A \quad \text{(compare (45))}\,,$$

$$C_2 \leqq \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-2}(t)\, (1 + R_0)^{-1}\, \mathrm{d}\operatorname{var} \varrho_\eta(t) = (1 + R_0)^{-1}\, \bar{A}\,,$$

where

$$\bar{A} = \int_{L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-2}(t)\, \mathrm{d}\operatorname{var} \varrho_\eta(t)\,. \tag{49}$$

Defining the measure $\bar{\nu}$ on Borel sets $B \subset \langle a, b\rangle$ by

$$\bar{\nu}(B) = \int_{B \cap L^r} \theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t)\, \mathrm{d}\operatorname{var} \varrho_\eta(t)$$

and repeating the argument used above for the estimate of A we obtain

$$\bar{A} = \int_a^b \varrho_\eta^{-1}(t)\, \mathrm{d}\bar{\nu}(t) = \bar{\nu}(L^r)\, R_0^{-1} + \int_r^{R_0} \tau^{-2}\, \bar{\nu}(M_\tau \backslash M_r)\, \mathrm{d}\tau\,,$$

$$\bar{\nu}(M_\tau \backslash M_r) \leqq \theta(\tau)\, u_\psi^q(\eta)\,, \quad \bar{\nu}(L^r) \leqq \theta(R_0)\, u_\psi^q(\eta)\,,$$

whence

$$\bar{A} \leqq u_\psi^q(\eta)\left[\theta(R_0)\,R_0^{-1} + \int_r^{R_0} \tau^{-2}\,\theta(\tau)\,\mathrm{d}\tau\right], \tag{50}$$

so that

$$C_2 \leqq (1 + R_0)^{-1}\,u_\psi^q(\eta)\left[\theta(R_0)\,R_0^{-1} + \int_r^{R_0} \tau^{-2}\,\theta(\tau)\,\mathrm{d}\tau\right]. \tag{51}$$

Let now J range over the system $\mathscr{S}_{R_0}(\eta)$ of all components of $M_{R_0} = \{t \in \langle a, b\rangle; \varrho_\eta(t) < R_0\}$ and put

$$J^r = L^r \cap J\,.$$

Each J is contained in a uniquely determined $I_J \in \mathscr{S}_\infty(\eta)$ and we put $\omega_z(t; J) = \omega_z(t; I_J)$, $t \in J$. Employing (47) and (34) we get by lemmas 2, 3

$$C_3 \leqq \int_{L^r} \theta(\varrho_\eta(t))\,q(t)\,\varrho_\eta^{-2}(t)\,(1 + R_0)^{-1}\,\mathrm{d}\,\mathrm{var}_t\,[\varrho_z^2(t) - \varrho_\eta^2(t)] \leqq$$

$$\leqq \sum_J 2r\left[\int_{J^r} \theta(\varrho_\eta(t))\,q(t)\,\varrho_\eta^{-2}(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_\eta(t) + \int_{J^r} \theta(\varrho_\eta(t))\,q(t)\,\varrho_\eta^{-1}(t)\,\mathrm{d}\,\mathrm{var}_t\,\omega_z(t; J)\right] =$$

$$= 2r\int_{L^r} \theta(\varrho_\eta(t))\,q(t)\,\varrho_\eta^{-2}(t)\,\mathrm{d}\,\mathrm{var}_t\,\varrho_\eta(t) +$$

$$+ 2\sum_J r\int_{J^r} \theta(\varrho_\eta(t))\,q(t)\,\varrho_\eta^{-1}(t)\,\mathrm{d}\,\mathrm{var}_t\,\omega_z(t; J)\,.$$

Recalling (49) we may write

$$C_3 \leqq 2r\bar{A} + 2\sum_J r\int_{J^r} \theta(\varrho_\eta(t))\,q(t)\,\varrho_\eta^{-1}(t)\,\mathrm{d}\,\mathrm{var}_t\,\omega_z(t; J)\,. \tag{52}$$

Define now the measures μ, μ_1 on Borel sets $B \subset J$ by

$$\mu(B) = \int_B \varrho_\eta(t)\,q(t)\,\mathrm{d}\,\mathrm{var}_t\,\omega_z(t; J)\,,$$

$$\mu_1(B) = \int_B \theta(\varrho_\eta(t))\,\varrho_\eta(t)\,q(t)\,\mathrm{d}\,\mathrm{var}_t\,\omega_z(t; J)\,,$$

so that

$$\int_{J^r} \theta(\varrho_\eta(t))\,q(t)\,\varrho_\eta^{-1}(t)\,\mathrm{d}\,\mathrm{var}_t\,\omega_z(t; J) = \int_{J^r} \varrho_\eta^{-2}(t)\,\mathrm{d}\mu_1(t)\,. \tag{53}$$

Put for $\tau > 0$

$$J_\tau^r = \{t \in J^r;\ \varrho_\eta^{-2}(t) > \tau\}$$

and observe that $J_\tau^r = \emptyset$ for $\tau > r^{-2}$ and

$$J_\tau^r = \{t \in J;\ r \leqq \varrho_\eta(t) < \tau^{-1/2}\} \quad \text{for} \quad 0 < \tau \leqq r^{-2}\,.$$

Hence

$$\mu_1(J_\tau^r) \leqq \theta(\tau^{-1/2})\, \mu(J_\tau^r)$$

and we get by lemma 2

$$\int_{J^r} \varrho_\eta^{-2}(t)\, \mathrm{d}\mu_1(t) = \int_0^{r^{-2}} \mu_1(J_\tau^r)\, \mathrm{d}\tau \leqq \int_0^{r^{-2}} \theta(\tau^{-1/2})\, \mu(J_\tau^r)\, \mathrm{d}\tau\,. \tag{54}$$

Since $\omega_z(t; J)$ differs only by an additive constant from a continuous argument $\vartheta_\eta(t; J)$ of $\psi(t) - \eta$ on J and

$$J_\tau^r \subset \{t \in J;\ \varrho_\eta(t) < x\} = J_x \quad \text{with} \quad x = \tau^{-1/2}\,,$$

we obtain by (27)

$$\sum_J \mu(J_\tau^r) \leqq k\tau^{-1/2}$$

(recall that

$$k = \sup_{u>0} u^{-1}\, v_{\psi u}^q(\eta))\,,$$

which together with (54), (53), (52) and (50) implies

$$C_3 \leqq 2r\,\bar{A} + 2kr \int_0^{r^{-2}} \theta(\tau^{-1/2})\, \tau^{-1/2}\, \mathrm{d}\tau \leqq$$

$$\leqq 2r \left[u_\psi^q(\eta)\, \theta(R_0)\, R_0^{-1} + (u_\psi^q(\eta) + k) \int_r^\infty \theta(x)\, x^{-2}\, \mathrm{d}x \right].$$

Combining this with (51), (48), (45), (46) and writing

$$U = \sup_{z \in \bar{S}} u_\psi^q(z)$$

(cf. proposition 12 and note also that $u_\psi^q(\eta) \leqq U$ in view of the lower-semicontinuity of $u_\psi^q(\ldots)$ established in lemma 7) we arrive at

$$|A_2| \leqq \varepsilon r \left[4U\, \theta(R_0)\, R_0^{-1} + (3U + k) \int_r^\infty \theta(x)\, x^{-2}\, \mathrm{d}x \right].$$

Adding this to the estimate (43) (see also (45)) we get by virtue of (41)

$$(55)\qquad \left|\int_{L^r} f_\varkappa \varrho_z^{-1}\,\mathrm{d}\varrho_z - \int_{L^r} f_\varkappa \varrho_\eta^{-1}\,\mathrm{d}\varrho_\eta\right| \leqq$$

$$\leqq \varepsilon r\left[5U\,\theta(R_0)\,R_0^{-1} + (4U + k)\int_r^\infty \theta(x)\,x^{-2}\,\mathrm{d}x\right].$$

Finally, consider the set

$$Z = \{t \in \langle a, b\rangle;\ \varrho_\eta(t) \geqq R_0\}$$

and note that

$$\int_Z f_\varkappa \varrho_z^{-1}\,\mathrm{d}\varrho_z - \int_Z f_\varkappa \varrho_\eta^{-1}\,\mathrm{d}\varrho_\eta = \operatorname{Re}\int_Z f_\varkappa(t)\left[\frac{1}{\psi(t) - z} - \frac{1}{\psi(t) - \eta}\right]\mathrm{d}\psi(t)\,.$$

Writting

$$m = \sup\{|f_\varkappa(t)|;\ t \in \langle a, b\rangle\}$$

one concludes easily that

$$(56)\qquad \left|\int_Z f_\varkappa \varrho_z^{-1}\,\mathrm{d}\varrho_z - \int_Z f_\varkappa \varrho_\eta^{-1}\,\mathrm{d}\varrho_\eta\right| \leqq \frac{mr\operatorname{var}\psi(\langle a, b\rangle)}{R_0(R_0 - r)}\,.$$

Let $T = \{t \in \langle a, b\rangle;\ \varrho_\eta(t) > 0\}$. On account of (30)

$$\int_a^b f_\varkappa \varrho_z^{-1}\,\mathrm{d}\varrho_z = \int_T f_\varkappa \varrho_z^{-1}\,\mathrm{d}\varrho_z\,,$$

which together with (37) yields for

$$D(z) = p_\psi f(z) - \varkappa\log\frac{|\psi(b) - z|}{|\psi(a) - z|} - \int_a^b f_\varkappa \varrho_\eta^{-1}\,\mathrm{d}\varrho_\eta$$

the estimate

$$|D(z)| \leqq \left|\int_{M_r} f_\varkappa \varrho_z^{-1}\,\mathrm{d}\varrho_z\right| + \left|\int_{M_r} f_\varkappa \varrho_\eta^{-1}\,\mathrm{d}\varrho_\eta\right| + \left|\int_{L^r} f_\varkappa \varrho_z^{-1}\,\mathrm{d}\varrho_z - \int_{L^r} f_\varkappa \varrho_\eta^{-1}\,\mathrm{d}\varrho_\eta\right| +$$

$$+ \left|\int_Z f_\varkappa \varrho_z^{-1}\,\mathrm{d}\varrho_z - \int_Z f_\varkappa \varrho_\eta^{-1}\,\mathrm{d}\varrho_\eta\right|.$$

Noting that

$$\theta(r)\,r^{-1} \leqq \int_r^\infty \theta(x)\,x^{-2}\,\mathrm{d}x$$

we infer from (39), (40), (55), (56) for arbitrary $z \in S$ with $|z - \eta| = r < R_0$

$$|D(z)| \leqq \varepsilon r(11U + k) \int_r^\infty \theta(x)\, x^{-2}\, dx + mr \operatorname{var} \psi(\langle a, b \rangle)\, R_0^{-1}(R_0 - r)^{-1},$$

which completes the proof, because $\varepsilon > 0$, $R_0 > 0$ are arbitrary constants fulfilling (38) and (22).

15. Remark. Suppose now that ψ is simple in the sense that for $t_1, t_2 \in \langle a, b \rangle$

$$0 < |t_1 - t_2| < b - a \Rightarrow \psi(t_1) \neq \psi(t_2)$$

and put

$$\psi(\langle a, b \rangle) = K$$

(in the introduction, the same symbol is used to denote the oriented curve described by ψ).

Let $Q \geqq 0$ be a bounded lower-semicontinuous function on K and put

$$q(t) = Q(\psi(t)), \quad t \in \langle a, b \rangle.$$

Defining $U_K^Q(\varrho, \eta)$ as in the introduction we have for $\varrho \notin \{|\psi(b) - \eta|, |\psi(a) - \eta|\}$

$$U_K^Q(\varrho, \eta) = u_\psi^q(\varrho, \eta),$$

whence

$$U_K^Q(\eta) = u_\psi^q(\eta).$$

Similarly,

$$V_{Kr}^Q(\eta) = v_{\psi r}^q(\eta).$$

Now it is easy to see that theorems 14, 9 imply the theorem stated in the introduction.

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