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ON A SHARP ESTIMATION IN THE THEORY OF BINARY RELATIONS ON A FINITE SET

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Dedicated to Professor L. Rédei *on the occasion of his seventieth birthday.*

Let $\Omega = \{a_1, a_2, \ldots, a_n\}$ be a finite set with $n > 1$ different elements. By a binary relation on Ω we mean a subset of $\Omega \times \Omega$. The empty relation will be denoted by z. The diagonal is denoted by $\Delta_\Omega = \Delta$.

Let B_Ω be the set of all binary relations on Ω. If $\varrho \in B_\Omega$, we denote

$$a_i\varrho = \{x \in \Omega \mid (a_i, x) \in \varrho\}\,,$$
$$\varrho a_i = \{y \in \Omega \mid (y, a_i) \in \varrho\}\,,$$
$$\mathrm{pr}_1(\varrho) = \bigcup_{i=1}^{n} \varrho a_i\,, \quad \mathrm{pr}_2(\varrho) = \bigcup_{i=1}^{n} a_i\varrho\,.$$

If M is a subset of Ω, then $M\varrho$ is defined as the union $\bigcup_{a_i \in M} a_i\varrho$. The relation $\bar{\varrho}^1$ is defined by the requirement $(a_i, a_j) \in \bar{\varrho}^1 \Leftrightarrow (a_j, a_i) \in \varrho$. Finally, we denote $\Pi(\varrho) = \mathrm{pr}_1(\varrho \cup \bar{\varrho}^1) = \mathrm{pr}_2(\varrho \cup \bar{\varrho}^1)$.

In B_Ω the usual multiplication of binary relations is introduced so that B_Ω becomes a semigroup containing z as the zero element and Δ as the unit element.

If $\varrho_1 \cap \varrho_2 = z$, we shall say that the relations ϱ_1, ϱ_2 are disjoint.

The following is easy to prove:

Lemma 1. *If* card $\Omega = n$, *then the transitive closure of* ϱ *is* $\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n$.

In other words: For any integer $t \geqq 1$ we have $\varrho^t \subset \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n$.

Let $\varrho \in B_\Omega$. Consider the sequence of powers

$$(1) \qquad \varrho, \varrho^2, \varrho^3, \ldots$$

This sequence contains only a finite number of different elements (relations). Let

$k = k(\varrho)$ be the least integer such that $\varrho^k = \varrho^l$ for some $l > k$. Let further $l = k + d$ be the least integer satisfying this equality. Then the sequence (1) is of the form

$$\varrho, \ldots, \varrho^{k-1} \mid \varrho^k, \ldots, \varrho^{k+d-1} \mid \varrho^k, \ldots, \varrho^{k+d-1} \mid \ldots$$

It is well known from the elements of the theory of semigroups that the set $G(\varrho) = \{\varrho^k, \ldots, \varrho^{k+d-1}\}$ is a cyclic group. The unit element of this group is ϱ^r, where $k \leqq r \leqq k + d - 1$. More precisely: Let β be the uniquely determined integer such that $k \leqq \beta d \leqq k + d - 1$. Then $r = \beta d$.

In a series of papers (see e.g. [6]–[8]) I have dealt with some properties of non negative matrices. If $A = (a_{ij})$ is an $n \times n$ non-negative matrix, we have studied the distribution of zeros and non-zeros in the sequence $A, A^2, A^3, \ldots$ To any non-negative A we can associate a "matrix" C_A with elements 0 and 1 by writing 0 or 1 on the place (i, j) according as $a_{ij} = 0$ or $a_{ij} > 0$. Such a "matrix" can be considered in an obvious meaning as the *"matrix representation"* of some binary relation on an indexed set $\Omega = \{a_1, \ldots, a_n\}$.

In the set of all such $0-1$ "matrices" a multiplication can be introduced which corresponds to the multiplication of the associated binary relations. The methods I used in the papers mentioned above have been of such a manner that many of the results obtained can be directly formulated as results concerning binary relations. We shall use therefore in the following a limited number of results obtained in these papers. (See also the forthcomming paper [9], where these results together with a series of new results are proved in terms of binary relations.)

Note, for further purposes, that an $n \times n$ non-negative matrix A is called *reducible* if there is an $n \times n$ permutation matrix P such that PAP^{-1} is of the form $\begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$. Otherwise it is called *irreducible.*

A non-negative $n \times n$ matrix A is irreducible iff $A + A^2 + \ldots + A^n$ is positive. This motivates the following definition.

Definition. Suppose that $\varrho \in B_\Omega$ and card $\Pi(\varrho) = n$. Then ϱ is called *irreducible* iff

$$\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n = \Omega \times \Omega .$$

It is the purpose of this paper to find a sharp estimation for the number $k = k(\varrho)$ in the case of an *irreducible relation.* This problem has been open for some years.

We note that for an irreducible ϱ we always have $1 \leqq d = d(\varrho) \leqq n$.

If $d = 1$ it is know that $k(\varrho) \leqq (n-1)^2 + 1$ and this result is the best possible. This goes back to a statement of H. Wielandt ([10]) and has been proved by several authors in the last ten years. (See e.g. [1].)

Suppose next $d > 1$ and write $n = \alpha d + s$, where $\alpha \geqq 1$ is an integer and $0 \leqq s \leqq d - 1$.

It follows from the results of JU. I. LJUBIČ ([5]) that $k(\varrho) \leqq n^2/d - 2n + 3d$.

I have proved in [8] that $k(\varrho) \leqq (\alpha^2 - 2\alpha + 3)\, d + s - 1$ if $\alpha \geqq 2$ (which implies $k(\varrho) \leqq n^2/d - 2n + 3d - s - 1$), and $k(\varrho) \leqq s + 1$ if $\alpha = 1$.

B. R. HEAP and M. S. LYNN ([3]) proved by graph-theoretical methods a slightly sharper result namely $k(\varrho) \leqq (\alpha^2 - 2\alpha + 2)\, d + 2s$.

In this paper we prove that $k(\varrho) \leqq (\alpha^2 - 2\alpha + 2)\,\mathrm{d} + s$ for $\alpha \geqq 2$, and $k(\varrho) \leqq \max(1, s)$ for $\alpha = 1$. This result is the best possible. It implies also an affirmative answer to my conjecture (see [8]) that for any irreducible relation ϱ with card $\Pi(\varrho) = n$ we have $k(\varrho) \leqq n^2/d - 2n + 2d$. The fact that this result cannot be sharpened follows from an example given in [5].

As to the result proved in this paper I should like to mention the following. At the International Congress in Moscow (1966) V. S. GRINBERG (Donetsk) annouced me (oral communication) that he obtained the same result. Since I am not aware of his proof (it has been never published) and my proof is most probably quite different it seems to me to be worth to publish it.

1. PRELIMINARIES

In this section we give some Lemmas the proofs of which can be found (in a somewhat other form) in the papers [6]–[9].

The group $G(\varrho)$ is cyclic. There exists therefore an integer u, $k \leqq u \leqq k + d - 1$ such that $G(\varrho) = \{\varrho^u, \varrho^{2u}, \ldots, \varrho^{du}\}$.

In what follows we shall choose $u = r + 1$. This is possible, since $(r + 1, d) = 1$. Further we denote $\delta = \varrho^{r+1}$.

We then have

$$\delta^2 = \delta^{2(r+1)} = \varrho^r \varrho^{r+2} = \varrho^{r+2},\ \delta^3 = \varrho^{r+3},\ \ldots,\ \delta^d = \varrho^r ,$$

so that

$$G(\varrho) = \{\delta, \delta^2, \ldots, \delta^d\} .$$

We now give some informations concerning the behaviour of the "rows" of ϱ.

Let a_i be any element $\in \Omega$ and consider the sequence

$$a_i\varrho,\ a_i\varrho^2,\ a_i\varrho^3, \ldots \tag{2}$$

The elements of this sequence are subsets of Ω (including eventually the empty set).

Denote by $k_i = k_i(\varrho)$ the least integer such that $a_i\varrho^{k_i}$ occurs in (2) more than once. Let further $d_i = d_i(\varrho)$ be the least integer $\geqq 1$ such that $a_i\varrho^{k_i} = a_i\varrho^{k_i + d_i}$. Then (2) is of the form

$$a_i\varrho, \ldots, a_i\varrho^{k_i - 1} \mid a_i\varrho^{k_i}, \ldots, a_i\varrho^{k_i + d_i - 1} \mid a_i\varrho^{k_i}, \ldots$$

Clearly $k_i \leqq k(\varrho)$, $d_i \leqq d(\varrho)$, and for any integers $u \geqq k_i$, $v \geqq k_i$ we have $a_i\varrho^u = a_i\varrho^v$ iff $u \equiv v \pmod{\mathrm{d}_i}$.

It follows immediately from the definition of k_i and d_i that

$$G_i = \{a_i\varrho^{k_i}, a_i\varrho^{k_i+1}, \ldots, a_i\varrho^{k_i+d_i-1}\} =$$
$$= \{a_i\varrho^{r+1}, a_i\varrho^{r+2}, \ldots, a_i\varrho^{r+d_i}\} = \{a_i\delta, a_i\delta^2, \ldots, a_i\delta^d\}.$$

We also have:

Lemma 2. *For any binary relation ϱ we have*

a) $k(\varrho) = \max_i k_i(\varrho)$;

b) $d(\varrho) = \text{l.c.m.}\ [d_1, d_2, \ldots, d_n]$.

Remark. All these results hold independently whether ϱ is reducible or irreducible. We now quote some results concerning irreducible relations.

Lemma 3. *A binary relation ϱ with $\Pi(\varrho) = \Omega$ is irreducible iff*

$$\delta \cup \delta^2 \cup \ldots \cup \delta^d = \Omega \times \Omega .$$

Lemma 4. *For an irreducible relation ϱ we have:*

a) $d_1(\varrho) = \ldots = d_n(\varrho) = d$;

b) $1 \leqq d \leqq n$;

c) $\varrho \subset \delta, \varrho^2 \subset \delta^2, \ldots, \varrho^d \subset \delta^d$;

d) *The relations $\delta, \delta^2, \ldots, \delta^d$ are pairwise disjoint. More generally, any d consecutive powers $\varrho^l, \varrho^{l+1}, \ldots, \varrho^{l+d-1}$ are pairwise disjoint*;

e) $\Delta \subset \delta^d$, *while* $\Delta \cap (\delta \cup \delta^2 \cup \ldots \cup \delta^{d-1}) = z$.

Remark. None of the properties a)–e) is necessarily true for a reducible relation.

Let now be $n = \alpha d + s$, where $\alpha \geqq 1$ is an integer and $0 \leqq s \leqq d - 1$. If $s = 0$, denote in the following Lemma 5 $\varrho^0 = \varrho^d$ and $\delta^0 = \delta^d$.

Using Lemma 4 and the fact that $\varrho \cup \ldots \cup \varrho^n = \delta \cup \ldots \cup \delta^d$ we obtain:

Lemma 5. *With the notations just introduced (for any irreducible relation ϱ) we have*:

$$\begin{array}{lllllll}
\varrho & \cup\, \varrho^{d+1} & \cup\, \varrho^{2d+1} & \cup \ldots \cup\, \varrho^{(\alpha-1)d+1} & \cup\, \varrho^{\alpha d+1} & = \delta, \\
\vdots & & & & & \vdots \\
\varrho^s & \cup\, \varrho^{d+s} & \cup\, \varrho^{2d+s} & \cup \ldots \cup\, \varrho^{(\alpha-1)d+s} & \cup\, \varrho^{\alpha d+s} & = \delta^s, \\
\varrho^{s+1} & \cup\, \varrho^{d+s+1} & \cup\, \varrho^{2d+s+1} & \cup \ldots \cup\, \varrho^{(\alpha-1)d+s+1} & & = \delta^{s+1}, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & & & & & \vdots \\
\varrho^d & \cup\, \varrho^{2d} & \cup\, \varrho^{3d} & \cup \ldots \cup\, \varrho^{\alpha d} & & = \delta^d .
\end{array} \tag{3}$$

With respect to Lemma 4d) the relations in two different rows of (3) have no element in common.

Taking account of the fact that for an irreducible ϱ we have

$$a_i\varrho \cup a_i\varrho^2 \cup \ldots \cup a_i\varrho^n = a_i\delta \cup a_i\delta^2 \cup \ldots \cup a_i\delta^d = \Omega\,,$$

we may state:

Lemma 6. *If ϱ is irreducible, then to any $a_i \in \Omega$ there exists a least integer h_i, $1 \leqq h_i \leqq n$, such that $a_i \in a_i\varrho^{h_i}$ and $d \mid h_i$ (hence $h_i \leqq \alpha d$).*

We shall also need the following

Lemma 7. *If ϱ is irreducible and M is a proper subset of $a_i\delta^l$, then $M\varrho^d$ contains at least one element $\in a_i\delta^l$ not contained in M.*

Proof. Let a_j be any element $\in M$. Then since $a_j \cup a_j\varrho \cup \ldots \cup a_j\varrho^n = \Omega$ and $a_j \in a_i\delta^l$, we have (with respect to Lemma 5) $a_j \cup a_j\varrho^d \cup a_j\varrho^{2d} \cup \ldots \cup a_j\varrho^{\alpha d} = a_i\delta^l$. The more

$$M \cup M\varrho^d \cup M\varrho^{2d} \cup \ldots \cup M\varrho^{\alpha d} = a_i\delta^l\,. \tag{4}$$

To prove our Lemma it is sufficient to show that $M\varrho^d \subset M$ cannot hold. If it were so, we would have $M \supset M\varrho^d \supset M\varrho^{2d} \supset \ldots \supset M\varrho^{\alpha d}$, hence [with respect to (4)] $M = a_i\delta^l$, a contradiction to our asumption.

2. PROOF OF THE THEOREM

In this section ϱ is supposed to be irreducible and $\Pi(\varrho) = \Omega$.

We begin the proof of our Theorem with the following almost trivial statement.

Lemma 8. *Let $a_i \in \Omega$. Then at least one of the sets $a_i\delta, a_i\delta^2, \ldots, a_i\delta^d$ contains $\leqq \alpha$ elements.*

For, if all sets contained at least $\alpha + 1$ elements, the union $\Omega = a_i\delta \cup \ldots \cup a_i\delta^d$ would contain $(\alpha + 1)\, d = \alpha d + d > \alpha d + s = n$ elements, which is impossible.

By Lemma 6 to any $a_i \in \Omega$ there is an integer h_i, $d \leqq h_i \leqq \alpha d$, such that $a_i \in a_i\varrho^{h_i}$. This inclusion immediately implies the validity of the following chains:

$$\begin{array}{llllll}
a_i & \subset a_i\varrho^{h_i} & \subset a_i\varrho^{2h_i} & \subset \ldots \subset a_i\varrho^{(\alpha-1)h_i} & \subset a_i\varrho^{\alpha h_i} & . \\
a_i\varrho & \subset a_i\varrho^{h_i+1} & \subset a_i\varrho^{2h_i+1} & \subset \ldots \subset a_i\varrho^{(\alpha-1)h_i+1} & \subset a_i\varrho^{\alpha h_i+1} & , \\
\vdots & \ldots & \ldots & \ldots & \ldots & \\
a_i\varrho^s & \subset a_i\varrho^{h_i+s} & \subset a_i\varrho^{2h_i+s} & \subset \ldots \subset a_i\varrho^{(\alpha-1)h_i+s} & \subset a_i\varrho^{\alpha h_i+s} & , \\
\vdots & \ldots & \ldots & \ldots & \ldots & \\
a_i\varrho^{d-1} & \subset a_i\varrho^{h_i+d-1} & \subset a_i\varrho^{2h_i+d-1} & \subset \ldots \subset a_i\varrho^{(\alpha-1)h_i+d-1} & \subset a_i\varrho^{\alpha h_i+d-1} & .
\end{array} \tag{5}$$

Denote in all what follows $a_i\delta^0 = a_i\delta^d$. With respect to Lemma 8 we shall distinguish (for a fixed a_i) two mutually excluding cases.

Condition $\mathbf{S_1}$.

$$\min\{\text{card}(a_i\delta^0), \text{card}(a_i\delta), \ldots, \text{card}(a_i\delta^{d-1})\} = \text{card}(a_i\delta^l) < \alpha,$$

for some l, $0 \leqq l \leqq d - 1$.

Condition $\mathbf{S_2}$.

$$\min\{\text{card}(a_i\delta^0), \text{card}(a_i\delta), \ldots, \text{card}(a_i\delta^{d-1})\} = \text{card}(a_i\delta^l) = \alpha,$$

for some l, $0 \leqq l \leqq d - 1$.

The following two lemmas are in certain sense essential.

Lemma 9. *Suppose that for a given a_i Condition S_2 is satisfied. Denote by $l = l_i$ the least integer l, $0 \leqq l \leqq d - 1$, for which $a_i\delta^{l_i}$ contains exactly α different elements $\in \Omega$. Then $l_i \leqq s$.*

Proof. If $l_i = 0$, there is nothing to prove so that we may suppose $l_i > 0$.

By definition of the number l_i (and by Lemma 4d) the set

$$a_i\delta^0 \cup a_i\delta^1 \cup \ldots \cup a_i\delta^{l_i-1}$$

contains at least $(\alpha + 1)\, l_i$ different elements $\in \Omega$, while

$$a_i\delta^{l_i} \cup \ldots \cup a_i\delta^{d-1}$$

contains at least $\alpha(d - l_i)$ different elements $\in \Omega$. Since $a_i\delta^0 \cup \ldots \cup a_i\delta^{d-1} = \Omega$, we have (by Lemma 4d)

$$(\alpha + 1)\, l_i + \alpha(d - l_i) \leqq n = \alpha d + s,$$

which implies $l_i \leqq s$, q.e.d.

It follows from Lemma 5 that with $l = l_i$ as defined in Lemma 9 we have:

$$a_i\varrho^{l_i} \cup a_i\varrho^{d+l_i} \cup \ldots \cup a_i\varrho^{(\alpha-1)d+l_i} \cup a_i\varrho^{\alpha d+l_i} = a_i\delta^{l_i}. \tag{6}$$

Lemma 10. *Suppose that* (for a given a_i) *Condition S_2 is satisfied and $h_i = \alpha d$, $\alpha \geqq 2$. Then*

$$a_i\varrho^{l_i} \cup \ldots \cup a_i\varrho^{(\alpha-2)d+l_i} \neq a_i\delta^{l_i}.$$

Remark. This means that in this (in some sense "worst") case the last two members on the left hand side of (6) cannot be omited.

Proof. Suppose for an indirect proof that

$$a_i\varrho^{l_i} \cup a_i\varrho^{l_i+d} \cup \ldots \cup a_i\varrho^{l_i+(\alpha-2)d} = a_i\delta^{l_i}.$$

Multiplying by $\varrho, \varrho^2, \ldots, \varrho^{d-1}$ we get:

$$\begin{aligned} a_i\varrho^{l_i+1} \quad &\cup a_i\varrho^{l_i+d+1} \cup \ldots \cup a_i\varrho^{l_i+(\alpha-2)d+1} &&= a_i\delta^{l_i+1}, \\ &\vdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_i\varrho^{l_i+d-1} &\cup a_i\varrho^{l_i+2d-1} \cup \ldots \cup a_i\varrho^{l_i+(\alpha-2)d+d-1} &&= a_i\delta^{l_i+d-1}. \end{aligned}$$

By summing we have

$$\Omega = a_i\delta^{l_i} \cup \ldots \cup a_i\delta^{l_i+d-1} = a_i\varrho^{l_i} \cup \ldots \cup a_i\varrho^{l_i+(\alpha-2)d+d-1}.$$

Now

$$l_i + (\alpha-2)\,d + d - 1 \leqq s + (\alpha-1)\,d - 1 = \alpha d + s - (d+1) < \alpha d.$$

This would imply that there is an integer h, $l_i \leqq h < \alpha d$, such that $a_i \in a_i\varrho^h$. This is a contradiction to $h_i = \alpha d$.

We first settle the case $\alpha = 1$ (in which case necessarily Condition S_2 holds).

Lemma 11. *If $n = d + s$, then $k(\varrho) \leqq \max(1, s)$.*

Proof. In this case we have $1 \leqq h_i \leqq d$, $d \mid h_i$, hence $h_i = d$ (for $i = 1, 2, \ldots, n$). By Lemma 8 (for every $a_i \in \Omega$) at least one of the sets $a_i\delta^0, a_i\delta, \ldots, a_i\delta^{d-1}$ contains exactly one element. Let l_i has the meaning defined in Lemma 9. If $l_i = 0$, $a_i\delta^d$ contains exactly one element and since [by (5)] $a_i \in a_i\varrho^d \subset a_i\delta^d$, we have $a_i = a_i\varrho^d$, hence $a_i\varrho = a_i\varrho^{d+1}$, i.e. $k_i = 1$.

If $l_i > 0$, $\varrho^{l_i} \subset \delta^{l_i}$ implies $a_i\varrho^{l_i} = a_i\delta^{l_i}$, so that (by Lemma 9) $k_i \leqq l_i \leqq s$. By Lemma 2 we finally have $k(\varrho) \leqq \max(1, s)$.

From now we shall suppose $\alpha > 1$ and we shall distinguish *three cases.*

Case 1. Suppose first that (for a given a_i) Condition S_1 holds.

Consider the chain

$$a_i\varrho^l \subset a_i\varrho^{h_i+l} \subset \ldots \subset a_i\varrho^{(\alpha-2)h_i+l} \subset a_i\varrho^{(\alpha-1)h_i+l},$$

$0 \leqq l \leqq d - 1$, where for $l = 0$ the symbol $a_i\varrho^l$ means the element a_i.

Then either there is an integer τ, $0 \leqq \tau < \alpha - 2$, such that $a_i\varrho^{\tau h_i+l} = a_i\varrho^{(\alpha-2)h_i+l}$ or $a_i\varrho^{(\alpha-2)h_i+l}$ contains exactly $\alpha - 1$ different elements $\in \Omega$. Hence in both cases $a_i\varrho^{(\alpha-2)h_i+l} = a_i\delta^l$. Therefore $k_i \leqq (\alpha-2)\,h_i + l \leqq (\alpha-2)\,\alpha d + d - 1 = (\alpha^2 - - 2\alpha + 1)\,d - 1$.

In the following Cases 2 and 3 the integer l_i is the same as defined in Lemma 9.

Case 2. Suppose that Condition S_2 is satisfied and $h_i < \alpha d$ [hence $h_i \leqq (\alpha - 1) d$]. Consider the chain

$$a_i\varrho^{l_i} \subset a_i\varrho^{l_i+h_i} \subset \ldots \subset a_i\varrho^{l_i+(\alpha-1)h_i},$$

where if $l_i = 0$, $a_i\varrho^{l_i}$ means the element a_i.

Here either there is a τ, $0 \leqq \tau < \alpha - 1$, such that $a_i\varrho^{\tau h_i+l_i} = a_i\varrho^{(\alpha-1)h_i+l_i}$ or $a_i\varrho^{l_i+(\alpha-1)h_i}$ contains exactly α different elements $\in \Omega$. In both cases we have $a_i\delta^{l_i} = a_i\varrho^{(\alpha-1)h_i+l_i}$. Therefore

$$k_i \leqq (\alpha - 1) h_i + l_i \leqq (\alpha - 1)(\alpha - 1) d + s = (\alpha^2 - 2\alpha + 1) d + s .$$

Case 3. Suppose that Condition S_2 is satisfied and $h_i = \alpha d$. Put again $a_i\varrho^{l_i} = a_i$ if $l_i = 0$.

A) We first show that in this case $a_i\varrho^{l_i}$ cannot have more than one element. For, suppose that $a_i\varrho^{l_i}$ contains at least two elements $\in \Omega$. By Lemma 7 if $a_i\varrho^{l_i} \neq a_i\delta^{l_i}$, then $a_i\varrho^{l_i} \cup a_i\varrho^{l_i+d_i}$ contains at least three elements $\in \Omega$. Again by Lemma 7 if $a_i\varrho^{l_i} \cup a_i\varrho^{l_i+d} \neq a_i\delta^{l_i}$, then $a_i\varrho^{l_i} \cup a_i\varrho^{l_i+d} \cup a_i\varrho^{l_i+2d}$ contains at least four different elements, etc. Repeating this argument we obtain that

$$a_i\varrho^{l_i} \cup a_i\varrho^{l_i+d} \cup \ldots \cup a_i\varrho^{l_i+(\alpha-2)d}$$

contains at least α elements $\in \Omega$. Hence this union is equal to $a_i\delta^{l_i}$. This is a contradiction to Lemma 10.

B) We next show that in this case $a_i\varrho^{l_i+td}$ (for $t = 1, 2, \ldots, \alpha - 1$) contains *exactly one element* not contained in $a_i\varrho^{l_i} \cup a_i\varrho^{l_i+d} \cup \ldots \cup a_i\varrho^{l_i+(t-1)d} = T_t$.

We know that $a_i\varrho^{l_i}$ contains exactly one element $\in \Omega$. If $a_i\varrho^{l_i} \neq a_i\delta^{l_i}$, then by Lemma 7 $a_i\varrho^{l_i} \cup a_i\varrho^{l_i+d}$ contains at least two different elements $\in \Omega$, etc. Repeating this argument we have: For any integer t, $1 \leqq t \leqq \alpha$, the set T_t contains *at least* t different elements $\in \Omega$. Suppose now that $a_i\varrho^{l_i+td}$ for some t (with $1 \leqq t \leqq \alpha - 1$) would contain at least two elements not contained in T_t. This would imply that $T_t \cup a_i\varrho^{l_i+td}$ contains at least $t + 2$ different elements $\in \Omega$. Repeating the same argument as sub A) we obtain that $a_i\varrho^{l_i} \cup a_i\varrho^{l_i+d} \cup \ldots \cup a_i\varrho^{l_i+(\alpha-2)d}$ contains α different elements $\in \Omega$, hence it is equal to $a_i\delta^{l_i}$. This is a contradiction to Lemma 10.

C) Consider now the finite sequence of sets

$$a_i\varrho^{l_i}, a_i\varrho^{l_i+d}, \ldots, a_i\varrho^{(\alpha-1)d+l_i}, a_i\varrho^{\alpha d+l_i}. \tag{7}$$

and recall that $T_\alpha = a_i\varrho^{l_i} \cup \ldots \cup a_i\varrho^{l_i+(\alpha-1)d} = a_i\delta^{l_i}$. We have to distinguish two cases:

a) Suppose that each member of the sequence (7) contains exactly one element. Then since $a_i\varrho^{l_i} \subset a_i\varrho^{\alpha d + l_i} = a_i\varrho^{h_i + l_i}$, we have $a_i\varrho^{l_i} = a_i\varrho^{\alpha d + l_i}$, whence $k_i \leqq \max(1, l_i) \leqq \max(1, s)$.

b) Let τ_0, $1 \leqq \tau_0 \leqq \alpha$, be the least integer such that $a_i\varrho^{l_i + \tau_0 d}$ contains more than one element.

If $\tau_0 = \alpha$, then since $a_i\varrho^{l_i} \subsetneqq a_i\varrho^{\alpha d + l_i}$ and

$$a_i\varrho^{\alpha d + l_i} \subset a_i\varrho^{l_i} \cup a_i\varrho^{l_i + d} \cup \ldots \cup a_i\varrho^{l_i + (\alpha - 1)d}$$

(where each summand contains a unique element $\in \Omega$), there is a β, $1 \leqq \beta < \alpha$, such that $a_i\varrho^{l_i + \beta d} \subset a_i\varrho^{l_i + \alpha d}$.

If $\tau_0 < \alpha$, then each summand of $T_{\tau_0} = a_i\varrho^{l_i} \cup \ldots \cup a_i\varrho^{l_i + (\tau_0 - 1)d}$ contains a unique element $\in \Omega$ while $a_i\varrho^{l_i + \tau_0 d}$ contains at least two elements $\in \Omega$ exactly one of which is not contained in T_{τ_0}. Hence there is a $\gamma \geqq 0$, $0 \leqq \gamma < \tau_0$, such that $a_i\varrho^{l_i + \gamma d} \subset a_i\varrho^{l_i + \tau_0 d}$.

In the first case we have

$$a_i\varrho^{l_i + \beta d} \subset a_i\varrho^{l_i + \alpha d} \subset a_i\varrho^{l_i + \alpha d + (\alpha - \beta)d} \subset \ldots \subset \varrho^{l_i + \alpha d + (\alpha - \beta)(\alpha - 2)d} \subset \ldots$$

This chain is of lenght at most α so that

$$k_i \leqq l_i + \alpha d + (\alpha - \beta)(\alpha - 2) d \leqq s + \alpha d + (\alpha - 1)(\alpha - 2) d = \\ = (\alpha^2 - 2\alpha + 2) d + s .$$

In the second case we get

$$a_i\varrho^{l_i + \gamma d} \subset a_i\varrho^{l_i + \tau_0 d} \subset a_i\varrho^{l_i + \tau_0 d + (\tau_0 - \gamma)d} \subset \ldots \subset a_i\varrho^{l_i + \tau_0 d + (\tau_0 - \gamma)(\alpha - 2)d} \subset \ldots,$$

and since again this chain cannot have more than α different members, we get

$$k_i \leqq l_i + \tau_0 d + (\tau_0 - \gamma)(\alpha - 2) d \leqq s + (\alpha - 1) d + (\alpha - 1)(\alpha - 2) d = \\ = (\alpha^2 - 2\alpha + 1) d + s .$$

By Lemma 2 (taking account of the results obtained in the Cases 1–3) we have for $\alpha \geqq 2$

$$k(\varrho) \leqq \max \{(\alpha^2 - 2\alpha + 1) d - 1, (\alpha^2 - 2\alpha + 1) d + s, 1, s, \\ (\alpha^2 - 2\alpha + 2) d + s, (\alpha^2 - 2\alpha + 1) d + s\} .$$

With respect to Lemma 11, we have proved:

Theorem 1. *Let ϱ be an irreducible relation with* card $\Pi(\varrho) = n$ *and* card $G(\varrho) = d$.

Write $n = \alpha d + s$, *where* $\alpha \geqq 1$ *is an integer and* $0 \leqq s \leqq d - 1$. *We then have*:

$$k(\varrho) \begin{cases} \leqq (\alpha^2 - 2\alpha + 2)\, d + s\,, & \textit{for} \quad \alpha \geqq 2\,, \\ \leqq \max(1, s)\,, & \textit{for} \quad \alpha = 1\,. \end{cases}$$

3. CONCLUDING REMARKS

A) There are examples which prove that the estimates of our Theorem cannot be sharpened.

Take, e.g., $\Omega = \{a_1, a_2, \ldots, a_7\}$ and the binary relation ϱ given by means of the oriented graph on Fig. 1.

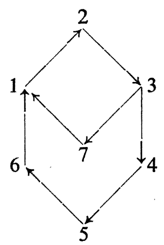

Fig. 1.

[This is a special case of a class of oriented graphs considered by B. R. Heap and M. S. Lynn in [3], [4].] Here $n = 7$, $d = 2$, so that $\alpha = 3$ and $s = 1$. By direct computation it can be shown that $k_4 = (\alpha^2 - 2\alpha + 2)\, d + s = 11$. The corresponding matrix representation of ϱ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have

$$\begin{array}{ll} a_4\varrho = a_5\,, & a_4\varrho^2 = a_6\,, \\ a_4\varrho^3 = a_1\,, & a_4\varrho^4 = a_2\,, \\ a_4\varrho^5 = a_3\,, & a_4\varrho^6 = \{a_4, a_7\}\,, \\ a_4\varrho^7 = \{a_1, a_5\}\,, & a_4\varrho^8 = \{a_2, a_6\}\,, \\ a_4\varrho^9 = \{a_1, a_3\}\,, & a_4\varrho^{10} = \{a_2, a_4, a_7\}\,, \\ a_4\varrho^{11} = \{a_1, a_3, a_5\}\,, & a_4\varrho^{12} = \{a_2, a_4, a_6, a_7\}\,, \\ a_4\varrho^{13} = a_4\varrho^{11}\,, & a_4\varrho^{14} = a_4\varrho^{12}\,. \end{array}$$

B) With respect to $\alpha = (n - s)/d$ we can write (for $\alpha \geqq 2$):

$$k(\varrho) \leqq (\alpha^2 - 2\alpha + 2)\, d + s = \frac{1}{d}(n - d)^2 + d - \frac{s}{d}[(2\alpha - 3)\, d + s] .$$

If $\alpha = 1$, we have $k(\varrho) \leqq \max(1, s) \leqq \max(1, d - 1) \leqq (n - d)^2/d + d$. If $\alpha \geqq 2$, $(2\alpha - 3)\, d + s \geqq d + s > 0$.

We can state therefore:

Theorem 2. *For any irreducible binary relation ϱ with* card $\Pi(\varrho) = n$ *and* card $G(\varrho) = d$, *we always have*

$$k(\varrho) \leqq (n - d)^2/d + d .$$

Here the sign of equality can hold only if $d \mid n$.

This result is again sharp. Take any n and d such that $d \mid n$. Put $n = \alpha d$. Denote by W the $0-1$ matrix of order α

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \end{pmatrix}$$

and construct the $0-1$ matrix of order $n = d\alpha$

$$(8) \qquad \begin{pmatrix} 0 & E & 0 & \dots & 0 \\ 0 & 0 & E & \dots & 0 \\ \\ 0 & 0 & 0 & \dots & E \\ W & 0 & 0 & \dots & 0 \end{pmatrix},$$

where E is the "unit matrix" of order α. Then a direct computation shows that the (irreducible) relation ϱ corresponding to the $0-1$ matrix (8) satisfies $k(\varrho) = (n - d)^2/d + d$. [This example has been considered by Ljubič [5].]

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