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FLOWS OF HEAT AND THE FOURIER PROBLEM

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INTRODUCTION

Let D be an arbitrary open set in R^m, the Euclidean m-space, and suppose that its boundary B is compact and non-void. Fix $T_1, T_2 \in R^1$, $T_1 < T_2$, and let

$$C = B \times \langle T_1, T_2), \quad E = D \times (T_1, T_2) .$$

By the term measure we shall usually mean a finite signed Borel measure in some Euclidean space. If μ is a measure and M is a Borel set in the domain of μ, then $|\mu| (M)$ will denote the variation of μ on M. Let $\mathscr{B}'(T_1, T_2) = \mathscr{B}'$ stand for the class of all measures μ in R^{m+1} with

$$|\mu| (R^{m+1} \setminus C) = 0 .$$

With each $\mu \in \mathscr{B}'$ we associate the corresponding thermal potential

$$U \mu(z) = \int G(z - \zeta) \, d\mu(\zeta) , \quad z \in E ,$$

where $G(z) = 0$ for $z = [z_1, \ldots, z_{m+1}]$ with $z_{m+1} \leqq 0$, while for $z_{m+1} > 0$

$$G(z) = z_{m+1}^{-m/2} \exp \left(- \sum_{j=1}^{m} z_j^2 / 4 z_{m+1}\right) .$$

Writing ∂_j for the derivative with respect to the j-th variable one easily verifies that $\partial_j U\mu$ are integrable over E for $j = 1, \ldots, m$. This makes it possible to introduce the functional $H\mu$ over the class $\mathscr{D}_{T_2}$ of all infinitely differentiable functions with compact support in $R^{m+1} \cap \{z; z_{m+1} < T_2\}$ defining

$$\langle \varphi, H\mu \rangle = \int_E \{[\sum_{j=1}^{m} \partial_j U \mu(z) \, . \, \partial_j \varphi(z)] - U \mu(z) \, . \, \partial_{m+1} \varphi(z)\} \, dz .$$

$H\mu$ will be termed the heat flow associated with μ. The reason for the terminology lies in the fact that, in the special case when the boundary B of D is a smooth hypersurface in R^m with the exterior normal $n(x) = [n_1(x), \ldots, n_m(x)]$ and the derivatives $\partial_j U\mu$ extend from E to continuous functions u_j on the closure of E, $\langle \varphi, H\mu \rangle$ transforms into

$$\int_{T_1}^{T_2} \left\{ \int_B \varphi(x, t) \left[\sum_{j=1}^{m} u_j(x, t)\, n_j(x) \right] \mathrm{d}\sigma_B(x) \right\} \mathrm{d}t\,,$$

where $\mathrm{d}\sigma_B$ is the area element on B (and, of course, $[x, t]$ stands for $[x_1, \ldots, x_m, t]$ whenever $x = [x_1, \ldots, x_m] \in R^m$ and $t \in R^1$). In general there is no measure ν_μ representing $H\mu$ over $\mathscr{D}_{T_2}$. In order to be able to formulate geometric conditions on D guaranteeing the existence of such a ν_μ for each $\mu \in \mathscr{B}'$ we adopt the following terminology introduced in [10]: Given $x \in R^m$, $r > 0$ and $\theta \in \Gamma = R^m \cap \{\theta; |\theta| = 1\}$, we call $y \in S_r(\theta, x) = \{x + \varrho\theta;\ 0 < \varrho < r\}$ a hit of $S_r(\theta, x)$ on D provided each ball

$$\Omega_\varrho(y) = R^m \cap \{v; |v - y| < \varrho\}$$

meets both $S_r(\theta, x) \setminus D$ and $S_r(\theta, x) \cap D$ in a set of positive linear measure. (Note that $\Omega_\varrho(y) \cap S_r(\theta, x) \cap D$ is open in $S_r(\theta, x)$; consequently, it is either void or it has a positive linear measure.) The number (possibly infinite) of all the hits of $S_r(\theta, x)$ on D will be denoted by $n_r(\theta, x)$. For fixed $r > 0$ and $x \in R^m$, $n_r(\theta, x)$ is a Baire function of the variable θ on Γ (see [10], proposition 1.6) and one may put

$$v_r(x) = \int_\Gamma n_r(\theta, x)\, \mathrm{d}\sigma_\Gamma(\theta)\,.$$

If $M \neq \emptyset$ is a subset of B we let

$$V_0(M) = \lim_{r \to 0+} \sup_{x \in M} v_r(x)\,;$$

finally, put $V_0(\emptyset) = 0$. With this notation we have now the following

Theorem 1. *In order that $H\mu$ be representable by means of a measure for each $\mu \in \mathscr{B}'$ it is necessary and sufficient that*

(1) $$V_0(B) < +\infty\,.$$

In what follows we always assume (1). For each $\mu \in \mathscr{B}'$ there is a uniquely determined measure ν_μ satisfying the following conditions (i), (ii):

(i) $$\varphi \in \mathscr{D}_{T_2} \Rightarrow \langle \varphi, H\mu \rangle = \int \varphi\, \mathrm{d}\nu_\mu\,,$$

(ii) $$|\nu_\mu|\, (R^m \times \langle T_2, +\infty)) = 0\,.$$

It is easily seen that the support of v_μ is contained in $B \times \langle T_1, T_2 \rangle$; in other words, $v_\mu \in \mathscr{B}'$ for each $\mu \in \mathscr{B}'$. Let us agree to write simply $v_\mu = H\mu$ and equip $\mathscr{B}'$ with the norm

$$\|\mu\| = |\mu| (R^{m+1}) = |\mu| (C) .$$

Then $H : \mu \to H\mu$ is a bounded operator on the Banach space $\mathscr{B}'$. Let us also quote here that (1) implies

$$\sup \{v_\infty(x); x \in R^m\} < +\infty .$$

Another consequence of (1) is the existence of the density

$$d_D(x) = \lim_{\varrho \to 0+} \frac{\text{volume}\,(\Omega_\varrho(x) \cap D)}{\text{volume}\,(\Omega_\varrho(x))}$$

at any $x \in R^m$.

Let now $\mathscr{B}(T_1, T_2) = \mathscr{B}$ be the Banach space of all continuous functions f on $B \times \langle T_1, T_2 \rangle$ such that $f(B \times \{T_2\}) = \{0\}$, with the norm

$$\|f\| = \sup \{|f(z)|; z \in B \times \langle T_1, T_2 \rangle\} .$$

We shall introduce an operator W_0 on $\mathscr{B}$ whose dual is $H : W_0' = H$. For this purpose we recall the following notation introduced in [10]. Given $x \in B$ and $\theta \in \Gamma$ we put for $r > 0$

$$s(r; x, \theta) = \varepsilon \quad (= \pm 1)$$

if there is a $\delta > 0$ such that

$$x + (r + \varepsilon\varrho)\, \theta \in D\,, \quad x + (r - \varepsilon\varrho)\, \theta \in R^m \setminus D$$

for almost every $\varrho \in (0, \delta)$; otherwise we set $s(r; x, \theta) = 0$. With each $f \in \mathscr{B}$, $t \in \langle T_1, T_2 \rangle$ and $\eta > 0$ we associate the sum

$$\sum f\left(x + r\theta, t + \frac{r^2}{4\eta}\right) s(r; x, \theta) = \sum_f([x, t]; \eta, \theta)$$

extended over $r \in (0, 2[\eta(T_2 - t)]^{1/2})$ (consequently, $\sum_f([x, t]; \eta, \theta) = 0$ if $t = T_2$). For fixed $\eta > 0$ and $z = [x, t] \in B \times \langle T_1, T_2 \rangle$, $\sum_f(z; \eta, \theta)$ is defined almost everywhere and integrable $d\sigma_\Gamma(\theta)$ on Γ and the integral

$$Wf(z) = \int_0^\infty e^{-\eta} \eta^{(m-1)/2} \left[\int_\Gamma \sum_f(z; \eta, \theta)\, d\sigma_\Gamma(\theta) \right] d\eta$$

is convergent. Writing $\hat{z} = [z_1, \ldots, z_m]$ for each $z = [z_1, \ldots, z_m, z_{m+1}] \in R^{m+1}$ we are now able to formulate the following

Theorem 2. *For each $f \in \mathscr{B}$ define*

$$W_0 f(z) = 2^{m-1}[Wf(z) + 2\pi^{m/2} d_D(\hat{z}) f(z)]\,, \quad z \in B \times \langle T_1, T_2 \rangle\,;$$

then $W_0 f \in \mathscr{B}$. The operator $W_0 : f \to W_0 f$ is bounded on $\mathscr{B}$ and H is dual to W_0.

Let I stand for the identity operator on $\mathscr{B}$ and consider the operators

$$W_\alpha = W_0 - 2^m \pi^{m/2} \alpha I\,, \quad \alpha \in R^1 \setminus \{0\}\,.$$

It is useful to evaluate the quantity

$$\omega W_\alpha = \inf \|W_\alpha - T\|\,,$$

T ranging over all compact operators acting on $\mathscr{B}$. In particular, in view of the equality

$$H = (2^m \pi^{m/2} \alpha I + W_\alpha)'\,,$$

it is important to know conditions on D guaranteeing the validity of the following estimate for $g(\alpha) = \omega W_\alpha / |\alpha|\, 2^m \pi^{m/2}$:

(2) $$a = \inf \{g(\alpha);\, \alpha \in R^1 \setminus \{0\}\} < 1\,.$$

They read as follows.

Theorem 3. *Let*

$$B_1 = B \cap \{x;\, d_D(x) = 1\}\,, \quad B_2 = B \cap \{x;\, d_D(x) = \tfrac{1}{2}\}$$

and write

$$A = 2\pi^{m/2} / \Gamma(\tfrac{1}{2}m)$$

for the area of the unit m-sphere Γ. Then (2) holds if and only if

(3) $$V_0(B_1) < A \quad \text{and} \quad V_0(B_2) < \tfrac{1}{2}A\,.$$

If these condition are fulfilled then γ yielding

$$a = g(\gamma)$$

is uniquely determined and one of the following cases (i)–(iii) must occur:

(i) $B_1 = \emptyset$,
(ii) $B_2 = \emptyset$ *or* $V_0(B_1) \geqq V_0(B_2) + \tfrac{1}{2}A$,
(iii) $B_1 \neq \emptyset \neq B_2$ *and* $|V_0(B_1) - V_0(B_2)| \leqq \tfrac{1}{2}A$.

The corresponding values of a and γ are then given as follows:

(i) $\Rightarrow Aa = 2V_0(B_2),\ \gamma = \frac{1}{2}$,

(ii) $\Rightarrow Aa = V_0(B_1),\ \gamma = 1$,

(iii) $$a = \frac{V_0(B_1) + V_0(B_2) + \frac{1}{2}A}{V_0(B_1) - V_0(B_2) + \frac{3}{2}A}, \quad \gamma = \frac{V_0(B_1) - V_0(B_2)}{2A} + \frac{3}{4}.$$

Since the equation

$$(2^m \pi^{m/2} \beta I + W_\alpha) f = 0$$

has only trivial solution in $\mathscr{B}$ provided $2^m \pi^{m/2} |\beta| > \omega W_\alpha$, the last theorem implies the following corollary:

Theorem 4. *If D fulfils (3) then H has a bounded inverse on $\mathscr{B}$.*

As a by-product one obtains also a theorem on integral representation of solutions of the first problem of Fourier for the equation

$$\sum_{j=1}^{m} \partial_j^2 u + \partial_{m+1} u = 0$$

(see theorem 3.11 below).

CHAPTER 1

In this chapter we shall prove several results related to theorem 1 announced in the introduction.

1.1. Notation. N is the set of all positive integers. If M is a subset in some Euclidean space (whose dimension will always be clear from the context) then the symbols $\operatorname{cl} M$, $\operatorname{int} M$, $\operatorname{fr} M$ and $\operatorname{diam} M$ will denote the closure, interior, boundary and diameter of M, respectively. Further let $H_k M$ stand for the outer Hausdorff k-dimensional measure of M defined by

(4) $$H_k M = 2^{-k} \alpha(k) \lim_{\varepsilon \to 0+} \inf \sum_n (\operatorname{diam} M_n)^k,$$

where

$$\alpha(k) = \pi^{k/2} / \Gamma(1 + \tfrac{1}{2}k)$$

is the volume of the unit k-ball and the infimum in (4) is taken over all sequences $\{M_n\}_{n \in N}$ of sets M_n with $\bigcup_n M_n = M$ such that $\operatorname{diam} M_n \leqq \varepsilon$ for all $n \in N$. If $M \subset R^k$ (= the Euclidean k-space), then $H_k M$ coincides with the outer Lebesgue measure of M. The support of a function f (with domain in some Euclidean space) will be denoted by $\operatorname{spt} f$.

The following simple remarks will be useful below.

1.2. Remarks. Fix an infinitely differentiable function ω in R^1 with spt $\omega \subset \subset (-1, 1)$ such that

$$\int_{R^1} \omega \, dH_1 = 1 \,, \quad \omega(-r) = \omega(r) \,, \quad r \in R^1 \,.$$

For each locally integrable function g in R^1 and each $n \in N$ define

$$A_n \, g(t) = n \int_{R^1} g(t - r) \, \omega(nr) \, dr \,.$$

Then $A_n g$ is infinitely differentiable and for each integrable function ψ with compact support in R^1

$$\int_{R^1} \psi A_n g \, dH_1 = \int_{R^1} g A_n \psi \, dH_1 \,.$$

Let now Z be a non-void set. For each function f on $R^1 \times Z$ and each $z \in Z$ define f_z on R^1 by

$$f_z(t) = f(t, z) \,, \quad t \in R^1 \,.$$

If f_z happens to be locally integrable for each $z \in Z$ we define $A_n f$ on $R^1 \times Z$ by

$$(A_n f)_z = A_n f_z \,, \quad z \in Z \,, \quad n \in N \,.$$

If the derivative $(f_z)'$ exists in R^1 for each $z \in Z$ then ∂f will denote the corresponding partial derivative in $R^1 \times Z$ given by

$$(\partial f)_z = (f_z)' \,, \quad z \in Z \,.$$

It is easily seen that, for each $n \in N$,

$$A_n \, \partial f = \partial A_n f$$

provided $(f_z)'$ is locally integrable in R^1 for each $z \in Z$.

Suppose now that $\mathbf{A}$ is a σ-algebra of subsets in Z and denote by $\mathbf{B}$ the σ-algebra of all Borel sets in R^1. If h is $\mathbf{B} \times \mathbf{A}$-measurable on $R^1 \times Z$ and h_z is integrable for each $z \in Z$ then the integral

$$\int_{R^1} h(t, z) \, dH_1(t)$$

represents an $\mathbf{A}$-measurable function of the variable $z \in Z$. Applying this to

$$h(t, z_1, z_2) = nf(z_1 - t, z_2) \, \omega(nt) \, (z_1, t \in R^1, z_2 \in Z)$$

with Z replaced by $R^1 \times Z$, z replaced by $[z_1, z_2]$ and $\mathbf{A}$ replaced by $\mathbf{B} \times \mathbf{A}$, one easily obtains that $A_n f$ is $\mathbf{B} \times \mathbf{A}$-measurable provided f is a $\mathbf{B} \times \mathbf{A}$-measurable function on $R^1 \times Z$ such that f_z is locally integrable for each $z \in Z$. Consequently, for such an f also $\partial A_n f$ is $\mathbf{B} \times \mathbf{A}$-measurable.

1.3. Lemma. *Let us keep the notation of* 1.2 *and let $\lambda \geqq 0$ be a measure on* $\mathbf{A}$. *For each $k \in N$ let Ψ_k be a class of $\mathbf{B} \times \mathbf{A}$-measurable functions on $R^1 \times Z$ enjoying the following properties*:

(P_1) $\Psi_k \subset \Psi_{k+1}$, $k \in N$.

(P_2) $\psi \in \Psi_k \Rightarrow -\psi \in \Psi_k$.

(P_3) *For each $\psi \in \Psi = \bigcup_{k \in N} \Psi_k$ both $\partial\psi$ and ψ are integrable $(H_1 \times \lambda)$ and, for each $z \in Z$, ψ_z is a continuously differentiable function with compact support in R^1.*

(P_4) *Given $k \in N$, there is a $n_k \in N$ such that*

$$(\psi \in \Psi_k,\ n \geqq n_k) \Rightarrow A_n\psi \in \Psi .$$

(P_5) *For each k there is a $G_k \in \mathbf{B} \times \mathbf{A}$ such that, for each bounded $\mathbf{B} \times \mathbf{A}$-measurable h on $R^1 \times Z$,*

$$\sup\left\{\int_{R^1 \times Z} h\psi \, \mathrm{d}(H_1 \times \lambda);\ \psi \in \Psi_k\right\} = \int_{G_k} |h| \, \mathrm{d}(H_1 \times \lambda) .$$

(P_6) *If g is a bounded $\mathbf{B}$-measurable function on R^1 then, for each $z \in Z$ and $k \in N$,*

$$\sup\left\{\int_{R^1} g\psi_z \, \mathrm{d}H_1;\ \psi \in \Psi_k\right\} = \int_{G_{kz}} |g| \, \mathrm{d}H_1 ,$$

where

$$G_{kz} = R^1 \cap \{t;\ [t, z] \in G_k\} .$$

Suppose now that f is a bounded $\mathbf{B} \times \mathbf{A}$-measurable function on $R^1 \times Z$ and let

$$F(z) = \sup\left\{\int_{R^1} f_z(\partial\psi)_z \, \mathrm{d}H_1;\ \psi \in \Psi\right\}, \quad z \in Z .$$

Then F is a non-negative $\mathbf{A}$-measurable function of the variable $z \in Z$ and

$$\int_Z F \, \mathrm{d}\lambda = \sup\left\{\int_{R^1 \times Z} f \, \partial\psi \, \mathrm{d}(H_1 \times \lambda);\ \psi \in \Psi\right\}.$$

Proof. Fix $z \in Z$. In view of (P_4), we have for $k \in N$ and $n \geqq n_k$

$$\sup\left\{\int_{R^1} f_z(\partial A_n\psi)_z \, \mathrm{d}H_1;\ \psi \in \Psi_k\right\} = F_{kn}(z) \leqq F(z),$$

whence it follows

$$\underline{F}_k(z) = \liminf_{n\to\infty} F_{kn}(z) \leqq \limsup_{n\to\infty} F_{kn}(z) = \bar{F}_k(z) \leqq F(z)\,.$$

In view of (P_1)

$$(5) \qquad k \in N \Rightarrow \underline{F}_k(z) \leqq \underline{F}_{k+1}(z)\,, \quad \bar{F}_k(z) \leqq \bar{F}_{k+1}(z)$$

and we conclude that

$$\lim_{k\to\infty} \bar{F}_k(z) \leqq F(z)\,.$$

On the other hand, if $c < F(z)$, then there is a $\psi \in \Psi$ with

$$\int_{R^1} f_z(\partial\psi)_z \,\mathrm{d}H_1 > c\,.$$

Noting that all the functions in $\{A_n(\partial\psi)_z\}_{n\in N}$ have support in a fixed compact subset of R^1 and converge uniformly to $(\partial\psi)_z$ as $n \to \infty$ (cf. (P_3)) we get for $k \in N$ with $\Psi_k \ni \psi$

$$\underline{F}_k(z) \geqq \int_{R^1} f_z(\partial\psi)_z \,\mathrm{d}H_1 > c\,.$$

We have thus proved

$$(6) \qquad F(z) = \lim_{k\to\infty} \underline{F}_k(z) = \lim_{k\to\infty} \bar{F}_k(z)\,.$$

Employing the remarks in 1.2 we see that, for $\psi \in \Psi_k$ and $n \geqq n_k$,

$$\int_{R^1} f_z(\partial A_n\psi)_z \,\mathrm{d}H_1 = \int_{R^1} f_z(A_n\,\partial\psi)_z \,\mathrm{d}H_1 = -\int_{R^1} (\partial A_n f)_z\,\psi_z \,\mathrm{d}H_1\,,$$

whence it follows by (P_6), (P_2) for $n \geqq n_k$

$$(7) \qquad F_{kn}(z) = \int_{G_{kz}} |(\partial A_n f)_z| \,\mathrm{d}H_1\,,$$

which is an **A**-measurable function of $z \in Z$. Consequently, also $\underline{F}_k$ and $F = \lim_{k\to\infty} \underline{F}_k$ are **A**-measurable non-negative functions. It remains to verify

$$\int_Z F \,\mathrm{d}\lambda \leqq \sup\left\{\int_{R^1\times Z} f\,\partial\psi \,\mathrm{d}(H_1 \times \lambda);\ \psi \in \Psi\right\} = K\,,$$

because the opposite inequality follows at once from the definition of F. We have by (5), (6)

$$\int_Z F \,\mathrm{d}\lambda = \lim_{k\to\infty} \int_Z \underline{F}_k \,\mathrm{d}\lambda \leqq \lim_{k\to\infty} \liminf_{n\to\infty} \int_Z F_{kn} \,\mathrm{d}\lambda\,.$$

Employing (7) and (P_5) we get

$$\int_Z F_{kn}\, d\lambda = \sup \left\{ \int_{R^1 \times Z} (\partial A_n f)\, \psi\, d(H_1 \times \lambda);\ \psi \in \Psi_k \right\}.$$

Now it is sufficient to observe that, for $\psi \in \Psi_k$ and $n \geqq n_k$

$$\int_{R^1 \times Z} (\partial A_n f)\, \psi\, d(H_1 \times \lambda) = - \int_{R^1 \times Z} f\, \partial A_n \psi\, d(H_1 \times \lambda) \leqq K$$

by 1.2, (P_4) and (P_2).

1.4. Remark. The above lemma, which is in fact an abstract version of lemma 1.10 in [10], is closely connected with investigations of functions whose partial derivatives are measures; see [1], [6], [9], [12], [15], [16], [18].

1.5. Notation. As in the introduction, $\Omega_\varrho(y)$ will denote the ball of center y and radius ϱ in R^m, and $\Gamma = \operatorname{fr} \Omega_1(0)$. It is convenient to adopt the following terminology introduced in [10], 1.5: If S is an open segment or half-line in R^m then $y \in S$ is termed a hit of S on D provided

$$H_1(\Omega_\varrho(y) \cap S \cap D) > 0 \quad \text{and} \quad H_1(S \cap \Omega_\varrho(y) \setminus D) > 0$$

for each $\varrho > 0$. The number of all hits of

$$S_r(x) = \{x + \varrho\theta;\ 0 < \varrho < r\} \quad (x \in R^m,\ \theta \in \Gamma)$$

on D will be denoted by $n_r(\theta, x)$ $(0 \leqq n_r(\theta, x) \leqq +\infty)$. According to 1.6 in [10], $n_r(\theta, x)$ is a Baire function of the variable $\theta \in \Gamma$ and we let

$$v(r; x) = \int_\Gamma n_r(\theta, x)\, dH_{m-1}(\theta)\,. \tag{8}$$

For the sake of brevity we shall sometimes write $v_r(x)$ instead of $v(r; x)$.

$\mathscr{D}$ will stand for the class of all infinitely differentiable functions with compact support in R^{m+1}. For $T \in (-\infty, +\infty\rangle$ let

$$R_T = R^m \times (-\infty, T) \tag{9}$$

and denote by $\mathscr{D}_T$ the class of all $\varphi \in \mathscr{D}$ with spt $\varphi \subset R_T$. The derivative with respect to the j-th variable will be denoted by ∂_j. The points $z = [z_1, \ldots, z_{m+1}] \in R^{m+1}$ will often be written in the form $[\hat{z}, z_{m+1}]$ with $\hat{z} = [z_1, \ldots, z_m] \in R^m$. We shall write

$$\hat{\nabla} = [\partial_1, \ldots, \partial_m]\,.$$

The Euclidean norm is denoted by $|\ldots|$. As in the introduction, we denote by G the well-known kernel connected with the heat equation, defining $G = 0$ on $\mathrm{cl}\, R_0$ and letting

$$G(x, t) = t^{-m/2} \exp\left(-|x|^2/4t\right) \quad \text{for} \quad [x, t] \in R^m \times (0, +\infty).$$

Simple calculation shows that for

$$R_{\alpha\beta} = R_\beta \setminus R_\alpha, \quad -\infty < \alpha < \beta < +\infty,$$

the following estimates hold:

$$\int_{R_{\alpha\beta}} |\partial_j G| \, \mathrm{d}H_{m+1} \leqq 2^{m+1}[\pi^{m-1}(\beta - \alpha)]^{1/2}, \quad 1 \leqq j \leqq m, \tag{10}$$

$$\int_{R_{\alpha\beta}} G \, \mathrm{d}H_{m+1} \leqq 2^m \pi^{m/2}(\beta - \alpha). \tag{11}$$

By the term measure we shall usually mean a finite signed Borel measure defined on the σ-algebra of all Borel subsets of a fixed Borel set in some Euclidean space. If μ is a measure and M is a Borel set in the domain of μ, then $|\mu|(M)$ denotes the variation of μ on M; spt μ will denote the support of μ.

Let $D \subset R^m$ be an open set with a compact boundary $B \neq \emptyset$. Fix now T_1, T_2, $-\infty < T_1 \leqq T_2 \leqq +\infty$, and put

$$E = D \times (T_1, T_2), \quad C = B \times \langle T_1, T_2).$$

Denote by $\mathscr{B}' = \mathscr{B}'(T_1, T_2)$ the Banach space of all measures μ in R^{m+1} with

$$|\mu|(R^{m+1} \setminus C) = 0;$$

the norm in $\mathscr{B}'$ is given by

$$\|\mu\| = |\mu|(C).$$

With each $\mu \in \mathscr{B}'$ associate the potential

$$U\mu(z) = \int G(z - \zeta) \, \mathrm{d}\mu(\zeta), \quad z \in R^{m+1} \setminus \mathrm{cl}\, C.$$

Then $U\mu$ is an infinitely differentiable function on $R^{m+1} \setminus \mathrm{cl}\, C$ satisfying there the heat equation

$$\sum_{j=1}^{m} \partial_j^2 U\mu = \partial_{m+1} U\mu. \tag{12}$$

Employing (10), (11) one obtains at once for

$$E_{\alpha\beta} = E \cap (R_\beta \setminus R_\alpha), \quad -\infty < \alpha < \beta < +\infty,$$

that

$$(13) \qquad \int_{E_{\alpha\beta}} |\partial_j U\mu| \, dH_{m+1} \leqq 2^{m+1}[\pi^{m-1}(\beta - \alpha)]^{1/2} \, \|\mu\| , \quad 1 \leqq j \leqq m ,$$

$$(14) \qquad \int_{E_{\alpha\beta}} |U\mu| \, dH_{m+1} \leqq 2^m \pi^{m/2}(\beta - \alpha) \, \|\mu\| .$$

Accordingly, we are justified to introduce the distribution $H\mu$ in R_{T_2} defining for $\varphi \in \mathscr{D}_{T_2}$

$$\langle \varphi, H\mu \rangle = \int_E (\hat{\nabla} U\mu \, . \, \hat{\nabla} \varphi - U\mu \, . \, \partial_{m+1}\varphi) \, dH_{m+1} .$$

As it is usual in distribution theory [23], we shall say that $H\mu$ is a measure provided there is a measure ν_μ in R^{m+1} such that

$$(15) \qquad \langle \varphi, H\mu \rangle = \int \varphi \, d\nu_\mu , \quad \varphi \in \mathscr{D}_{T_2} .$$

It is easily seen that (15) together with

$$(16) \qquad |\nu_\mu| \, (R^{m+1} \setminus R_{T_2}) = 0$$

determine ν_μ uniquely and that each ν_μ enjoying (15), (16) satisfies

$$(17) \qquad \operatorname{spt} \nu_\mu \subset \operatorname{cl} C .$$

Indeed, if $\varphi \in \mathscr{D}_{T_2}$ and $\operatorname{spt} \varphi \cap \operatorname{cl} C = \emptyset$, then there is a bounded open set $\tilde{D} \subset R^m$ with $\operatorname{cl} \tilde{D} \subset D$ such that the boundary of $\tilde{D}$ is a smooth hypersurface $\tilde{B}$ and

$$E \cap \operatorname{spt} \varphi \subset \tilde{D} \times (T_1, T_2) = \tilde{E} .$$

Taking into account (12) (note also that $U\mu$ vanishes on $D \times \{T_1\}$ and φ vanishes on $D \times \{T_2\}$) and writing $\tilde{n}$ for the exterior normal of $\tilde{D}$ we obtain by the Gauss-Green theorem

$$\langle \varphi, H\mu \rangle = \int_{\tilde{E}} (\hat{\nabla} U\mu \, . \, \hat{\nabla} \varphi - U\mu \, . \, \partial_{m+1}\varphi) \, dH_{m+1} =$$

$$= \int_{T_1}^{T_2} dt \int_{\tilde{B}} \varphi(x, t) \, \tilde{n}(x) \, . \, \hat{\nabla} U\mu(x, t) \, dH_{m-1}(x) = 0 .$$

We conclude from (16), (17) that $\nu_\mu \in \mathscr{B}'$.

1.6. Lemma. *Given* $\zeta = [\xi, \tau] \in R^{m+1}$ *and* $\varphi \in \mathscr{D}$ *let*

$$\tilde{W} \varphi(\zeta) = \int_E [\hat{\nabla} G(z - \zeta) \, . \, \hat{\nabla} \varphi(z) - G(z - \zeta) \, \partial_{m+1} \varphi(z)] \, dz$$

and define $S\varphi$ *on* $(0, +\infty) \times (0, +\infty) \times \Gamma$ *by*

$$(18) \qquad S\varphi(\varrho, \eta, \theta) = \varphi(\xi + \varrho\theta, \tau + \varrho^2/4\eta)\,, \quad \varrho, \eta \in (0, +\infty)\,, \quad \theta \in \Gamma\,.$$

If $\tau \in \langle T_1, T_2)$ *then*

$$\tilde{W}\varphi(\zeta) = -2^{m-1} \int_\Gamma \mathrm{d}H_{m-1}(\theta) \int_0^\infty \mathrm{e}^{-\eta} \eta^{m/2-1}\,\mathrm{d}\eta \int_{D_*} \partial_1 S\varphi(\varrho, \eta, \theta)\,\mathrm{d}\varrho\,,$$

where

$$(19) \qquad D_* = \{\varrho;\ 0 < \varrho < 2[\eta(T_2 - \tau)]^{1/2},\ \xi + \varrho\theta \in D\}\,.$$

Proof. Simple calculation yields

$$\tilde{W}\varphi(\zeta) = -\frac{1}{2}\int_\tau^{T_2} (t - \tau)^{-m/2-1}\,\mathscr{I}(t)\,\mathrm{d}t\,,$$

where

$$\mathscr{I}(t) = \int_D \mathrm{e}^{-|x-\xi|^2/4(t-\tau)}\,[(x - \xi)\,.\,\hat{\nabla}\varphi(x, t) + 2(t - \tau)\,\partial_{m+1}\varphi(x, t)]\,\mathrm{d}x\,.$$

Let us now introduce the variables $r \in (0, +\infty)$ and $\theta \in \Gamma$ by

$$x = \xi + r\theta\,.$$

Then $\mathrm{d}x = r^{m-1}\,\mathrm{d}r\,\mathrm{d}H_{m-1}(\theta)$ and $\mathscr{I}(t)$ transforms into

$$\int_\Gamma \mathscr{K}(t, \theta)\,\mathrm{d}H_{m-1}(\theta) = \mathscr{I}(t)\,,$$

where $\mathscr{K}(t, \theta)$ denotes the integral extended over

$$D_\theta = \{r;\ r > 0,\ \xi + r\theta \in D\}$$

given by

$$\mathscr{K}(t, \theta) = \int_{D_\theta} \mathrm{e}^{-r^2/4(t-\tau)} \left[r\,\frac{\partial\varphi(\xi + r\theta, t)}{\partial r} + 2(t - \tau)\,\frac{\partial\varphi(\xi + r\theta, t)}{\partial t}\right] r^{m-1}\,\mathrm{d}r\,.$$

Consequently,

$$\tilde{W}\varphi(\zeta) = -\frac{1}{2}\int_\Gamma \mathscr{L}(\theta)\,\mathrm{d}H_{m-1}(\theta)\,,$$

where

$$\mathscr{L}(\theta) = \int_\tau^{T_2} (t - \tau)^{-m/2-1}\,\mathscr{K}(t, \theta)\,\mathrm{d}t = \iint \ldots\,\mathrm{d}r\,\mathrm{d}t$$

may be considered as a double integral extended over $[r, t] \in D_\theta \times (\tau, T_2)$. Employing the change of variables

$$r = \varrho\,, \quad t = \tau + \frac{\varrho^2}{4\eta}$$

we get after simple calculation

$$\mathscr{L}(\theta) = 2^m \int_0^\infty e^{-\eta}\eta^{m/2-1}\,d\eta \int_{D_*} \partial_1 S\varphi(\varrho, \eta, \theta)\,d\varrho\,,$$

which completes the proof.

1.7. Remark. Let δ_ζ denote the unit point mass (= Dirac measure) concentrated at ζ. Noting that

$$G(z - \zeta) = U\,\delta_\zeta(z)$$

we observe that

$$\tilde{W}\varphi(\zeta) = \langle\varphi, H\delta_\zeta\rangle \tag{20}$$

provided $\varphi \in \mathscr{D}_{T_2}$ and $\zeta \in C$.

1.8. Proposition. *Let* $\zeta = [\xi, \tau] \in R^{m+1}$, $T_1 \leqq \tau < T_2$, *fix* $R > 0$, $\varepsilon > 0$ *and put*

$$\mathscr{D}^1 = \mathscr{D}_{T_2} \cap \{\varphi;\ |\varphi| \leqq 1,\ \operatorname{spt}\varphi \subset [\Omega_R(\xi)\setminus\{\xi\}] \times (\tau, \tau + \varepsilon)\}\,, \tag{21}$$

$$r(\eta) = \min\{R, 2[\eta \min(\varepsilon, T_2 - \tau)]^{1/2}\}\,, \quad \eta > 0\,. \tag{22}$$

Then

$$2^{m-1}\int_0^\infty e^{-\eta}\eta^{m/2-1} v(r(\eta); \xi)\,d\eta = \sup\{\tilde{W}\varphi(\zeta);\ \varphi \in \mathscr{D}^1\}\,.$$

Proof. We shall apply lemma 1.3. Define the measure μ by

$$d\mu(\eta) = 2^{m-1}e^{-\eta}\eta^{m/2-1}\,dH_1(\eta)$$

and consider the product measure $\lambda = \mu \times H_{m-1}$ on the σ-algebra of all Borel subsets of $Z = (0, \infty) \times \Gamma$. It is easily seen that the mapping

$$\Phi : [\varrho, \eta, \theta] \to \left[\xi + \varrho\theta, \tau + \frac{\varrho^2}{4\eta}\right]$$

maps $(0, \infty) \times (0, \infty) \times \Gamma = (0, \infty) \times Z$ homeomorphically onto

$$[R^m \setminus \{\xi\}] \times (\tau, \infty)\,.$$

Let

$$c = \min(\varepsilon, T_2 - \tau),\ \tilde{E} = [\Omega_R(\xi) \setminus \{\xi\}] \times (\tau, \tau + c),$$

define $r(\eta)$ by (22) and put

$$G = \Phi^{-1}(\tilde{E}) = \{[\varrho, \eta];\ \eta > 0,\ 0 < \varrho < r(\eta)\} \times \Gamma .$$

Fix a decreasing sequence of positive numbers $\{\varepsilon_k\}_{k=1}^{\infty}$ such that

$$2\varepsilon_1 < R,\quad \lim_{k \to \infty} \varepsilon_k = 0$$

and define

$$G_k = \{[\varrho, \eta];\ \eta > \varepsilon_k^2 c^{-1},\ \varepsilon_k < \varrho < r(\eta) - \varepsilon_k\} \times \Gamma .$$

Denote by Ψ_k the class of all functions ψ with domain $X = R^1 \times Z$ for which there is a $\varphi \in \mathscr{D}^1$ (depending on ψ) such that

$$\operatorname{spt} S\varphi \subset G_k,\quad \psi = S\varphi \text{ in } G$$

and

$$\psi(X \setminus G) = \{0\} .$$

Then the class of all (point-wise) limits of sequences of elements of Ψ coincides with the class of all the functions g of the first class of Baire on X such that

$$|g| \leqq 1,\quad X \setminus G_k \subset g^{-1}(0) .$$

Hence we conclude that the conditions (P_6), (P_5) in 1.3 are satisfied. Fix now $n_k > \varepsilon_k^{-1}$. If $\psi \in \Psi_k$, $\psi = S\varphi$ in G and $A_n\psi$ with $n \geqq n_k$ is defined by 1.2, then $A_n\psi$ has a compact support contained in G. Simple calculation shows that the value attained by $(A_n\psi) \circ \Phi^{-1}$ (= the composite of Φ^{-1} and $A_n\psi$) at $[x, t] \in [R^m \setminus \{\xi\}] \times (\tau, \infty)$ is given by the integral

$$\int_{R^1} n\varphi(h(u, x)(x - \xi) + \xi,\ h^2(u, x)(t - \tau) + \tau)\, \omega(nu)\, du , \tag{23}$$

where ω has the meaning described in 1.2 and

$$h(u, x) = \frac{|x - \xi| - u}{|x - \xi|} .$$

Defining $\tilde{\varphi}(x, t)$ by (23) for $[x, t] \in \tilde{E}$ and letting

$$\tilde{\varphi}(R^{m+1} \setminus \tilde{E}) = \{0\}$$

we see that $\tilde{\varphi} \in \mathscr{D}^1$,

$$A_n\psi = \tilde{\varphi} \circ \Phi = S\tilde{\varphi} \quad \text{in} \quad G .$$

Consequently, $A_n\psi \in \Psi = \bigcup_{k=1}^{\infty} \Psi_k$ and (P_4) is verified. The conditions (P_1), (P_2), (P_3) being obviously fulfilled we are justified to apply 1.3 to the characteristic function f of G. Employing 1.6 we get

$$\sup\{\tilde{W}\varphi(\zeta);\ \varphi \in \mathscr{D}^1\} =$$

$$\sup\left\{\int_X f\,\partial_1\psi\,\mathrm{d}(H_1 \times \lambda);\ \psi \in \Psi\right\} = \int_Z F\,\mathrm{d}\lambda$$

where, for fixed $z = [\eta, \theta] \in Z$,

$$F(z) = \sup\left\{\int_{R^1} f_z(\partial_1\psi)_z\,\mathrm{d}H_1;\ \psi \in \Psi\right\}.$$

Note that $\{\psi_z;\ \psi \in \Psi\}$ coincides with the class of all infinitely differentiable functions γ in R^1 with

$$|\gamma| \leqq 1\,, \quad \operatorname{spt}\gamma \subset (0, r(\eta))\,.$$

Taking into account that f_z is the characteristic function of $D_* \cap (0, r(\eta))$, where D_* is defined by (19), we conclude from 1.9 in [10] that $F(z)$ equals the number of hits of $(0, r(\eta))$ on D_*. In other words, $F(z)$ is the number of hits of

$$\{\xi + \varrho\theta;\ 0 < \varrho < r(\eta)\}$$

on D and, consequently,

$$\int_\Gamma F(\eta, \theta)\,\mathrm{d}H_{m-1}(\theta) = v(r(\eta); \xi)\,,$$

which completes the proof.

1.9. Lemma. *Fix* $\zeta = [\xi; \tau] \in C$. *If* $H\delta_\zeta$ *is a measure then*

$$v_\infty(\xi) < \infty\,. \tag{24}$$

Conversely, if (24) *holds, then* $H\delta_\zeta$ *may be identified with an element of* $\mathscr{B}'$ *and its norm admits the estimates*

$$\|H\delta_\zeta\| \leqq 2^{m-1}[v_\infty(\xi)\,\Gamma(\tfrac{1}{2}m) + 2\pi^{m/2}]\,, \tag{25_1}$$

$$\|H\delta_\zeta\| \geqq 2^{m-1}\,v_\infty(\xi)\int_b^\infty e^{-\eta}\eta^{m/2-1}\,\mathrm{d}\eta\,, \tag{25_2}$$

where

$$b = \frac{(\operatorname{diam} B)^2}{4(T_2 - \tau)}\,.$$

Proof. Let $R = +\infty = \varepsilon$ and define $\mathscr{D}^1$ by (21) for this particular choice of R and ε. Suppose first that $H\delta_\zeta$ is a measure. Then $H\delta_\zeta \in \mathscr{B}'$ (see 1.5) and

$$\|H\delta_\zeta\| \geqq \sup \{\langle \varphi, H\delta_\zeta \rangle;\ \varphi \in \mathscr{D}^1\} =$$

$$= 2^{m-1} \int_0^\infty e^{-\eta} \eta^{m/2-1} v(2[\eta(T_2 - \tau)]^{1/2}; \xi)\, d\eta \geqq 2^{m-1}\, v_\infty(\xi) \int_b^\infty e^{-\eta} \eta^{m/2-1}\, d\eta\,,$$

because $v_\varrho(\xi) = v_\infty(\xi)$ for $\varrho > \operatorname{diam} B$.

Assume now (24) and consider $\varphi \in \mathscr{D}_{T_2}$, $|\varphi| \leqq 1$. Fix $\theta \in \Gamma$, $\eta > 0$ and define D_*, $S\varphi$ as in 1.6. We shall show that

$$\left| \int_{D_*} \partial_1 S\varphi(\varrho, \eta, \theta)\, d\varrho \right| \leqq 1 + n_\infty(\theta, \xi)\,. \tag{26}$$

It is sufficient to consider the case when $n_\infty(\theta, \xi) < +\infty$. Let us agree to write simply $S(\varrho) = S\varphi(\varrho, \eta, \theta)$, so that $S'(\varrho) = \partial_1 S\varphi(\varrho, \eta, \theta)$. Put $r = 2[(T_2 - \tau)\, \eta]^{1/2}$ and let $\varrho_1 < \ldots < \varrho_n$ be all the hits of $(0, r)$ on D_*. Further put $\varrho_{n+1} = r$, $\varrho_0 = 0$. Since D_* is open and $(\varrho_{i-1}, \varrho_i)$ contains no hits on D_*,

$$D_i = D_* \cap (\varrho_{i-1}, \varrho_i) \quad \text{is either void and} \quad \int_{D_i} S'(\varrho)\, d\varrho = 0\,,$$

or else $H_1(D_i) = \varrho_i - \varrho_{i-1}$, in which case

$$\int_{D_i} S'(\varrho)\, d\varrho = S(\varrho_i) - S(\varrho_{i-1})\,.$$

Noting that $S(\varrho_i) \leqq 1$ for $0 \leqq i \leqq n$ and $S(\varrho_{n+1}) = 0$, we conclude that

$$\left| \int_{D_*} S'(\varrho)\, d\varrho \right| \leqq n + 1\,.$$

The inequality (26) together with 1.6, 1.7 yields

$$\sup \{\langle \varphi, H\delta_\zeta \rangle;\ \varphi \in \mathscr{D}_{T_2},\ |\varphi| \leqq 1\} \leqq 2^{m-1}[2\pi^{m/2} + \Gamma(\tfrac{1}{2}m)\, v_\infty(\xi)]$$

and the proof is complete.

1.10. Remark. If $\mu \in \mathscr{B}'$ and $\varphi \in \mathscr{D}_{T_2}$, then

$$\langle \varphi, H\mu \rangle = \int_C \langle \varphi, H\delta_\zeta \rangle\, d\mu(\zeta)\,. \tag{27}$$

Proof. Taking into account (10), (11) and applying Fubini's theorem to

$$\iint_{E\times C} [\hat{\nabla} G(z-\zeta) \,.\, \hat{\nabla}\varphi(z) - G(z-\zeta)\,\partial_{m+1}\,\varphi(z)]\,\mathrm{d}H_{m+1}(z)\,\mathrm{d}\mu(\zeta)$$

one obtains (27) (see also 1.7, 1.6).

A reasoning similar to that used in the proof of theorem 1.13 in [10] permits now to establish the following

1.11. Theorem. *$H\mu$ is a measure for each $\mu \in \mathscr{B}'$ if and only if*

$$V = \sup\{v_\infty(\xi);\ \xi \in B\} < \infty\,. \tag{28}$$

If (28) holds then, for each $\mu \in \mathscr{B}'$, $H\mu$ may be identified with a uniquely determined element of $\mathscr{B}'$, the operator $H : \mu \to H\mu$ is bounded on $\mathscr{B}'$ and

$$\|H\| \leqq 2^{m-1}[\Gamma(\tfrac{1}{2}m)\,V + 2\pi^{m/2}]\,.$$

Proof. For each $\varphi \in \mathscr{D}_{T_2}$ define the functional L_φ on $\mathscr{B}'$ by

$$\langle \mu, L_\varphi\rangle = \langle \varphi, H\mu\rangle\,, \quad \mu \in \mathscr{B}'\,.$$

Let spt $\varphi \subset R_\beta \setminus R_\alpha$, $-\infty < \alpha < \beta < \infty$ and put

$$c = 2^m \max\{2[\pi^{m-1}(\beta-\alpha)]^{1/2},\ \pi^{m/2}(\beta-\alpha)\}\,, \quad k(\varphi) = c\sup\sum_{i=1}^{m+1}|\partial_i\varphi|\,.$$

We get from the definition of $\langle\varphi, H\mu\rangle$ and (13), (14)

$$|\langle\mu, L_\varphi\rangle| \leqq k(\varphi)\,\|\mu\|\,,$$

so that each functional L_φ is bounded on $\mathscr{B}'$.

Let

$$\mathscr{A} = \mathscr{D}_{T_2} \cap \{\varphi;\ |\varphi| \leqq 1\}\,.$$

If $H\mu$ is measure for each $\mu \in \mathscr{B}'$, then the class of functionals $\{L_\varphi\}_{\varphi\in\mathscr{A}}$ is pointwise bounded on $\mathscr{B}'$. Hence it follows by the uniform boundedness principle

$$\sup\{\|L_\varphi\|;\ \varphi \in \mathscr{A}\} = K < +\infty\,.$$

In particular, for each $\zeta \in C$,

$$\|H\delta_\zeta\| = \sup\{\langle\delta_\zeta, L_\varphi\rangle;\ \varphi \in \mathscr{A}\} \leqq K\,.$$

Consider now an arbitrary $\xi \in B$ and let $\zeta = [\xi, T_1] \in C$. Defining

$$b = \frac{(\operatorname{diam} B)^2}{4(T_2 - T_1)}$$

we obtain from (25_2)

$$v_\infty(\xi) \leqq 2^{1-m}\left(\int_b^\infty e^{-\eta}\eta^{m/2-1}\,d\eta\right)^{-1} . K$$

and (28) is verified. Conversely, if (28) holds, then 1.10 and 1.9 imply

$$\sup\{|\langle\varphi, H\mu\rangle|;\ \varphi \in \mathscr{A}\} \leqq 2^{m-1}[V\Gamma(\tfrac{1}{2}m) + 2\pi^{m/2}]\,\|\mu\|$$

for each $\mu \in \mathscr{B}'$. This completes the proof.

Remark. In connection with the above reasonings we wish to mention here the work of G. FICHERA [5] on applications of functional analysis to boundary value problems.

CHAPTER 2

We are now going to investigate more closely the function $\tilde{W}\varphi$ which has appeared in 1.6, 1.7 for the special case when $\varphi \in \mathscr{D}_{T_2}$.

2.1. Lemma. *Fix $\xi \in R^m$ with $v_\infty(\xi) < +\infty$ and define $s(\varrho; \xi, \theta)$ for $\varrho > 0$ and $\theta \in \Gamma$ as follows (compare 2.4 in [10]):*

$$s(\varrho; \xi, \theta) = \sigma(= \pm 1)$$

if there is a $\delta > 0$ such that

$$\xi + (\varrho + \sigma u)\,\theta \in D\,, \quad \xi + (\varrho - \sigma u)\,\theta \in R^m \setminus D$$

for almost every $u \in (0, \delta)$; otherwise we set $s(\varrho; \xi, \theta) = 0$. Further fix $\tau \in \langle T_1, T_2\rangle$; put $\zeta = [\xi, \tau]$ and associate with each bounded Baire function f on C the function $\sum_f(\zeta; \eta, \theta)$ defined for $\theta \in \Gamma$ and $\eta > 0$ as follows: If $n_\infty(\theta, \xi) < +\infty$ then

$$\textstyle\sum_f(\zeta; \eta, \theta) = \sum_\varrho f\left(\xi + \varrho\theta, \tau + \frac{\varrho^2}{4\eta}\right) s(\varrho; \xi, \theta)\,,$$

the sum on the right — hand side being extended over ϱ satisfying

$$0 < \varrho < 2[\eta(T_2 - \tau)]^{1/2}, \quad s(\varrho; \xi, \theta) \neq 0\,;$$

if $n_\infty(\theta, \xi) = +\infty$, we set

$$\textstyle\sum_f(\zeta; \eta, \theta) = 0\,.$$

Then $\sum_f(\zeta; \eta, \theta)$ is integrable $dH_{m-1}(\theta)$ over Γ for each $\eta > 0$. Besides that,

$$V_f(\zeta; \eta) = \int_\Gamma \textstyle\sum_f(\zeta; \eta, \theta)\, dH_{m-1}(\theta)$$

is a bounded Baire function of the variable $\eta > 0$. *We are thus justified to define*

$$Wf(\zeta) = 2^{m-1} \int_0^\infty \mathrm{e}^{-\eta} V_f(\zeta; \eta) \, \eta^{m/2-1} \, \mathrm{d}\eta .$$

Remark. If F is a function whose domain contains C, then WF is taken to mean Wf, where f is the restriction of F to C.

Proof of lemma 2.1. Denote by K_ξ the set of those $\theta \in \Gamma$, for which there is an $\varepsilon = \varepsilon(\theta) > 0$ such that

$$H_1(\{\xi + \varrho\theta;\ 0 < \varrho < \varepsilon\} \setminus D) = 0 ,$$

and consider first

$$\theta \in K_\xi , \quad n_\infty(\theta, \xi) < +\infty . \tag{29}$$

Fix $\eta > 0$, put

$$r(\eta) = 2[\eta(T_2 - \tau)]^{1/2}$$

and define

$$D_* = \{\varrho; 0 < \varrho < r(\eta),\ \xi + \varrho\theta \in D\} .$$

If $\varrho_1 < \ldots < \varrho_n$ are all the hits of $(0, r(\eta))$ on D_*, then

$$s(\varrho_{j+1}; \xi, \theta) = -s(\varrho_j; \xi, \theta) \quad \text{for} \quad 1 \leqq j < n , \tag{30}$$

$$s(\varrho_1; \xi, \theta) = -1 . \tag{31}$$

Letting $\varrho_0 = 0$ we conclude for

$$S(\varrho) = S\varphi(\varrho, \eta, \theta)$$

defined by (18) that

$$\int_{D^*} S'(\varrho) \, \mathrm{d}\varrho = \sum_{j=0}^{n} (-1)^{j-1} S(\varrho_j) = -\varphi(\zeta) - \textstyle\sum_\varphi(\zeta; \eta, \theta) .$$

We have thus shown for θ satisfying (29) that

$$\int_{D^*} \partial_1 S_\varphi(\varrho, \eta, \theta) \, \mathrm{d}\varrho = -\varphi(\zeta) - \textstyle\sum_\varphi(\zeta; \eta, \theta) .$$

A similar reasoning shows for θ satisfying

$$\theta \in \Gamma \setminus K_\xi , \quad n_\infty(\theta, \xi) < +\infty \tag{32}$$

that

$$\int_{D^*} \partial_1 S\varphi(\varrho, \eta, \theta)\, \mathrm{d}\varrho = -\sum_\varphi(\zeta; \eta, \theta)\,.$$

Clearly,

$$\left|\sum_\varphi(\zeta; \eta, \theta)\right| \leqq n_\infty(\theta, \xi)\,.\, \sup|\varphi|\,.$$

Let us recall that K_ξ is measurable (H_{m-1}) by 2.6 in [10]. Noting that $\partial_1 S\varphi(\varrho, \eta, \theta)$ is a continuous function on $(0, \infty) \times (0, \infty) \times \Gamma$ and taking into account that

$$H_{m-1}(\Gamma \cap \{\theta; \eta_\infty(\theta, \xi) = +\infty\}) = 0$$

we conclude that $\sum_\varphi(\zeta; \eta, \theta)$ is measurable (H_{m-1}) on Γ and

$$\text{(33)} \qquad V_\varphi(\zeta; \eta) = -\varphi(\zeta)\, H_{m-1}(K_\xi) - \int_\Gamma \mathrm{d}H_{m-1}(\theta) \int_{D^*} \partial_1\, S_\varphi(\varrho, \eta, \theta)\, \mathrm{d}\varrho$$

is a Baire function of the variable $\eta > 0$ satisfying the inequality

$$|V_\varphi(\zeta; \eta)| \leqq v_\infty(\xi) \sup|\varphi|\,.$$

Consider now the class $\mathscr{F}$ of all bounded Baire functions f on C for which $\sum_f(\zeta; \eta, \theta)$ is integrable $\mathrm{d}H_{m-1}(\theta)$ over Γ for each $\eta > 0$ and $V_f(\zeta; \eta)$ is a bounded Baire function of $\eta > 0$. We have just seen that $\mathscr{F}$ contains restriction to C of any $\varphi \in \mathscr{D}_{T_2}$. If $\{f_k\}_{k=1}^\infty$ is a sequence of elements of $\mathscr{F}$ with

$$\lim_{k\to\infty} f_k = f$$

such that, for suitable $K \in R^1$,

$$\sup|f_k| \leqq K\,, \quad k \in N\,,$$

then

$$\lim_{k\to\infty} \sum_{f_k}(\zeta; \eta, \theta) = \sum_f(\zeta; \eta, \theta)$$

and

$$\left|\sum_{f_k}(\zeta; \eta, \theta)\right| \leqq K n_\infty(\theta, \xi)$$

for all $k \in N$. By the Lebesgue dominated convergence theorem also

$$\text{(34)} \qquad \lim_{k\to\infty} V_{f_k}(\zeta, \eta) = V_f(\zeta, \eta)\,.$$

Since

$$|V_f(\zeta, \eta)| \leqq K v_\infty(\xi)\,,$$

we see that $f \in \mathscr{F}$. Consequently, $\mathscr{F}$ contains all bounded Baire functions on C and the proof is complete.

2.2. Corollary. *Let* $\zeta = [\xi, \tau] \in R_{T_2} \setminus R_{T_1}$ (see (9)) *and denote by* $d_D(\xi)$ *the* m-*dimensional density of* D *at* ξ. *Let* $\varphi \in \mathscr{D}_{T_2}$ *and define* $\tilde{W}\varphi(\zeta)$ *by* 1.6. *Then*

$$W\varphi(\zeta) = \tilde{W}\varphi(\zeta) - 2^m \pi^{m/2}\, d_D(\xi)\, \varphi(\zeta)\,.$$

Proof. Let us keep the notation from the above proof. According to 2.6 in [10]

$$H_{m-1}(K_\xi) = d_D(\xi)\, 2\pi^{m/2}/\Gamma(\tfrac{1}{2}m)\,. \tag{35}$$

Now it is sufficient to employ (33) and 1.6.

The following corollary was actually proved in the course of the proof of lemma 2.1.

2.3. Corollary. *Let* $\zeta = [\xi, \tau] \in R_{T_2} \setminus R_{T_1}$ *and suppose that*

$$v_\infty(\xi) < +\infty\,.$$

If F *is a bounded Baire function on* C *then*

$$|WF(\zeta)| \leqq 2^{m-1}\, \Gamma(\tfrac{1}{2}m)\, v_\infty(\xi)\,.\,\sup |F|\,. \tag{36}$$

If $\{f_k\}$ *is a pointwise convergent sequence of bounded Baire functions on* C *such that for suitable* $K \in R^1$,

$$k \in N \Rightarrow |f_k| \leqq K$$

and

$$\lim_{k\to\infty} f_k = f\,,$$

then

$$\lim_{k\to\infty} Wf_k(\zeta) = Wf(\zeta)\,.$$

Proof. The inequality (36) follows from the estimate

$$|V_F(\zeta, \eta)| \leqq v_\infty(\xi) \sup |F|\,, \quad \eta > 0\,. \tag{37}$$

Employing (37) with $F = f_k$ we get

$$|V_{f_k}(\zeta, \eta)| \leqq K\, v_\infty(\xi)\,, \quad \eta > 0\,.$$

Now it is sufficient to use (34) and refer to the Lebesgue dominated convergence theorem.

2.4. Remark. Let us recall that a unit vector $\theta \in \Gamma$ is called the exterior normal of D at $y \in R^m$ in the sense of H. Federer provided the symmetric difference of D and the half-space

$$R^m \cap \{x; (x - y)\,.\,\theta < 0\}$$

has m-dimensional density 0 at y. In what follows we shall put $n(y) = \theta$ if $\theta \in \Gamma$ is the exterior normal of D at y (which is easily seen to be uniquely determined) and we denote by $n(y)$ the zero vector if there is no exterior normal $\theta \in \Gamma$ at y in the above mentioned sense. The set $\hat{B} = R^m \cap \{y; |n(y)| \neq 0\}$ will be termed the reduced boundary of D.

The following assertion is a consequence of proposition 2.10 in [10] and results of E. De Giorgi and H. Federer (see [2], [3] and 2.11. in [10]):

Proposition. *Suppose there is an $(m+1)$-tuple of points $x^1, \ldots, x^{m+1} \in R^m$ in general position (i.e., not situated on a single hyperplane) such that*

$$\sum_{i=1}^{m+1} v_\infty(x^i) < \infty .$$

Then $H_{m-1}(\hat{B}) < +\infty$. If $w = [w_1, \ldots, w_m]$ is a vector-valued function with m components $w_j \in \mathcal{D}$, then

$$\int_B w(y)\, n(y)\, \mathrm{d}H_{m-1}(y) = \int_D \operatorname{div} w(x)\, \mathrm{d}x . \tag{38}$$

In the rest of this chapter we shall always assume that

$$\sup \{v_\infty(y);\, y \in B\} = V < \infty . \tag{39}$$

As shown in theorem 2.13 in [10], (39) implies

$$\sup \{v_\infty(x);\, x \in R^m\} \leqq V + H_{m-1}(\Gamma) . \tag{40}$$

Consequently, (38) is valid for each w satisfying the assumptions of the above proposition. This makes it possible to derive another useful integral representation for Wf.

2.5. Lemma. *If f is a bounded Baire function on C then, for each $\zeta = [\xi, \tau] \in R_{T_2} \setminus R_{T_1}$,*

$$Wf(\zeta) = \int_{T_1}^{T_2} \mathrm{d}t \int_B f(x, t)\, n(x) \,.\, \hat{\nabla} G(x - \xi, t - \tau)\, \mathrm{d}H_{m-1}(x) , \tag{41}$$

where $n(x)$ is the exterior normal of D at x as defined in 2.4.

Proof. Fix $\zeta = [\xi, \tau] \in R_{T_2} \setminus R_{T_1}$ and suppose first that $f \in \mathcal{D}_{T_2}$ and f vanishes in some neighbourhood of ζ. Choose $\tilde{G} \in \mathcal{D}$ so that

$$\tilde{G}(z) = G(z - \zeta)$$

for all z in some neighborhood of spt f and fix $t \in (T_1, T_2)$. Then (38) applies to $w = [w_1, \ldots, w_m]$ defined by

$$w_j(y) = f(y, t) \,.\, \partial_j \tilde{G}(y, t)$$

and we get for

$$I(t) = \int_B f(y, t)\, n(y) \,.\, \hat{\nabla} G(y - \xi, t - \tau)\, \mathrm{d}H_{m-1}(y) =$$

$$= \int_D \sum_{j=1}^{m} [\partial_j f(x, t)\, \partial_j \tilde{G}(x, t) + f(x, t)\, \partial_j^2 \tilde{G}(x, t)]\, \mathrm{d}x \,.$$

Noting that

$$f \sum_{j=1}^{m} \partial_j^2 \tilde{G} = f \partial_{m+1} \tilde{G}$$

we obtain finally

$$\int_{T_1}^{T_2} I(t)\, \mathrm{d}t = \int_{T_1}^{T_2} \mathrm{d}t \int_D [\hat{\nabla} f(x, t) . \hat{\nabla} G(x - \xi, t - \tau) - G(x - \xi, t - \tau)\, \partial_{m+1} f(x, t)]\, \mathrm{d}x =$$

$$= \tilde{W} f(\zeta) \leqq W f(\zeta) \,.$$

Letting

$$\mathscr{D}^1 = \mathscr{D}_{T_2} \cap \{f; |f| \leqq 1, \zeta \notin \operatorname{spt} f\}$$

we conclude from proposition 1.8 that

$$\int_{T_1}^{T_2} \mathrm{d}t \int_B |n(x) \,.\, \hat{\nabla} G(x - \xi, t - \tau)|\, \mathrm{d}H_{m-1}(x) = \tag{42}$$

$$= \sup \{\tilde{W} f(\zeta); f \in \mathscr{D}^1\} \leqq 2^{m-1} \Gamma(\tfrac{1}{2}m)\, v_\infty(\xi) < \infty \,.$$

We have so far verified that (41) is valid for $f \in \mathscr{D}_{T_2}$ vanishing near ζ. Using corollary 2.3 and (42) one easily shows that (41) holds for an arbitrary bounded Baire function f on C.

We shall now investigate the behavior of $Wf(z)$ for z approaching C. The following result is an analogue of theorem 2.15 in [10].

2.6. Theorem. *Let*

$$D^i = R^m \cap \{x; d_D(x) = i\} \,, \quad i = 0, 1 \,.$$

Fix $\zeta = [\xi, \tau] \in C$ *and suppose that* f *is a bounded Baire function on* C *such that*

$$\lim_{\substack{z \to \zeta \\ z \in C}} f(z) = \alpha \,. \tag{43}$$

Then, for $i = 0, 1$,

$$(z \in D^i \times \langle T_1, T_2), z \to \zeta) \Rightarrow Wf(z) \to Wf(\zeta) + \alpha[d_D(\xi) - i]\, 2^m \pi^{m/2} \,. \tag{44}$$

Proof. We shall assume that $T_2 = +\infty$ (for we may always extend f to $B \times \times \langle T_1, +\infty)$ defining $f(y, t) = 0$ for $t \in \langle T_2, +\infty)$ and $y \in B$). We are going to evaluate $Wf(\zeta)$ for any $\zeta = [\xi, \tau] \in R^m \times \langle T_1, +\infty)$ assuming that

$$f = 1 \quad \text{on} \quad C = B \times \langle T_1, +\infty) . \tag{45}$$

Define K_ξ as in the proof of lemma 2.1. Fix $\theta \in \Gamma$ with $n_\infty(\theta, \xi) < +\infty$ and let

$$D_\theta = \{\varrho; \varrho > 0, \xi + \varrho\theta \in D\} .$$

If $\varrho_1 < \ldots < \varrho_n$ are all the hits of $(0, +\infty)$ on D_θ, then (30) holds. Besides that,

$$\theta \in K_\xi \Rightarrow s(\varrho_1; \xi, \theta) = -1 ,$$
$$\theta \in \Gamma \setminus K_\xi \Rightarrow s(\varrho_1; \xi, \theta) = 1$$

and $s(\varrho_n; \xi, \theta) = -1$ or $s(\varrho_n; \xi, \theta) = 1$ according as D is bounded or not. We thus conclude for bounded D

$$(\theta \in K_\xi, \ \eta > 0) \Rightarrow \textstyle\sum_f(\zeta; \eta, \theta) = -1 ,$$
$$(\theta \in \Gamma \setminus K_\xi, \ \eta > 0) \Rightarrow \textstyle\sum_f(\zeta, \eta, \theta) = 0 ,$$

while for unbounded D

$$\theta \in K_\xi \quad \Rightarrow \textstyle\sum_f(\zeta; \eta, \theta) = 0 \quad \text{for all} \quad \eta > 0 ,$$
$$\theta \in \Gamma \setminus K_\xi \Rightarrow \textstyle\sum_f(\zeta; \eta, \theta) = 1 \quad \text{for all} \quad \eta > 0 .$$

Employing (35) we obtain for bounded D

$$Wf(\zeta) = -2^m d_D(\xi)\, \pi^{m/2} ,$$

while for unbounded D

$$Wf(\zeta) = +2^m[1 - d_D(\xi)]\, \pi^{m/2} .$$

Since $\xi \in R^m$ was arbitrary, we see that (44) holds with $\alpha = 1$ for f satisfying (45). It remains to verify (44) provided (43) holds with $\alpha = 0$. We may clearly assume that $f(\zeta) = 0$, too. Then, for any $\varepsilon > 0$, there is a decomposition $f = f_\varepsilon + g_\varepsilon$ such that g_ε is a bounded Baire function vanishing in some nieghborhood of ζ in C and $|f_\varepsilon| \leqq \varepsilon$ on C. It follows from lemma 2.5 that

$$\lim_{z \to \xi} Wg_\varepsilon(z) = Wg_\varepsilon(\zeta) .$$

On the other hand, (36) together with (40) imply

$$|Wf_\varepsilon(z)| \leqq 2^{m-1}\varepsilon[\Gamma(\tfrac{1}{2}m)\, V + 2\pi^{m/2}]$$

for all $z \in R^m \times \langle T_1, +\infty)$. Since $\varepsilon > 0$ can be chosen as small as we want, we conclude that in this case

$$Wf(\zeta) = \lim_{z \to \zeta} Wf(z), \quad z \in R^m \times \langle T_1, +\infty),$$

and the proof is complete.

2.7. Definition. Let $-\infty < T_1 < T_2 < +\infty$ and denote by $\mathscr{B} = \mathscr{B}(T_1, T_2)$ the Banach space of all continuous functions on $B \times \langle T_1, T_2 \rangle$ vanishing on $B \times \{T_2\}$, equipped with the supremum norm. Given $f \in \mathscr{B}$ and $\alpha \in R^1$ define $W_\alpha f$ on $B \times \times \langle T_1, T_2 \rangle$ letting for $\xi \in B$

$$W_\alpha f(\xi, T_2) = 0,$$

$$W_\alpha f(\xi, \tau) = Wf(\xi, \tau) + 2^m \pi^{m/2} [d_D(\xi) - \alpha] f(\xi, \tau), \quad T_1 \leqq \tau < T_2.$$

2.8. Lemma. *Fix $\alpha \in R^1$. Then*

$$f \in \mathscr{B} \Rightarrow W_\alpha f \in \mathscr{B}.$$

The operator $W_\alpha : f \to W_\alpha f$ is bounded on $\mathscr{B}$ and

$$\|W_\alpha\| \leqq [V\Gamma(\tfrac{1}{2}m) + (1 + |\alpha|)\, 2\pi^{m/2}]\, 2^{m-1}. \tag{46}$$

If I' stands for the identity operator on $\mathscr{B}'$, then the operator

$$H_\alpha = H - \alpha 2^m \pi^{m/2} I'$$

(acting on $\mathscr{B}'$) is dual to W_α.

Proof. Fix $f \in \mathscr{B}$ and define F on $B \times \langle T_1, +\infty)$ so that $F = f$ on $B \times \langle T_1, T_2 \rangle$, $F = 0$ on $B \times (T_2, +\infty)$. Then F is continuous on

$$C_\infty = B \times \langle T_1, +\infty).$$

According to theorem 2.6 (where now C is replaced by C_∞),

$$L(\xi, \tau) = \lim WF(x, t)\, (x \in D,\, x \to \xi;\, t > T_1,\, t \to \tau)$$

is defined for $[\xi, \tau] \in C_\infty$ and

$$WF(\xi, \tau) + 2^m \pi^{m/2} d_D(\xi)\, F(\xi, \tau) = 2^m \pi^{m/2} F(\xi, \tau) + L(\xi, \tau)$$

is a continuous function of $[\xi, \tau] \in C_\infty$ vanishing on $B \times \langle T_2, +\infty)$. Noting that, for $[\xi, \tau] \in B \times \langle T_1, T_2 \rangle = \operatorname{cl} C$, $W_\alpha f(\xi, \tau)$ coincides with

$$WF(\xi, \tau) + 2^m \pi^{m/2} [d_D(\xi) - \alpha]\, F(\xi, \tau),$$

we conclude that $W_\alpha f \in \mathscr{B}$. The estimate (46) follows at once from the definition of W_α and (36).

If F is a function with domain containing cl C such that $f = F \mid \text{cl}\, C$ (= the restriction of F to cl C) belongs to $\mathscr{B}$, we agree to use $W_\alpha F$ to denote $W_\alpha f$.

Consider now $\varphi \in \mathscr{D}_{T_2}$. It follows from 2.7 and 2.2 that, for $\zeta \in C$,

$$W_\alpha\, \varphi(\zeta) = \tilde{W}\, \varphi(\zeta) - 2^m \pi^{m/2} \alpha\, \varphi(\zeta)\,.$$

Employing (20) and (27) we conclude that

$$\langle W_\alpha \varphi, \mu \rangle = \langle \varphi, H_\alpha \mu \rangle\,, \quad \mu \in \mathscr{B}'\,, \quad \varphi \in \mathscr{D}_{T_2}\,.$$

If $f \in \mathscr{B}$, then there is a sequence $\varphi_n \in \mathscr{D}_{T_2}$ $(n = 1, 2, \ldots)$ such that $\varphi_n \to f$ uniformly on $B \times \langle T_1, T_2 \rangle$ as $n \to \infty$. Hence it follows that

$$(\mu \in \mathscr{B}', f \in \mathscr{B}) \Rightarrow \langle W_\alpha f, \mu \rangle = \langle f, H_\alpha \mu \rangle$$

and the proof is complete.

Remark. Let us denote by I the identity operator acting on $\mathscr{B}$. It follows from 2.8 that the operator H is dual to

$$\alpha\, 2^m \pi^{m/2} I + W_\alpha\,.$$

Accordingly, the following simple result appears to be useful in connection with investigations of the range of H.

2.9. Proposition. *Fix $\alpha, \beta \in R^1$ and denote by $\mathscr{B}_{\alpha\beta}$ the class of all $f \in \mathscr{B}$ satisfying*

$$(\beta I + W_\alpha) f = 0\,.$$

Then $\mathscr{B}_{\alpha\beta}$ is a subspace of $\mathscr{B}$ which is either trivial (i.e., the function vanishing identically on cl C *is the only element of $\mathscr{B}_{\alpha\beta}$) or infinite dimensional.*

Proof. For $\varepsilon \geqq 0$ and $f \in \mathscr{B}$ define $T^\varepsilon f$ as follows. Given $\xi \in B$, let $\mathscr{J} = \langle T_1, T_2 \rangle$ and put

$$T^\varepsilon f(\xi, t) = 0 \quad \text{for} \quad t \in \mathscr{J} \cap \langle T_2 - \varepsilon, +\infty)\,,$$
$$T^\varepsilon f(\xi, t) = f(\xi, t + \varepsilon) \quad \text{for} \quad t \in \mathscr{J} \cap (-\infty, T_2 - \varepsilon)\,.$$

Clearly, $T^\varepsilon(\mathscr{B}) \subset \mathscr{B}$ for each $\varepsilon > 0$. It follows easily from the definition of W_α and 2.5 (note also that $G(z) = 0$ for $z \in \text{cl}\, R_0$) that

$$W_\alpha T^\varepsilon f = T^\varepsilon W_\alpha f\,, \quad f \in \mathscr{B}\,.$$

Consequently, also $\mathscr{B}_{\alpha\beta}$ is translation invariant in the sense that $T^\varepsilon(\mathscr{B}_{\alpha\beta}) \subset \mathscr{B}_{\alpha\beta}$ for each $\varepsilon > 0$. Now it is sufficient to employ the following elementary lemma:

Let g be a continuous function on $\mathscr{J}$, $g(T_2) = 0$, and define for each $\varepsilon \geqq 0$

$$T^{\varepsilon} g(t) = 0 \quad \text{for} \quad t \in \mathscr{J} \cap \langle T_2 - \varepsilon, +\infty) ,$$
$$T^{\varepsilon} g(t) = g(t + \varepsilon) \quad \text{for} \quad t \in \mathscr{J} \cap (-\infty, T_2 - \varepsilon) .$$

If

$$\tau = \inf \{t;\, t \in \mathscr{J},\, g(t) = 0\} > T_1 ,$$

then, for each choice of $n \in N$ and $\varepsilon > 0$ with

$$T_1 + n\varepsilon < \tau ,$$

the functions in $\{T^{(k-1)\varepsilon} g\}_{k=1}^{n}$ are linearly independent.

Indeed, for $k = 1, \ldots, n$, $T^{(k-1)\varepsilon} g$ does not vanish identically on

$$\mathscr{J}_{k\varepsilon} = \langle \tau - k\varepsilon, \tau - (k-1)\,\varepsilon \rangle ,$$

while all $T^{j\varepsilon} g$ with $j \geqq k$ do vanish on $\mathscr{J}_{k\varepsilon}$. The rest is obvious.

CHAPTER 3

Unless the contrary is explicitly stated, in this chapter we always assume that

$$\sup \{v_{\infty}(\xi);\, \xi \in B\} = V < +\infty . \tag{47}$$

We proceed to investigate the dual equations

$$H\mu = v \;(\text{over } \mathscr{B}') , \quad W_0 f = g \;(\text{over } \mathscr{B})$$

associated with the Fourier problem. The methods usually used when B is a sufficiently smooth hypersurface are no longer applicable under the general assumption (47). (Under appropriate smoothness assumptions on B the resolvent of the resulting integral equation can be evaluated in the form of a series; cf. [20], where also further references to the work of E. Holmgren, E. Levi, M. Gevrey, H. Müntz, S. G. Michlin, A. N. Tichonov may be found. See also [7], [8], [11], [13], [14], [17].)

We consider the decompositions

$$H = 2^m \pi^{m/2} \alpha I' + H_\alpha , \quad W_0 = 2^m \pi^{m/2} \alpha I + W_\alpha$$

and evaluate the Fredholm radius of W_α, which is the reciprocal of the quantity

$$\omega W_\alpha = \inf_{Q} \|W - Q\| ,$$

where Q ranges over all compact operators acting on $\mathscr{B}$. It appears that ωW_α can be expressed in geometric terms connected with D. This makes it possible to find the optimal value γ of the parameter α in dependence on the shape of D and establish conditions on D guaranteeing

$$\omega W_\gamma < 2^m \pi^{m/2} |\gamma| .$$

The Riesz-Schauder theory together with proposition 2.9 then yield the desired result concerning the Fourier problem.

Remark. We shall see that the optimal value of the parameter α, for which $\omega W_\alpha / |\alpha|$ attains its minimum, equals $\frac{1}{2}$ if $d_D(x) = \frac{1}{2}$ for all $x \in B$. This naturally occurs if B is a smooth hypersurface. It is interesting to observe that under the assumption (47) the optimal value of the parameter may be different from $\frac{1}{2}$ (see 3.9 below).

It should be noted here that already J. RADON considered the quantity corresponding to ωW_α for special choice of α in his investigations of the logarithmic potential; he evaluated it for plane domains bounded by curves with bounded rotation (see [21], [22]). Compare also [10], [24] treating boundary value problems for Newtonian potentials in n-space.

3.1. Notation. Throughout this chapter we assume that $-\infty < T_1 < T_2 < +\infty$. Given $\varepsilon, \delta > 0$ and $\zeta = [\xi, \tau] \in B \times \langle T_1, T_2 \rangle = \operatorname{cl} C$, we denote by $\chi_\zeta^{\varepsilon\delta}$ the characteristic function of

$$M_\zeta(\varepsilon, \delta) = R^{m+1} \setminus [\Omega_\varepsilon(\xi) \times (\tau - \delta, \tau + \delta)] .$$

$\hat{B}$ and n will denote the reduced boundary and exterior normal of D, respectively, as defined in 2.4. For $0 < r < \varepsilon$ put

$$q_\varepsilon(r) = \sup_{x \in B} H_{m-1}\{\hat{B} \cap [\Omega_{\varepsilon+r}(x) \setminus \Omega_{\varepsilon-r}(x)]\} . \tag{48}$$

We define for each bounded Baire function f on cl C and $\zeta = [\xi, \tau] \in \operatorname{cl} C$

$$W^{\varepsilon\delta} f(\zeta) = \int_{T_1}^{T_2} \mathrm{d}t \int_B \chi_\zeta^{\varepsilon\delta}(x, t) f(x, t)\, n(x) \,.\, \hat{\nabla} G(x - \xi, t - \tau)\, \mathrm{d}H_{m-1}(x) .$$

3.2. Lemma. *Fix $\varepsilon, \delta > 0$. Then there is a positive constant $c \in R^1$ such that*

$$|W^{\varepsilon\delta} f(\zeta) - W^{\varepsilon\delta} f(\bar{\zeta})| \leqq c[q_\varepsilon(|\zeta - \bar{\zeta}|) + |\zeta - \bar{\zeta}|] \tag{49}$$

for each Baire function f satisfying

$$\sup \{|f(z)|; z \in \operatorname{cl} C\} \leqq 1 \tag{50}$$

and each couple of points $\zeta = [\xi, \tau]$, $\bar{\zeta} = [\bar{\xi}, \bar{\tau}]$ in cl C *satisfying*

$$|\xi - \bar{\xi}| < \tfrac{1}{2}\varepsilon , \quad |\tau - \bar{\tau}| < \tfrac{1}{2}\delta . \tag{51}$$

Proof. It is easy to see that there is a $c_1 \in R^1$ such that

$$|\hat{\nabla} G| \leqq c_1$$

in

$$M_0(\tfrac{1}{2}\varepsilon, \tfrac{1}{2}\delta) = \{z; |z_{m+1}| > \tfrac{1}{2}\delta\} \cup \{z; |\hat{z}| > \tfrac{1}{2}\varepsilon\}$$

and

$$|\hat{\nabla} G(u) - \hat{\nabla} G(\bar{u})| \leqq c_1 |u - \bar{u}|$$

for each couple of points $u, \bar{u}$ in $M_0(\tfrac{1}{2}\varepsilon, \tfrac{1}{2}\delta)$. Consider now $\zeta = [\xi, \tau]$, $\bar{\zeta} = [\bar{\xi}, \bar{\tau}] \in R^{m+1}$ satisfying (51) and suppose that f is a Baire function on cl C satisfying (50). Writing $z = [x, t]$,

$$J_1 = \int_{T_1}^{T_2} \mathrm{d}t \int_B \chi_\zeta^{\varepsilon\delta}(z) f(z)\, n(x) \,.\, [\hat{\nabla} G(z - \zeta) - \hat{\nabla} G(z - \bar{\zeta})] \,\mathrm{d}H_{m-1}(x)\,,$$

$$J_2 = \int_{T_1}^{T_2} \mathrm{d}t \int_B [\chi_\zeta^{\varepsilon\delta}(z) - \chi_{\bar{\zeta}}^{\varepsilon\delta}(z)] f(z)\, n(x) \,.\, \hat{\nabla} G(z - \bar{\zeta}) \,\mathrm{d}H_{m-1}(x)\,,$$

we have

(52) $$|W^{\varepsilon\delta} f(\zeta) - W^{\varepsilon\delta} f(\bar{\zeta})| \leqq |J_1| + |J_2|\,.$$

If z is in $M_\zeta(\varepsilon, \delta)$ then, in view of (51), both $z - \zeta$ and $z - \bar{\zeta}$ belong to $M_0(\tfrac{1}{2}\varepsilon, \tfrac{1}{2}\delta)$, so that

(53) $$|J_1| \leqq c_1 |\zeta - \bar{\zeta}|\, H_{m-1}(\hat{B})\,(T_2 - T_1)\,.$$

Put

$$R = \varepsilon + |\xi - \bar{\xi}|\,, \quad r = \varepsilon - |\xi - \bar{\xi}|\,.$$

Then the symmetric difference of $M_\zeta(\varepsilon, \delta)$ and $M_{\bar{\zeta}}(\varepsilon, \delta)$ is contained in the union of

$$[\Omega_R(\xi) \setminus \Omega_r(\xi)] \times R^1$$

and

$$R^m \times \{t; \delta - |\tau - \bar{\tau}| \leqq |t - \tau| < \delta + |\tau - \bar{\tau}|\}\,.$$

Hence it follows

(54) $$|J_2| \leqq c_1 q_\varepsilon(|\xi - \bar{\xi}|)\,(T_2 - T_1) + 4 c_1 H_{m-1}(\hat{B})\, |\tau - \bar{\tau}|\,.$$

Combining (52), (53) and (54) we get (49) with

$$c = \max\{c_1(T_2 - T_1),\, c_1 H_{m-1}(\hat{B})\,[(T_2 - T_1) + 4]\}\,.$$

3.3. Lemma. *Given* $\varepsilon, \delta > 0$ *and* $\zeta = [\xi, \tau] \in R^{m+1}$, denote by $\varkappa_\zeta^{\varepsilon\delta}$ *the characteristic function of*

$$\Omega_\varepsilon(\xi) \times (\tau, \tau + \delta)$$

and define

$$v^{\varepsilon\delta}(\zeta) = \int_{T_1}^{T_2} \mathrm{d}t \int_B \varkappa_\zeta^{\varepsilon\delta}(x, t) \, |n(x) \, . \, \hat{\nabla} G(x - \xi, t - \tau)| \, \mathrm{d}H_{m-1}(x) \, .$$

If $M \subset C$ *is dense in* C, *then*

(55)
$$\omega W_\alpha \leqq \sup_{\zeta \in M} \{2^m \pi^{m/2} |d_D(\hat{\zeta}) - \alpha| + v^{\varepsilon\delta}(\zeta)\} = \\ = \sup_{\zeta \in \zeta} \{2^m \pi^{m/2} |d_D(\hat{\zeta}) - \alpha| + v^{\varepsilon\delta}(\zeta)\}$$

for all $\alpha \in R^1$ *and* $\varepsilon, \delta > 0$.

Proof. Fix $\alpha \in R^1$ and $\varepsilon, \delta > 0$. Noting that $v^{\varepsilon\delta}(\zeta)$ is a non-decreasing function of the variable $\varepsilon > 0$, we may assume for the proof of (55) that

(56)
$$x \in B \Rightarrow H_{m-1}[\hat{B} \cap \mathrm{fr}\, \Omega_\varepsilon(x)] = 0 \, .$$

Indeed, the set of those $\varepsilon > 0$ for which (56) is violated is at most countable, because

$$\sum_{i=1}^{n} H_{m-1}(\hat{B} \cap S_i) \leqq H_{m-1}(\hat{B}) < +\infty$$

for each choice of spheres $S_i = \mathrm{fr}\, \Omega_{\varepsilon_i}(x^i)$ with mutually different radii $\varepsilon_1 > \ldots > \varepsilon_n$ and arbitrary $n \in N$.

Defining q_ε by (48) we conclude from (56) that

$$\lim_{r \to 0+} q_\varepsilon(r) = 0 \, .$$

It follows from lemma 3.2 that all the functions in

(57)
$$\{W^{\varepsilon\delta} f; f \in \mathscr{B}, \|f\| \leqq 1\}$$

are equicontinuous on cl C. Employing (36) one easily sees that (57) is contained in $\mathscr{B} \cap \{g; \|g\| \leqq 2^{m-1} \Gamma(\frac{1}{2}m) V\}$ (see also (47)). Consequently, $W^{\varepsilon\delta}: f \to W^{\varepsilon\delta}$ is a compact operator on $\mathscr{B}$ and

(58)
$$\omega W_\alpha \leqq \|W_\alpha - W^{\varepsilon\delta}\| \, .$$

Noting that G vanishes on cl R_0 we get from (41) for $f \in \mathscr{B}$ and $\zeta = [\xi, \tau] \in$ cl C

$$(W_\alpha - W^{\varepsilon\delta}) f(\zeta) = 2^m \pi^{m/2} [d_D(\xi) - \alpha] f(\zeta) + \\ + \int_{T_1}^{T_2} \mathrm{d}t \int_B f(x, t) \, \varkappa_\zeta^{\varepsilon\delta}(x, t) \, n(x) \, . \, \hat{\nabla} G(x - \xi, t - \tau) \, \mathrm{d}H_{m-1}(x) \, ,$$

whence it follows that

$$2^m \pi^{m/2} |d_D(\hat{\zeta}) - \alpha| + v^{\varepsilon\delta}(\zeta) =$$
$$= \sup \{(W_\alpha - W^{\varepsilon\delta}) f(\zeta); f \in \mathscr{B}, \|f\| \leqq 1\}$$

is a lower-semicontinuous function of $\zeta \in \operatorname{cl} C$ and

$$\|W_\alpha - W^{\varepsilon\delta}\| = \sup_{\zeta \in M} \{2^m \pi^{m/2} |d_D(\hat{\zeta}) - \alpha| + v^{\varepsilon\delta}(\zeta)\} =$$
$$= \sup_{\zeta \in C} \{2^m \pi^{m/2} |d_D(\hat{\zeta}) - \alpha| + v^{\varepsilon\delta}(\zeta)\} .$$

This together with (58) completes the proof.

The following slight modification of a known result due to J. RADON will be needed below.

3.4. Lemma. *If Q is a compact operator on $\mathscr{B}$ then, for every $\varepsilon > 0$, there exist $f_1, \ldots, f_s \in \mathscr{B}$ and $\mu_1, \ldots, \mu_s \in \mathscr{B}'$ such that the operator*

(59) $$Q_\varepsilon : f \to \sum_{i=1}^{s} \langle f, \mu_i \rangle f_i , \quad f \in \mathscr{B} ,$$

satisfies

(60) $$\|Q - Q_\varepsilon\| \leqq \varepsilon .$$

Proof. For $z \in \operatorname{cl} C$ define $\Phi(z) \in \mathscr{B}'$ by

$$\langle f, \Phi(z) \rangle = Q f(z) , \quad f \in \mathscr{B} .$$

Accordingly,

$$(z \in \operatorname{cl} C,\ z_{m+1} = T_2) \Rightarrow \|\Phi(z)\| = 0 .$$

Since Q is a compact operator on $\mathscr{B}$, Φ is a continuous map on cl B to $\mathscr{B}'$ (compare [22], chap. V, n°90, p. 218). Consequently, we may fix $T \in (T_1, T_2)$ such that

$$(z \in C,\ z_{m+1} > T) \Rightarrow \|\Phi(z)\| < \varepsilon .$$

Further choose open sets $U_1, \ldots, U_s$ with

$$C \cap \operatorname{cl} R_T \subset \bigcup_{i=1}^{s} U_i \subset R_{T_2}$$

such that, for $i = 1, \ldots, s$,

$$(z, \bar{z} \in C \cap U_i) \Rightarrow \|\Phi(z) - \Phi(\bar{z})\| < \varepsilon .$$

Put

$$U_0 = R^{m+1} \cap \{z;\ z_{m+1} > T\} ,$$

so that

$$U_0, U_1, \ldots, U_s \tag{61}$$

is an open covering of cl C. Associate with (61) the decomposition of unity formed by continuous non-negative functions $f_0, f_1, \ldots, f_s$ on cl C such that

$$\operatorname{spt} f_j \subset U_j (0 \leqq j \leqq s), \quad \sum_{j=0}^{s} f_j = 1 \quad \text{on} \quad \text{cl}\, C .$$

Fix $z^i \in C \cap U_i$, put $\mu_i = \Phi(z^i)$ $(1 \leqq i \leqq s)$ and define Q_ε by (59). Consider now an arbitrary $f \in \mathscr{B}$ with $\|f\| \leqq 1$. We have

$$(Q - Q_\varepsilon) f(z) = \langle f, \Phi(z) \rangle f_0(z) + \sum_{i=1}^{s} \langle f, \Phi(z) - \Phi(z^i) \rangle f_i(z) . \tag{62}$$

Since f_0 vanishes outside U_0 and $\|\Phi(z)\| < \varepsilon$ for $z \in U_0 \cap \text{cl}\, C$, we have

$$|\langle f, \Phi(z) \rangle f_0(z)| \leqq \varepsilon f_0(z) . \tag{63}$$

Note that $f_1, \ldots, f_s$ vanish outside $\bigcup_{i=1}^{s} U_i$, while

$$\|\Phi(z) - \Phi(z^i)\| < \varepsilon$$

for z in

$$U_i \cap \text{cl}\, C \supset \{z; f_i(z) \neq 0\} , \quad i = 1, \ldots, s .$$

Consequently,

$$\left|\sum_{i=1}^{s} \langle f, \Phi(z) - \Phi(z^i) \rangle f_i(z)\right| \leqq \varepsilon[1 - f_0(z)] . \tag{64}$$

Combining (64), (63) and (62) we get (60).

3.5. Lemma. *Let us keep the notation from lemma 3.3. Then*

$$\omega W_\alpha \geqq \lim_{\varepsilon, \delta \to 0+} \sup_{\zeta \in C} \{2^m \pi^{m/2} |d_D(\zeta) - \alpha| + v^{\varepsilon\delta}(\zeta)\}$$

for every $\alpha \in R^1$.

Proof. Fix $\alpha \in R^1$ and let

$$k > \omega W_\alpha . \tag{65}$$

According to lemma 3.4, there are $f_1, \ldots, f_s \in \mathscr{B}$ and $\mu_1, \ldots, \mu_s \in \mathscr{B}'$ such that the operator

$$Q : f \to \sum_{i=1}^{s} \langle f, \mu_i \rangle f_i , \quad f \in \mathscr{B} ,$$

satisfies

$$k > \|W_\alpha - Q\| . \tag{66}$$

Writing c_M for the characteristic function of M we associate with each $\zeta = [\xi, \tau] \in C$ the measure ν_ζ defined on the system of Borel sets $M \subset R^{m+1}$ by

$$\nu_\zeta(M) = \int_{T_1}^{T_2} \mathrm{d}t \int_B c_M(x, t)\, n(x) \,.\, \hat{\nabla} G(x - \xi, t - \tau)\, \mathrm{d}H_{m-1}(x) \,.$$

Clearly, $\nu_\zeta \in \mathscr{B}'$. Denoting by δ_ζ the Dirac measure (= unit point mass) concentrated at ζ we have by (41)

$$W_\alpha f(\zeta) = \langle f, \nu_\zeta + 2^m \pi^{m/2} [d_D(\zeta) - \alpha]\, \delta_\zeta \rangle \,, \quad \zeta \in C \,, \quad f \in \mathscr{B} \,.$$

Let

$$C_0 = C \cap \{\zeta; \sum_{i=1}^{s} |\mu_i| (\{\zeta\}) = 0\} \,.$$

Clearly, $C \setminus C_0$ is at most countable. Consequently,

$$|\nu_\zeta| (C \setminus C_0) = 0 = |\nu_\zeta| (\{\zeta\}) \quad \text{for every} \quad \zeta \in C \,. \tag{67}$$

For $i = 1, \ldots, s$ consider the decomposition

$$\mu_i = \mu_i^1 + \mu_i^2 \,,$$

where

$$\mu_i^1, \mu_i^2 \in \mathscr{B}' \,,$$

$$z \in R^{m+1} \Rightarrow |\mu_i^1| (\{z\}) = 0 \,, \tag{68}$$

$$|\mu_i^2| (C_0) = 0 \,. \tag{69}$$

In view of (66)–(69) we have then for each $\zeta \in C_0$

$$\begin{aligned} k > \|W_\alpha - Q\| &> \|\nu_\zeta + 2^m \pi^{m/2} [d_D(\zeta) - \alpha]\, \delta_\zeta - \sum_{i=1}^{s} f_i(\zeta)\, \mu_i\| = \\ &= \|\nu_\zeta - \sum_{i=1}^{s} f_i(\zeta)\, \mu_i^1\| + 2^m \pi^{m/2} |d_D(\zeta) - \alpha| + \\ &+ \|\sum_{i=1}^{s} f_i(\zeta)\, \mu_i^2\| \geqq \|\nu_\zeta - \sum_{i=1}^{s} f_i(\zeta)\, \mu_i^1\| + 2^m \pi^{m/2} |d_D(\zeta) - \alpha| \,. \end{aligned} \tag{70}$$

Put

$$K = \max \{|f_i(z)|;\ z \in \mathrm{cl}\, C,\ 1 \leqq i \leqq s\}$$

and define $\varkappa_\zeta^{\varepsilon\delta}$ as in lemma 3.3. For $\mu \in \mathscr{B}'$, $\varkappa_\zeta^{\varepsilon\delta}\mu \in \mathscr{B}'$ is defined by $\langle f, \varkappa_\zeta^{\varepsilon\delta}\mu\rangle = \langle f\varkappa_\zeta^{\varepsilon\delta}, \mu\rangle$ $(f \in \mathscr{B})$, as usual. Then

$$\left\|v_\zeta - \sum_{i=1}^{s} f_i(\zeta)\,\mu_i\right\| \geqq \left\|\varkappa_\zeta^{\varepsilon\delta}\left[v_\zeta - \sum_{i=1}^{s} f_i(\zeta)\,\mu_i^1\right]\right\| \geqq$$
$$\geqq \left\|\varkappa_\zeta^{\varepsilon\delta}v_\zeta\right\| - K\sum_{i=1}^{s} \sup\left\{\left\|\varkappa_\zeta^{\varepsilon\delta}\mu_i^1\right\|;\ \zeta \in \operatorname{cl} C\right\},$$

whence we conclude by (70)

$$k > \sup_{\zeta\in C_0}\left\{2^m\pi^{m/2}\left|d_D(\zeta) - \alpha\right| + \left\|\varkappa_\zeta^{\varepsilon\delta}v_\zeta\right\|\right\} - K\sum_{i=1}^{s}\sup\left\{\left\|\varkappa_\zeta^{\varepsilon\delta}\mu_i^1\right\|;\ \zeta \in \operatorname{cl} C\right\}.$$

Noting that C_0 is dense in C and $\left\|\varkappa_\zeta^{\varepsilon\delta}v_\zeta\right\| = v^{\varepsilon\delta}(\zeta)$ as introduced in lemma 3.3, we get by (55)

$$\sup_{\zeta\in C_0}\left\{2^m\pi^{m/2}\left|d_D(\zeta) - \alpha\right| + \left\|\varkappa_\zeta^{\varepsilon\delta}v_\zeta\right\|\right\} = \sup_{\zeta\in C}\left\{2^m\pi^{m/2}\left|d_D(\zeta) - \alpha\right| + v^{\varepsilon\delta}(\zeta)\right\}.$$

Since cl C is compact, (68) implies

$$\lim_{\varepsilon,\delta\to 0+}\ \sum_{i=1}^{s}\sup\left\{\left\|\varkappa_\zeta^{\varepsilon\delta}\mu_i^1\right\|;\ \zeta\in\operatorname{cl} C\right\} = 0.$$

Consequently,

$$k \geqq \lim_{\varepsilon,\delta\to 0+}\ \sup_{\zeta\in C}\left\{2^m\pi^{m/2}\left|d_D(\zeta) - \alpha\right| + v^{\varepsilon\delta}(\zeta)\right\},$$

which completes the proof, because k was an arbitrary number satisfying (66).

Combining lemmas 3.3 and 3.5 we obtain at once the following

3.6. Proposition. *If $M \subset C$ is dense in C, then*

$$\omega W_\alpha = \lim_{\varepsilon,\delta\to 0+}\ \sup_{\zeta\in M}\left\{2^m\pi^{m/2}\left|d_D(\zeta) - \alpha\right| + v^{\varepsilon\delta}(\zeta)\right\}.$$

Remark. The above proposition (as well as its proof) is a complete analogue of 3.6 in [10].

The following lemma will enable us to derive a geometric expression for ωW_α.

3.7. Lemma. *Define $v(r; x)$ $(r > 0,\ x \in R^m)$ by (8) and let for $L \subset B$, $L \neq \emptyset$,*

$$V_0(L) = \lim_{r\to 0+}\ \sup_{x\in L} v(r; x).$$

Then

$$2^{m-1}\,\Gamma(\tfrac{1}{2}m)\,V_0(L) = \lim_{\varepsilon,\delta\to 0+}\ \sup\left\{v^{\varepsilon\delta}(\zeta);\ \zeta\in L\times\langle T_1, T_2)\right\},$$

where $v^{\varepsilon\delta}(\zeta)$ has the meaning described in 3.3.

Proof. Fix $L \subset B$, $L \neq \emptyset$. Note that $v^{\varepsilon\delta}(\zeta)$ is a non-decreasing function of each of the variables $\varepsilon, \delta > 0$ separately and

$$\sup \{v^{\varepsilon\delta}(\xi, \tau);\ \tau \in \langle T_1, T_2)\} = v^{\varepsilon\delta}(\xi, T_1), \quad \xi \in B.$$

Hence

$$(71) \qquad \lim_{\varepsilon,\delta\to 0+} \sup \{v^{\varepsilon\delta}(\zeta);\ \zeta \in L \times \langle T_1, T_2)\} =$$

$$\lim_{\varepsilon\to 0+} \sup \{v^{\varepsilon\delta}(\xi, T_1);\ \xi \in L,\ \delta = \tfrac{1}{4}\varepsilon^4\}.$$

Put

$$r(\eta, \varepsilon) = \min(\varepsilon, \varepsilon^2 \sqrt{\eta}), \quad \eta > 0,$$

and consider ε, δ satisfying

$$0 < \varepsilon < [4(T_2 - T_1)]^{1/4}, \quad \delta = \tfrac{1}{4}\varepsilon^4.$$

Then

$$(72) \qquad v^{\varepsilon\delta}(\xi, T_1) = 2^{m-1} \int_0^\infty e^{-\eta}\eta^{m/2-1} v(r(\eta, \varepsilon); \xi)\, d\eta,$$

as it follows from (41), 2.2 and 1.8. Since

$$v(r; \xi) \leqq V < +\infty$$

for any $\xi \in B$ and $r > 0$, we get

$$\lim_{\varepsilon\to 0+} \sup_{\xi\in L} \int_{\varepsilon^{-2}}^\infty e^{-\eta}\eta^{m/2-1}\, v(r(\eta, \varepsilon); \xi)\, d\eta = 0.$$

Taking into account that $r(\eta, \varepsilon) = \varepsilon^2 \sqrt{\eta}$ for $0 \leqq \eta \leqq \varepsilon^{-2}$ we obtain

$$(73) \qquad \lim_{\varepsilon\to 0+} \sup_{\xi\in L} \int_0^\infty e^{-\eta}\eta^{m/2-1}\, v(r(\eta, \varepsilon); \xi)\, d\eta =$$

$$= \lim_{\varepsilon\to 0+} \sup_{\xi\in L} \int_0^\infty e^{-\eta}\eta^{m/2-1}\, v(\varepsilon^2 \sqrt{(\eta)}; \xi)\, d\eta.$$

Note now that

$$\eta \geqq \varepsilon^2 \Rightarrow v(\varepsilon^2 \sqrt{(\eta)}; \xi) \geqq v(\varepsilon^3; \xi),$$

$$\eta \leqq \varepsilon^{-2} \Rightarrow v(\varepsilon^2 \sqrt{(\eta)}; \xi) \leqq v(\varepsilon; \xi).$$

Hence we conclude

$$V_0(L)\,\Gamma(\tfrac{1}{2}m) = \limsup_{\varepsilon\to 0+}\ \sup_{\xi\in L} v(\varepsilon^3;\xi)\int_{\varepsilon^2}^{\varepsilon^{-2}} e^{-\eta}\eta^{m/2-1}\,d\eta \leqq$$

$$\leqq \limsup_{\varepsilon\to 0+}\ \sup_{\xi\in L}\int_0^{\varepsilon^{-2}} e^{-\eta}\eta^{m/2-1}\,v(\varepsilon^2\sqrt{(\eta)};\xi)\,d\eta \leqq$$

$$\leqq \limsup_{\varepsilon\to 0+}\ \sup_{\xi\in L} v(\varepsilon;\xi)\int_0^{\varepsilon^{-2}} e^{-\eta}\eta^{m/2-1}\,d\eta = V_0(L)\,\Gamma(\tfrac{1}{2}m)\,.$$

We see that

$$\limsup_{\varepsilon\to 0+}\ \sup_{\xi\in L}\int_0^{\varepsilon^{-2}} e^{-\eta}\eta^{m/2-1}\,v(\varepsilon^2\sqrt{(\eta)};\xi)\,d\eta = V_0(L)\,\Gamma(\tfrac{1}{2}m)\,,$$

which together with (73), (72), (71) completes the proof.

Remark. It is useful to extend the definition of $V_0(L)$ letting $V_0(\emptyset) = 0$. Now we are in position to evaluate ωW_α as follows.

3.8. Theorem. *Put*

$$B_1 = B \cap \{x;\, d_D(x) = 1\}\,,\quad B_2 = R^m \cap \{x;\, d_D(x) = \tfrac{1}{2}\}$$

and write, for the sake of brevity,

$$\omega_\alpha = \frac{\omega W_\alpha}{2^{m-1}\,\Gamma(\tfrac{1}{2}m)}\,,$$

(74) $$A = 2\pi^{m/2}/\Gamma(\tfrac{1}{2}m)\,,$$

(75) $$V_i = V_0(B_i)\,,\quad i = 1, 2$$

(*see also* 3.7). *Let us distinguish the following three cases*:

(i) $B_1 = \emptyset$ *or* $V_2 \geqq \frac{1}{2}A + V_1$,
(ii) $B_2 = \emptyset$ *or* $V_1 \geqq \frac{1}{2}A + V_2$,
(iii) $B_1 \neq \emptyset \neq B_2$ *and* $|V_1 - V_2| \leqq \frac{1}{2}A$.

Then

(76) $$\omega_\alpha = A|\alpha - \tfrac{1}{2}| + V_2 \quad \textit{in the case (i)}\,,$$

(77) $$\omega_\alpha = A|\alpha - 1| + V_1 \quad \textit{in the case (ii)}\,,$$

while in the case (iii)

(78) $$\omega_\alpha = \tfrac{1}{4}A + \tfrac{1}{2}(V_1 + V_2) + A|\alpha - [\tfrac{3}{4} + (V_1 - V_2)/2A]|\,.$$

Proof. If $x \in B \setminus B_1$, then $d_D(x) < 1$ and each ball $\Omega_r(x)$ $(r > 0)$ meets $R^m \setminus D$ in a set of positive m-measure; since D is open, also

$$H_m(\Omega_r(x) \cap D) > 0 .$$

Hence it follows by the relative isoperimetric inequality for sets with finite perimeter (see Theorem (4.3) in [16] or isoperimetric inequalities for currents established in [4], § 6) that

$$H_{m-1}(\Omega_r(x) \cap \hat{B}) > 0 ,$$

where $\hat{B} \subset B_2$ is the reduced boundary of D as defined in 2.4. In particular, $x \in \text{cl}\, B_2$. We have thus shown that $B_1 \cup B_2$ is dense in B. Put $L_i = B_i \times \langle T_1, T_2 \rangle$, $M = L_1 \cup L_2$, so that M is dense in C and we obtain from 3.6 that

$$\omega W_\alpha = \lim_{\varepsilon,\delta \to 0+} \sup_{\zeta \in M} \{2^m \pi^{m/2} |d_D(\hat{\zeta}) - \alpha| + v^{\varepsilon\delta}(\zeta)\} . \tag{79}$$

If $B_1 = \emptyset$ then $d_D(\hat{\zeta}) = \frac{1}{2}$ for each $\zeta \in M = L_2$ and 3.7 yields

$$\omega W_\alpha = 2^m \pi^{m/2} |\tfrac{1}{2} - \alpha| + 2^{m-1}\, \Gamma(\tfrac{1}{2}m)\, V_2 ,$$

which is in accordance with (76). Similarly, $B_2 = \emptyset$ implies that $d_D(\hat{\zeta}) = 1$ for each $\zeta \in M = L_1$, whence we conclude by 3.7

$$\omega W_\alpha = 2^m \pi^{m/2}\, |1 - \alpha| + 2^{m-1}\, \Gamma(\tfrac{1}{2}m)\, V_1 ,$$

which is the formula occuring in (77). Consider now the case when $B_1 \neq \emptyset \neq B_2$ and let

$$m_i = \lim_{\varepsilon,\delta \to 0+} \sup_{\zeta \in L_i} \{2^m \pi^{m/2} |d_D(\hat{\zeta}) - \alpha| + v^{\varepsilon\delta}(\zeta)\} ,$$

so that (79) implies

$$\omega W_\alpha = \max\, \{m_1, m_2\} .$$

Since $d_D(\hat{\zeta}) = 2^{1-i}$ for each $\zeta \in L_i$ $(i = 1, 2)$, we have by 3.7

$$m_i = 2^m \pi^{m/2} |2^{1-i} - \alpha| + 2^{m-1}\, \Gamma(\tfrac{1}{2}m)\, V_i ,$$

whence it follows after simple calculation that

$$\omega_\alpha = \frac{\max\, \{m_1, m_2\}}{2^{m-1}\, \Gamma(\frac{1}{2}m)}$$

attains the value given by (76), (77) and (78), according as

$$V_2 \geqq \tfrac{1}{2}A + V_1 , \quad V_1 \geqq \tfrac{1}{2}A + V_2 \quad \text{and} \quad |V_1 - V_2| \leqq \tfrac{1}{2}A ,$$

respectively. This completes the proof.

Remark. It would be interesting and useful to evaluate ωW_α^n, where W_α^n is the n-th power of the operator W_α (see [25], chap. X).

In connection with the operator

$$H = (2^m \pi^{m/2} \alpha I + W_\alpha)'$$

it is important to investigate

(80) $$g(\alpha) = \frac{\omega W_\alpha}{2^m \pi^{m/2} |\alpha|} = \frac{\omega_\alpha}{A|\alpha|}, \quad \alpha \neq 0,$$

and evaluate

(81) $$a = \inf \{g(\alpha); \alpha \neq 0\}.$$

Indeed, the condition $a < 1$ permits to apply the Riesz-Schauder theory to H. The above theorem enables us to establish the following corollary.

3.9. Theorem. *Define a by (81), (80), A by (74) and V_i by (75). Then*

(82) $$a < 1$$

holds if and only if

(83) $$V_1 < A \quad \text{and} \quad V_2 < \tfrac{1}{2}A.$$

If the conditions (83) are fulfilled then

(84) $$g(\gamma) = a$$

determines γ uniquely and one of the following three cases must occur:

(i*) $B_1 = \emptyset$,
(ii) $B_2 = \emptyset$ *or* $V_1 \geqq \frac{1}{2}A + V_2$,
(iii) $B_1 \neq \emptyset \neq B_2$ *and* $|V_1 - V_2| \leqq \frac{1}{2}A$.

The corresponding values of a and γ are then given as follows:

$$Aa = 2V_2, \quad \gamma = \tfrac{1}{2} \quad \text{in the case (i*)},$$
$$Aa = V_1, \quad \gamma = 1 \quad \text{in the case (ii)},$$

while in the case (iii)

$$a = \frac{V_1 + V_2 + \frac{1}{2}A}{V_1 - V_2 + \frac{3}{2}A}, \quad \gamma = \frac{3}{4} + \frac{V_1 - V_2}{2A}.$$

Proof. We shall distinguish the cases (i) – (iii) occuring in 3.8. Consider first the case (i). According to 3.8 we have in this case

$$(85) \qquad Ag(\alpha) = \frac{\omega_\alpha}{|\alpha|} = \frac{A + V_2}{\alpha} - A \quad \text{for} \quad 0 < \alpha \leqq \tfrac{1}{2},$$

$$(86) \qquad Ag(\alpha) = A - \frac{\frac{1}{2}A + V_2}{\alpha} \quad \text{for} \quad \alpha < 0,$$

$$(87) \qquad Ag(\alpha) = A + \frac{V_2 - \frac{1}{2}A}{\alpha} \quad \text{for} \quad \alpha \geqq \tfrac{1}{2}.$$

Hence we see that $a < 1$ implies $V_2 < \frac{1}{2}A$, which together with (i) means that $B_1 = \emptyset$ and $V_1 = 0$, so that (83), (i*) are fulfilled. Conversely, if (83) holds, then (85) – (87) show that $a = 2V_2/A < 1$ and a is attained by g at $\gamma = \frac{1}{2}$ only. Now we shall examine the case (ii). As it follows from 3.8,

$$Ag(\alpha) = \frac{V_1 + A}{\alpha} - A \quad \text{for} \quad 0 < \alpha \leqq 1,$$

$$Ag(\alpha) = A - \frac{A + V_1}{\alpha} \quad \text{for} \quad \alpha < 0,$$

$$Ag(\alpha) = A + \frac{V_1 - A}{\alpha} \quad \text{for} \quad \alpha \geqq 1.$$

We see that in this case (82) holds if and only if $V_1 < A$, which is now just the same as (83). If (83) holds, then $a = V_1/A$ is attained by g at $\gamma = 1$ only. Finally, consider the case (iii). Employing 3.8 we get

$$Ag(\alpha) = \frac{V_1 + A}{\alpha} - A \quad \text{for} \quad 0 < \alpha \leqq \frac{V_1 - V_2}{2A} + \frac{3}{4},$$

$$Ag(\alpha) = A - \frac{V_1 + A}{\alpha} \quad \text{for} \quad \alpha < 0,$$

$$Ag(\alpha) = A + \frac{V_2 - \frac{1}{2}A}{\alpha} \quad \text{for} \quad \alpha \geqq \frac{V_1 - V_2}{2A} + \frac{3}{4}.$$

Hence we conclude after simple calculation that $a < 1$ if and only if $V_2 < \frac{1}{2}A$, which is now equivalent with (83). If (83) holds, then g attains its minimum at

$$\gamma = \frac{3}{4} + \frac{V_1 - V_2}{2A}$$

only and

$$g(\gamma) = a = \frac{V_1 + V_2 + \frac{1}{2}A}{V_1 - V_2 + \frac{3}{2}A}.$$

The proof is complete.

Now it is easy to prove the following theorem concerning the second Fourier problem.

3.10. Theorem. *Assume* (83). *Then for each* $\nu \in \mathscr{B}'$ *there is a uniquely determined* $\mu \in \mathscr{B}'$ *such that* $H\mu = \nu$.

Proof. Fix $\gamma \in R^1$ satisfying (84) so that, by theorem 3.9,

$$\frac{\omega W_\gamma}{2^m \pi^{m/2}|\gamma|} = g(\gamma) < 1 .$$

Noting that

$$H = (2^m \pi^{m/2} \gamma I + W_\gamma)'$$

(see 2.8, 2.7) and writing $\beta = 2^m \pi^{m/2} \gamma$ we conclude by the Riesz-Schauder theory that

$$\mathscr{B} \cap \{f;(\beta I + W_\gamma) f = 0\} = \mathscr{B}_{\gamma\beta}$$

is finite dimensional and

$$H(\mathscr{B}') = \mathscr{B}' \cap \{\nu; \langle \mathscr{B}_{\gamma\beta}, \nu\rangle = 0\} .$$

Since $\mathscr{B}_{\gamma\beta}$ is trivial by 2.9, the same must hold of $\mathscr{B}' \cap \{\mu; H\mu = 0\}$ and the proof is complete.

Remark. We see that weak characterization of the normal derivative by means of the functional H permits application of potentialtheoretic methods to the Fourier problem for general domains satisfying (83). (Note that the boundary of such a domain need not be a hypersurface.) As it is well known, weak characterizations of boundary values occur frequently in the literature (see also [26], [10] for further references). We wish to note here that already J. Radon [21] referred to a related concept termed "Plemeljsche Randströmung" when treating the boundary value problems for logarithmic potentials in plane domains bounded by curves with bounded rotation. Unfortunately, the corresponding work of Plemelj [19] has not been available to the present author. Main results of this paper have been announced without proofs in [11].

Employing the integral representation derived in 2.5 one easily verifies that, for every $f \in \mathscr{B}$, $Wf = u$ satisfies the equation

(88) $$\sum_{j=1}^{m} \partial_j^2 u + \partial_{m+1} u = 0$$

in $R^{m+1} \setminus \mathrm{cl}\, C$. By duality based on the Riesz-Schauder theory we obtain thus the following result concerning the first problem of Fourier for the equation (88).

3.11. Theorem. *Assume* (83), *define* $D^0 = R^m \cap \{x; d_D(x) = 0\}$ *and put* $E^0 = D^0 \times \langle T_1, T_2 \rangle$. *Given* $h \in \mathscr{B}$ *there is an* $f \in \mathscr{B}$ *such that, for each* $\zeta \in C \cap \mathrm{cl}\, E^0$,

$$h(\zeta) = \lim Wf(z), \quad z \to \zeta, \quad z \in E^0.$$

If $B \subset \mathrm{cl}\, D^\circ$ *then such an* f *is uniquely determined.*

Proof. Define γ by (84) and put $\beta = 2^m \pi^{m/2} \gamma$. As we have seen in the proof of 3.10, $\mathscr{B}_{\gamma\beta}$ is trivial. Hence we conclude that, given $h \in \mathscr{B}$, there is a uniquely determined $f \in \mathscr{B}$ such that

$$(\beta I + W_\gamma) f = h.$$

Now it is sufficient to employ theorem 2.6 (see also 2.7) showing that for $\zeta \in C \cap \mathrm{cl}\, E^0$

$$(\beta I + W_\gamma) f(\zeta) = \lim Wf(z), \quad z \to \zeta, \quad z \in E^0.$$

Remark. It is easy to see that the assumption (47) introduced in the beginning of this chapter is a consequence of the weaker assumption

(89) $$V_0(B) < +\infty.$$

Indeed, let us drop (47) and assume (89). Then there is an $r > 0$ with

(90) $$\sup_{x \in B} v_r(x) < +\infty.$$

Given $x \in B$ and $\varrho = \frac{1}{2}r$, there is a finite constant K such that

(91) $$\int_{D \cap \Omega_\varrho(x)} \mathrm{div}\, v(x)\, \mathrm{d}x \leq K$$

for every vector-valued function $v = [v_1, \ldots, v_m]$ with infinitely differentiable components v_j satisfying $\mathrm{spt}\, v_j \subset \Omega_\varrho(x)$ $(j = 1, \ldots, m)$,

$$\sum_{j=1}^m v_j^2 \leq 1.$$

This is obvious if $B \cap \Omega_\varrho(x)$ is contained in a hyperplane. In the opposite case one may fix points $x^1, \ldots, x^{m+1} \in B \cap \Omega_\varrho(x)$ that are not situated on a single hyperplane and employ the reasoning described in the proof of 2.10 in [10] to get a finite constant K (depending on the quantities $v_r(x^j)$ and on mutual position of the points $x^1, \ldots, x^{m+1}$) such that (91) holds for all v described above (note also that $\Omega_\varrho(x) \subset$

$\subset \Omega_r(x^j)$, $1 \leqq j \leqq m + 1$). Since B is compact, we conclude that D has finite perimeter and

$$v_\infty(x) = \int_B \frac{|n(y) . (y - x)|}{|y - x|^m} \, \mathrm{d}H_{m-1}(y),$$

$$v_r(x) = \int_{B \cap \Omega_r(x)} \frac{|n(y) . (y - x)|}{|y - x|^m} \, \mathrm{d}H_{m-1}$$

(see 2.12 and 2.8 in [10]). Now it is easy to see that (90) implies (47).

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